

Symbolic Dynamics and Dynamical System Models:

Topological, Categorical and Quantum Dynamics

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Symbol

A **symbol** is something such as an particular mark that represents some piece of information. For example, a red octagon may be a symbol for "STOP". On maps, tables would mean campsite. Numerals are symbols for numbers (amounts). All language consists of symbols. Personal names are symbols representing individuals.

Psychoanalysis and archetypes

Swiss psychoanalyst Carl Jung, who studied archetypes, proposed an alternative definition of symbol, distinguishing it from the term *sign*. In Jung's view, a sign stands for something known, as a word stands for its referent. He contrasted this with symbol, which he used to stand for something that is unknown and that cannot be made clear or precise. An example of a symbol in this sense is Christ as a symbol of the archetype called *self*.^[1] For example, written languages are composed of a variety of different symbols that create words. Through these written words, humans communicate with each other. Kenneth Burke described *Homo sapiens* as a "symbol-using, symbol making, and symbol misusing animal" to indicate that a person creates symbols in her or his life as well as misuses them. One example he uses to indicate his meaning behind symbol misuse is the story of a man who, when told a particular food item was whale blubber, could barely keep from throwing it up. Later, his friend discovered it was actually just a dumpling. But the man's reaction was a direct consequence of the symbol of "blubber" representing something inedible in his mind. In addition, the symbol of "blubber" for the man was created by him through various kinds of learning. Burke emphasizes that humans gain this type of learning that helps us create symbols by seeing various print sources, our life experiences, and symbols about the past.

Burke goes on to describe symbols as also being derived from Sigmund Freud's work on condensation and displacement further stating that they are not just relevant to the theory of dreams, but also to "normal symbol systems". He says they are related through "substitution" where one word, phrase, or symbol is substituted for another in order to change the meaning. In other words, if a person does not understand a certain word or phrase, another person may substitute a synonym or symbol in order to get the meaning of the original word or phrase across. However, when faced with that new way of interpreting a specific symbol, a person may change their already formed ideas to incorporate the new information based on how the symbol is expressed to the person.

Jean Dalby Clift says that people not only add their own interpretations to symbols, they also create personal symbols that represent their own understanding of their lives: what she calls "core images" of the person. She argues that symbolic work with these personal symbols or core images can be as useful as working with dream symbols in psychoanalysis or counseling.^[2]

Etymology

The word *symbol* came to the English language by way of Middle English, from Old French, from Latin, from the Greek σύμβολον (*sýmbolon*) from the root words συν- (*syn-*), meaning "together," and βολή (*bolē*), "a throw", having the approximate meaning of "to throw together", literally a "co-incidence", also "sign, ticket, or contract". The earliest attestation of the term is in the Homeric Hymn to Hermes where Hermes on seeing the tortoise exclaims σύμβολον ἤδη μοι μέγ' ὀνήσιμον "*symbolon* [symbol/sign/portent/encounter/chance find?] of joy to me!" before turning it into a lyre.

Role of context in symbolism

A symbol's meaning may be modified by various factors including popular usage, history, and contextual intent.

Historical meaning

This history of a symbol is one of many factors in determining a particular symbol's apparent meaning. Consequently, symbols with emotive power carry problems analogous to false etymologies.

For example, the Rebel Flag of the American South predates the American Civil War. An early variant of the crossed bars resembled the Scottish Flag.^[3]

Juxtaposition

Juxtaposition^[4] further complicates the matter. Similar five-pointed stars might signify a law enforcement officer or a member of the armed services, depending the uniform.

Notes

[1] *Psychological Types*, C. G. Jung, (trans. Baynes), p. 601.

[2] Jean Dalby Clift, *Core Images of the Self: A Symbolic Approach to Healing and Wholeness*. Crossroad, 1992.

[3] http://books.google.com/books?hl=en&lr=&id=ERsyiUOYI4kC&oi=fnd&pg=PA15&dq=confederate+flag+extremist+groups+ku+klux+klan&ots=7IgVGROsTS&sig=YhygSgtlzsU_ffVgxzTHN11ZPUI

[4] <http://en.wiktionary.org/wiki/juxtaposition>

External links

- symbols.com (<http://www.symbols.com/>)
 - Symbolism Wiki
 - Symbols & signs (<http://www.digiden.nl/en/symbols-and-signs/>)
 - Ancient Symbolism (<http://www.ancient-symbols.com/>)
 - Numericana (<http://www.numericana.com/answer/symbol.htm>)
 - logo search (<http://www.logo-search.com>)
-

Systemics

In the context of systems science and systems philosophy, the term **systemics** refers to an initiative to study systems from a holistic point of view. It is an attempt at developing logical, mathematical, engineering and philosophical paradigms and frameworks in which physical, technological, biological, social, cognitive, and metaphysical systems can be studied and modeled.

The term "systemics" was coined in the 1970s by Mario Bunge and others, as an alternative paradigm for research related to general systems theory and systems science.^[1]

References

[1] Mario Bunge (1979). *A world of systems*. Dordrecht; Boston, Reidel.

Further reading

- Mario Bunge (1979), *A world of systems*. Dordrecht; Boston, Reidel.
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- Watson, D. E., G. E. Schwartz, L. G. S. Russek (1998), The Theory of Enformed Systems - A Paradigm of Organization and Holistic Systems ([http://www.enformy.com/\\$wsr02.html](http://www.enformy.com/$wsr02.html))
- Donald E. Watson (2005), Systemics: The Most Basic Science ([http://www.enformy.com/\\$system.html](http://www.enformy.com/$system.html)).
- Frederic Vester (2008), *The Art of interconnected thinking: Tools and concepts for a new approach to tackling complexity*; Munich, MCB.

External links

- A Taste of Systemics (<http://www.iss.org/taste.html>) By Béla H. Bánáthy
 - Journal of Systemics, Cybernetics and Informatics (<http://www.iiisci.org/Journal/SCI/Home.asp>)
 - Computational Philosophy of Science (<http://mitpress.mit.edu/catalog/item/default.asp?ttype=2&tid=5526>) - The MIT Press
-

Systems science

Systems science is an interdisciplinary field of science that studies the nature of complex systems in nature, society, and science. It aims to develop interdisciplinary foundations, which are applicable in a variety of areas, such as engineering, biology, medicine and social sciences.

Systems sciences covers formal sciences fields like complex systems, cybernetics, dynamical systems theory, and systems theory, and applications in the field of the natural and social sciences and engineering, such as control theory, operations research, social systems theory, systems biology, systems dynamics, systems ecology, systems engineering and systems psychology.

Theories

Since the emergence of the General Systems Research in the 1950s systems thinking and systems science has been developed into all kinds of theoretical frameworks. The following overview will only show the most basic types.

Systems analysis

Systems analysis is the interdisciplinary branch of science, dealing with analysis of systems, often prior to their automation as computer systems, and the interactions within those systems. This field is closely related to operations research.

Systems design

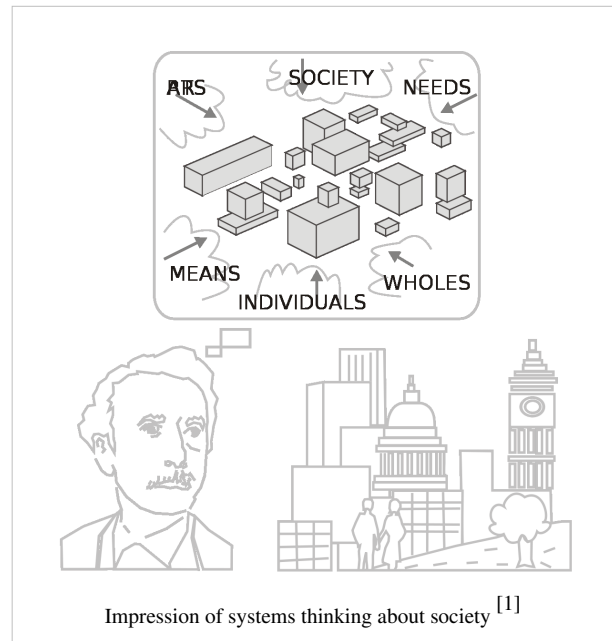
In computing systems design is the process or art of defining the hardware and software architecture, components, modules, interfaces, and data for a computer system to satisfy specified requirements. One could see it as the application of systems theory to computing. Some overlap with the discipline of systems analysis appears inevitable.

System dynamics

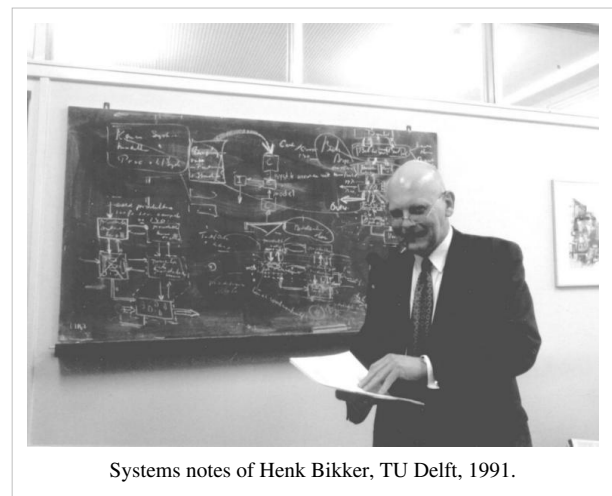
System dynamics is an approach to understanding the behaviour of complex systems over time. It deals with internal feedback loops and time delays that affect the behaviour of the entire system.^[2] What makes using system dynamics different from other approaches to studying complex systems is the use of feedback loops and stocks and flows. These elements help describe how even seemingly simple systems display baffling nonlinearity.

Systems engineering

Systems Engineering (SE) is an interdisciplinary field of engineering, that focuses on the development and organization of complex artificial systems. Systems engineering has emerged into all kinds of sciences, and



Impression of systems thinking about society ^[1]



Systems notes of Henk Bikker, TU Delft, 1991.

universities nowadays offer all kinds of specialized academic programs.^[3]

Systems Methodologies

There are several types of Systems Methodologies, that is, disciplines for analysis of systems. For example:

- Soft Systems Methodology (SSM) : in the field of organizational studies is an approach to organisational process modelling and it can be used both for general problem solving and in the management of change. It was developed in England by academics at the University of Lancaster Systems Department through a ten year Action Research programme.
- System Development Methodology (SDM) in the field of IT development is a general term applied to a variety of structured, organized processes for developing information technology and embedded software systems.

Systems theories

Systems theory is an interdisciplinary field of science. It studies the nature of complex systems in nature, society, and science. More specifically, it is a framework by which one can analyze and/or describe any group of objects that work in concert to produce some result.

Systems science

Systems sciences are scientific disciplines partly based on systems thinking such as Chaos theory, Complex systems, Control theory, Cybernetics, Sociotechnical systems theory, Systems biology, Systems ecology, Systems psychology and the already mentioned Systems dynamics, Systems engineering and Systems theory.

Fields

Systems sciences covers formal sciences fields like dynamical systems theory and applications in the field of the natural and social sciences and engineering, such as social systems theory and systems dynamics.

- | | | |
|---|--|---|
| • Chaos theory | • Operations research | • Systems psychology <ul style="list-style-type: none"> • Ergonomics • Family systems theory • Systemic therapy |
| • Complex systems | • Systems biology <ul style="list-style-type: none"> • Computational systems biology • Synthetic biology • Systems immunology | • Systems theory <ul style="list-style-type: none"> • Biochemical systems theory • Ecological systems theory • Developmental systems theory • General systems theory • Living systems theory • LTI system theory • Sociotechnical systems theory • Mathematical system theory • World-systems theory |
| • Complexity theory | • System dynamics <ul style="list-style-type: none"> • Social dynamics | • Systems theory in sociology <ul style="list-style-type: none"> • Talcott Parsons • Niklas Luhmann |
| • Cybernetics <ul style="list-style-type: none"> • Biocybernetics • Engineering cybernetics • Management cybernetics • Medical cybernetics • New Cybernetics • Second-order cybernetics | • Systems ecology <ul style="list-style-type: none"> • Ecosystem ecology | |

- Control theory
 - Affect control theory
 - Control engineering
 - Control systems
 - Dynamical systems
 - Perceptual control theory
- Systems engineering
 - Biological systems engineering
 - Earth systems engineering and management
 - Enterprise systems engineering
 - Systems analysis
- Systems theory in anthropology

Systems scientists

General systems scientists can be divided into three generations. The founders of the systems movement like Ludwig von Bertalanffy, Kenneth Boulding, Ralph Gerard, James Grier Miller, George J. Klir, and Anatol Rapoport were all born between 1900 and 1920. They all came from different natural and social science disciplines and joined forces in the 1950s to establish the general systems theory paradigm. Along with the organization of their efforts a first generation of systems scientists rose.

Among them were other scientists like Ackoff, Ashby and Churchman, who popularized the systems concept in the 1950s and 1960s. These scientists inspired and educated a second generation with more notable scientists like Ervin Laszlo (1932) and Fritjof Capra (1939), who wrote about systems theory in the 1970s and 1980s. Others got acquainted and started studying these works in the 1980s and started writing about it since the 1990s. Debora Hammond can be seen as a typical representative of these third generation of general systems scientists.

Organizations

The International Society for the Systems Sciences (ISSS) is an organisation for interdisciplinary collaboration and synthesis of systems sciences. The ISSS is unique among systems-oriented institutions in terms of the breadth of its scope, bringing together scholars and practitioners from academic, business, government, and non-profit organizations. Based on fifty years of tremendous interdisciplinary research from the scientific study of complex systems to interactive approaches in management and community development. This society was initially conceived in 1954 at the Stanford Center for Advanced Study in the Behavioral Sciences by Ludwig von Bertalanffy, Kenneth Boulding, Ralph Gerard, and Anatol Rapoport.

In the field of systems science the International Federation for Systems Research (IFSR) is an international federation for global and local societies in the field of systems science. This federation is a non-profit, scientific and educational agency founded in 1981, and constituted of some thirty member organizations from various countries. The overall purpose of this Federation is to advance cybernetic and systems research and systems applications and to serve the international systems community.

The best known research institute in the field is the Santa Fe Institute (SFI) located in Santa Fe, New Mexico, United States, dedicated to the study of complex systems. This institute was founded in 1984 by George Cowan, David Pines, Stirling Colgate, Murray Gell-Mann, Nick Metropolis, Herb Anderson, Peter A. Carruthers, and Richard Slansky. All but Pines and Gell-Mann were scientists with Los Alamos National Laboratory. SFI's original mission was to disseminate the notion of a separate interdisciplinary research area, complexity theory referred to at SFI as complexity science.

References

- [1] Illustration is made by Marcel Douwe Dekker (2007) based on an own standard and Pierre Malotiaux (1985), "Constructie van de menselijke samenwerking", in BB5 Colledictaat TU Delft, pp. 120-147.
- [2] MIT System Dynamics in Education Project (SDEP) (<http://sysdyn.clexchange.org>)
- [3] See for further details: List of systems engineering at universities

Further reading

- B. A. Bayraktar, *Education in Systems Science*, 1979, 369 pp.
- Kenneth D. Bailey, "Fifty Years of Systems Science: Further Reflections", *Systems Research and Behavioral Science*, 22, 2005, pp. 355–361.
- Robert L. Flood, Ewart R Carson, *Dealing with Complexity: An Introduction to the Theory and Application of Systems Science*, 1988.
- George J. Klir, *Facets of Systems Science*, Plenum Press, 1991.
- Ervin László, *Systems Science and World Order: Selected Studies*, 1983.
- Anatol Rapoport (ed.), *General Systems: Yearbook of the Society for the Advancement of General Systems Theory*, Society for General Systems Research, Vol 1., 1956.
- Li D. Xu, "The contributions of Systems Science to Information Systems Research", *Systems Research and Behavioral Science*, 17, 2000, pp. 105–116.
- Graeme Donald Snooks, "A general theory of complex living systems: Exploring the demand side of dynamics", *Complexity*, vol. 13, no. 6, July/August 2008.
- John N. Warfield, "A proposal for Systems Science", *Systems Research and Behavioral Science*, 20, 2003, pp. 507–520.

External links

- Principia Cybernetica Web (<http://pespmc1.vub.ac.be/>)
 - International Society for the System Sciences (<http://isss.org/world/>)
 - UK Systems Society (<http://www.ukss.org.uk>)
-

Systems theory

Systems theory is the transdisciplinary study of systems in general, with the goal of elucidating principles that can be applied to all types of systems in all fields of research. The term does not yet have a well-established, precise meaning, but systems theory can reasonably be considered a specialization of systems thinking and a generalization of systems science. The term originates from Bertalanffy's General System Theory (GST) and is used in later efforts in other fields, such as the action theory of Talcott Parsons and the system-theory of Niklas Luhmann.

In this context the word "systems" is used to refer specifically to self-regulating systems, i.e. that are self-correcting through feedback. Self-regulating systems are found in nature, including the physiological systems of our body, in local and global ecosystems, and in climate.

Overview

Contemporary ideas from systems theory have grown with diversified areas, exemplified by the work of Béla H. Bánáthy, ecological systems with Howard T. Odum, Eugene Odum and Fritjof Capra, organizational theory and management with individuals such as Peter Senge, interdisciplinary study with areas like Human Resource Development from the work of Richard A. Swanson, and insights from educators such as Debora Hammond and Alfonso Montuori. As a transdisciplinary, interdisciplinary and multiperspectival domain, the area brings together principles and concepts from ontology, philosophy of science, physics, computer science, biology, and engineering as well as geography, sociology, political science, psychotherapy (within family systems therapy) and economics among others. Systems theory thus serves as a bridge for interdisciplinary dialogue between autonomous areas of study as well as within the area of systems science itself.



Margaret Mead was an influential figure in systems theory.

In this respect, with the possibility of misinterpretations, von Bertalanffy^[1] believed a general theory of systems "should be an important regulative device in science," to guard against superficial analogies that "are useless in science and harmful in their practical consequences." Others remain closer to the direct systems concepts developed by the original theorists. For example, Ilya Prigogine, of the Center for Complex Quantum Systems at the University of Texas, Austin, has studied emergent properties, suggesting that they offer analogues for living systems. The theories of autopoiesis of Francisco Varela and Humberto Maturana are a further development in this field. Important names in contemporary systems science include Russell Ackoff, Béla H. Bánáthy, Anthony Stafford Beer, Peter Checkland, Robert L. Flood, Fritjof Capra, Michael C. Jackson, Edgar Morin and Werner Ulrich, among others.

With the modern foundations for a general theory of systems following the World Wars, Ervin Laszlo, in the preface for Bertalanffy's book *Perspectives on General System Theory*, maintains that the translation of "general system theory" from German into English has "wrought a certain amount of havoc".^[2] The preface explains that the original concept of a general system theory was "*Allgemeine Systemtheorie* (or *Lehre*)", pointing out the fact that "Theorie" (or "Lehre") just as "Wissenschaft" (translated Scholarship), "has a much broader meaning in German than the closest English words 'theory' and 'science'".^[2] With these ideas referring to an organized body of knowledge and "any systematically presented set of concepts, whether they are empirical, axiomatic, or philosophical, "Lehre" is associated with theory and science in the etymology of general systems, but also does not translate from the German very well; "teaching" is the "closest equivalent", but "sounds dogmatic and off the mark".^[2] While many of the root meanings for the idea of a "general systems theory" might have been lost in the translation and many were led to

believe that the systems theorists had articulated nothing but a pseudoscience, systems theory became a nomenclature that early investigators used to describe the interdependence of relationships in organization by defining a new way of thinking about science and scientific paradigms.

A system from this frame of reference is composed of regularly interacting or interrelating groups of activities. For example, in noting the influence in organizational psychology as the field evolved from "an individually oriented industrial psychology to a systems and developmentally oriented organizational psychology," it was recognized that organizations are complex social systems; reducing the parts from the whole reduces the overall effectiveness of organizations.^[3] This is different from conventional models that center on individuals, structures, departments and units separate in part from the whole instead of recognizing the interdependence between groups of individuals, structures and processes that enable an organization to function. Laszlo^[4] explains that the new systems view of organized complexity went "one step beyond the Newtonian view of organized simplicity" in reducing the parts from the whole, or in understanding the whole without relation to the parts. The relationship between organizations and their environments became recognized as the foremost source of complexity and interdependence. In most cases the whole has properties that cannot be known from analysis of the constituent elements in isolation. Béla H. Bánáthy, who argued—along with the founders of the systems society—that "the benefit of humankind" is the purpose of science, has made significant and far-reaching contributions to the area of systems theory. For the Primer Group at ISSS, Bánáthy defines a perspective that iterates this view:

The systems view is a world-view that is based on the discipline of SYSTEM INQUIRY. Central to systems inquiry is the concept of SYSTEM. In the most general sense, system means a configuration of parts connected and joined together by a web of relationships. The Primer group defines system as a family of relationships among the members acting as a whole. Von Bertalanffy defined system as "elements in standing relationship.

___^[5]

Similar ideas are found in learning theories that developed from the same fundamental concepts, emphasizing how understanding results from knowing concepts both in part and as a whole. In fact, Bertalanffy's organismic psychology paralleled the learning theory of Jean Piaget.^[6] Interdisciplinary perspectives are critical in breaking away from industrial age models and thinking where history is history and math is math, the arts and sciences specialized and separate, and where teaching is treated as behaviorist conditioning.^[7] The influential contemporary work of Peter Senge^[8] provides detailed discussion of the commonplace critique of educational systems grounded in conventional assumptions about learning, including the problems with fragmented knowledge and lack of holistic learning from the "machine-age thinking" that became a "model of school separated from daily life." It is in this way that systems theorists attempted to provide alternatives and an evolved ideation from orthodox theories with individuals such as Max Weber, Émile Durkheim in sociology and Frederick Winslow Taylor in scientific management, which were grounded in classical assumptions.^[9] The theorists sought holistic methods by developing systems concepts that could be integrated with different areas.

The contradiction of reductionism in conventional theory (which has as its subject a single part) is simply an example of changing assumptions. The emphasis with systems theory shifts from parts to the organization of parts, recognizing interactions of the parts are not "static" and constant but "dynamic" processes. Conventional closed systems were questioned with the development of open systems perspectives. The shift was from absolute and universal authoritative principles and knowledge to relative and general conceptual and perceptual knowledge,^[10] still in the tradition of theorists that sought to provide means in organizing human life. Meaning, the history of ideas that preceded were rethought not lost. Mechanistic thinking was particularly critiqued, especially the industrial-age mechanistic metaphor of the mind from interpretations of Newtonian mechanics by Enlightenment philosophers and later psychologists that laid the foundations of modern organizational theory and management by the late 19th century.^[11] Classical science had not been overthrown, but questions arose over core assumptions that historically influenced organized systems, within both social and technical sciences.

History

Timeline
<p>Precursors</p> <ul style="list-style-type: none"> Saint-Simon (1760–1825), Karl Marx (1817–1883), Herbert Spencer (1820–1903), Rudolf Clausius (1822–1888), Vilfredo Pareto (1848–1923), Émile Durkheim (1858–1917), Alexander Bogdanov (1873–1928), Nicolai Hartmann (1882–1950), Stafford Beer (1926–2002), Robert Maynard Hutchins (1929–1951), among others <p>Pioneers</p> <ul style="list-style-type: none"> 1946-1953 Macy conferences 1948 Norbert Wiener publishes <i>Cybernetics or Control and Communication in the Animal and the Machine</i> 1954 Ludwig von Bertalanffy, Anatol Rapoport, Ralph W. Gerard, Kenneth Boulding establish Society for the Advancement of General Systems Theory, in 1956 renamed to Society for General Systems Research. 1955 W. Ross Ashby publishes <i>Introduction to Cybernetics</i> 1968 Ludwig von Bertalanffy publishes <i>General System theory: Foundations, Development, Applications</i> <p>Developments</p> <ul style="list-style-type: none"> 1970-1980s Second-order cybernetics developed by Heinz von Foerster, Gregory Bateson, Humberto Maturana and others 1971-1973 Cybersyn, rudimentary internet and cybernetic system for democratic economic planning developed in Chile under Allende government by Stafford Beer 1970s Catastrophe theory (René Thom, E.C. Zeeman) Dynamical systems in mathematics. 1977 Ilya Prigogine received the Nobel Prize for his works on self-organization, conciliating important <i>systems theory</i> concepts with system thermodynamics. 1980s Chaos theory David Ruelle, Edward Lorenz, Mitchell Feigenbaum, Steve Smale, James A. Yorke 1986 Context theory, Anthony Wilden 1988 International Society for Systems Science 1990 Complex adaptive systems (CAS), John H. Holland, Murray Gell-Mann, W. Brian Arthur <p>Contemporary applications</p> <ul style="list-style-type: none"> 2010 Elements of the public policy of science, technology and innovation ^[12], Julio E. Rubio

Whether considering the first systems of written communication with Sumerian cuneiform to Mayan numerals, or the feats of engineering with the Egyptian pyramids, systems thinking in essence dates back to antiquity. Differentiated from Western rationalist traditions of philosophy, C. West Churchman often identified with the I Ching as a systems approach sharing a frame of reference similar to pre-Socratic philosophy and Heraclitus.^[13] Von Bertalanffy traced systems concepts to the philosophy of G.W. von Leibniz and Nicholas of Cusa's *coincidentia oppositorum*. While modern systems are considerably more complicated, today's systems are embedded in history.

An important step to introduce the *systems approach*, into (rationalist) hard sciences of the 19th century, was the energy transformation, by figures like James Joule and Sadi Carnot. Then, the Thermodynamic of this century, with Rudolf Clausius, Josiah Gibbs and others, built the *system* reference model, as a formal scientific object.

Systems theory as an area of study specifically developed following the World Wars from the work of Ludwig von Bertalanffy, Anatol Rapoport, Kenneth E. Boulding, William Ross Ashby, Margaret Mead, Gregory Bateson, C. West Churchman and others in the 1950s, specifically catalyzed by the cooperation in the Society for General Systems Research. Cognizant of advances in science that questioned classical assumptions in the organizational sciences, Bertalanffy's idea to develop a theory of systems began as early as the interwar period, publishing "An Outline for General Systems Theory" in the *British Journal for the Philosophy of Science*, Vol 1, No. 2, by 1950. Where assumptions in Western science from Greek thought with Plato and Aristotle to Newton's *Principia* have historically influenced all areas from the hard to social sciences (see David Easton's seminal development of the "political system" as an analytical construct), the original theorists explored the implications of twentieth century advances in terms of systems.

Subjects like complexity, self-organization, connectionism and adaptive systems had already been studied in the 1940s and 1950s. In fields like cybernetics, researchers like Norbert Wiener, William Ross Ashby, John von

Neumann and Heinz von Foerster examined complex systems using mathematics. John von Neumann discovered cellular automata and self-reproducing systems, again with only pencil and paper. Aleksandr Lyapunov and Jules Henri Poincaré worked on the foundations of chaos theory without any computer at all. At the same time Howard T. Odum, the radiation ecologist, recognised that the study of general systems required a language that could depict energetics, thermodynamic and kinetics at any system scale. Odum developed a general systems, or Universal language, based on the circuit language of electronics to fulfill this role, known as the Energy Systems Language. Between 1929-1951, Robert Maynard Hutchins at the University of Chicago had undertaken efforts to encourage innovation and interdisciplinary research in the social sciences, aided by the Ford Foundation with the interdisciplinary Division of the Social Sciences established in 1931.^[14] Numerous scholars had been actively engaged in ideas before (Tectology of Alexander Bogdanov published in 1912-1917 is a remarkable example), but in 1937 von Bertalanffy presented the general theory of systems for a conference at the University of Chicago.

The systems view was based on several fundamental ideas. First, all phenomena can be viewed as a web of relationships among elements, or a system. Second, all systems, whether electrical, biological, or social, have common patterns, behaviors, and properties that can be understood and used to develop greater insight into the behavior of complex phenomena and to move closer toward a unity of science. System philosophy, methodology and application are complementary to this science.^[2] By 1956, the Society for General Systems Research was established, renamed the International Society for Systems Science in 1988. The Cold War affected the research project for systems theory in ways that sorely disappointed many of the seminal theorists. Some began to recognize theories defined in association with systems theory had deviated from the initial General Systems Theory (GST) view.^[15] The economist Kenneth Boulding, an early researcher in systems theory, had concerns over the manipulation of systems concepts. Boulding concluded from the effects of the Cold War that abuses of power always prove consequential and that systems theory might address such issues.^[16] Since the end of the Cold War, there has been a renewed interest in systems theory with efforts to strengthen an ethical view.

Developments in system theories

General systems research and systems inquiry

Many early systems theorists aimed at finding a general systems theory that could explain all systems in all fields of science. The term goes back to Bertalanffy's book titled "*General System theory: Foundations, Development, Applications*" from 1968.^[6] According to Von Bertalanffy, he developed the "allgemeine Systemlehre" (general systems teachings) first via lectures beginning in 1937 and then via publications beginning in 1946.^[17]

Von Bertalanffy's objective was to bring together under one heading the organismic science that he had observed in his work as a biologist. His desire was to use the word "system" to describe those principles which are common to systems in general. In GST, he writes:

...there exist models, principles, and laws that apply to generalized systems or their subclasses, irrespective of their particular kind, the nature of their component elements, and the relationships or "forces" between them. It seems legitimate to ask for a theory, not of systems of a more or less special kind, but of universal principles applying to systems in general.^[18]

Ervin Laszlo^[19] in the preface of von Bertalanffy's book *Perspectives on General System Theory*:^[20]

Thus when von Bertalanffy spoke of Allgemeine Systemtheorie it was consistent with his view that he was proposing a new perspective, a new way of doing science. It was not directly consistent with an interpretation often put on "general system theory", to wit, that it is a (scientific) "theory of general systems." To criticize it as such is to shoot at straw men. Von Bertalanffy opened up something much broader and of much greater significance than a single theory (which, as we now know, can always be falsified and has usually an ephemeral existence): he created a new paradigm for the development of theories.

Ludwig von Bertalanffy outlines systems inquiry into three major domains: Philosophy, Science, and Technology. In his work with the Primer Group, Béla H. Bánáthy generalized the domains into four integratable domains of systemic inquiry:

Domain	Description
Philosophy	the ontology, epistemology, and axiology of systems;
Theory	a set of interrelated concepts and principles applying to all systems
Methodology	the set of models, strategies, methods, and tools that instrumentalize systems theory and philosophy
Application	the application and interaction of the domains

These operate in a recursive relationship, he explained. Integrating Philosophy and Theory as Knowledge, and Method and Application as action, Systems Inquiry then is knowledgeable action.^[21]

Cybernetics

The term cybernetics derives from a Greek word which meant steersman, and which is the origin of English words such as "govern". Cybernetics is the study of feedback and derived concepts such as communication and control in living organisms, machines and organisations. Its focus is how anything (digital, mechanical or biological) processes information, reacts to information, and changes or can be changed to better accomplish the first two tasks.

The terms "systems theory" and "cybernetics" have been widely used as synonyms. Some authors use the term *cybernetic* systems to denote a proper subset of the class of general systems, namely those systems that include feedback loops. However Gordon Pask's differences of eternal interacting actor loops (that produce finite products) makes general systems a proper subset of cybernetics. According to Jackson (2000), von Bertalanffy promoted an embryonic form of general system theory (GST) as early as the 1920s and 1930s but it was not until the early 1950s it became more widely known in scientific circles.

Threads of cybernetics began in the late 1800s that led toward the publishing of seminal works (e.g., Wiener's *Cybernetics* in 1948 and von Bertalanffy's *General Systems Theory* in 1968). Cybernetics arose more from engineering fields and GST from biology. If anything it appears that although the two probably mutually influenced each other, cybernetics had the greater influence. Von Bertalanffy (1969) specifically makes the point of distinguishing between the areas in noting the influence of cybernetics: "Systems theory is frequently identified with cybernetics and control theory. This again is incorrect. Cybernetics as the theory of control mechanisms in technology and nature is founded on the concepts of information and feedback, but as part of a general theory of systems;" then reiterates: "the model is of wide application but should not be identified with 'systems theory' in general", and that "warning is necessary against its incautious expansion to fields for which its concepts are not made." (17-23). Jackson (2000) also claims von Bertalanffy was informed by Alexander Bogdanov's three volume *Tectology* that was published in Russia between 1912 and 1917, and was translated into German in 1928. He also states it is clear to Gorelik (1975) that the "conceptual part" of general system theory (GST) had first been put in place by Bogdanov. The similar position is held by Mattessich (1978) and Capra (1996). Ludwig von Bertalanffy never even mentioned Bogdanov in his works, which Capra (1996) finds "surprising".

Cybernetics, catastrophe theory, chaos theory and complexity theory have the common goal to explain complex systems that consist of a large number of mutually interacting and interrelated parts in terms of those interactions. Cellular automata (CA), neural networks (NN), artificial intelligence (AI), and artificial life (ALife) are related fields, but they do not try to describe general (universal) complex (singular) systems. The best context to compare the different "C"-Theories about complex systems is historical, which emphasizes different tools and methodologies, from pure mathematics in the beginning to pure computer science now. Since the beginning of chaos theory when Edward Lorenz accidentally discovered a strange attractor with his computer, computers have become an indispensable source of information. One could not imagine the study of complex systems without the use of

computers today.

Complex adaptive systems

Complex adaptive systems are special cases of complex systems. They are *complex* in that they are diverse and made up of multiple interconnected elements and *adaptive* in that they have the capacity to change and learn from experience. The term *complex adaptive systems* was coined at the interdisciplinary Santa Fe Institute (SFI), by John H. Holland, Murray Gell-Mann and others. However, the approach of the complex adaptive systems does not take into account the adoption of information which enables people to use it.

CAS ideas and models are essentially evolutionary. Accordingly, the theory of complex adaptive systems bridges developments of the system theory with the ideas of 'generalized Darwinism', which suggests that Darwinian principles of evolution help explain a wide range of phenomena.

Applications of system theories

Living systems theory

Living systems theory is an offshoot of von Bertalanffy's general systems theory, created by James Grier Miller, which was intended to formalize the concept of "life". According to Miller's original conception as spelled out in his magnum opus *Living Systems*, a "living system" must contain each of 20 "critical subsystems", which are defined by their functions and visible in numerous systems, from simple cells to organisms, countries, and societies. In *Living Systems* Miller provides a detailed look at a number of systems in order of increasing size, and identifies his subsystems in each.

James Grier Miller (1978) wrote a 1,102 pages volume to present his living systems theory. He constructed a general theory of living systems by focusing on concrete systems—nonrandom accumulations of matter-energy in physical space-time organized into interacting, interrelated subsystems or components. Slightly revising the original model a dozen years later, he distinguished eight "nested" hierarchical levels in such complex structures. Each level is "nested" in the sense that each higher level contains the next lower level in a nested fashion.

Organizational theory

The systems framework is also fundamental to organizational theory as organizations are complex dynamic goal-oriented processes. One of the early thinkers in the field was Alexander Bogdanov, who developed his Tectology, a theory widely considered a precursor of von Bertalanffy's GST, aiming to model and design human organizations (see Mattessich 1978, Capra 1996). Kurt Lewin was particularly influential in developing the systems perspective within organizational theory and coined the term "systems of ideology", from his frustration with behavioral psychologies that became an obstacle to sustainable work in psychology.^[22] Jay Forrester with his work in dynamics and management alongside numerous theorists including Edgar Schein that followed in their tradition since the Civil Rights Era have also been influential.



Kurt Lewin attended the Macy conferences and is commonly identified as the founder of the movement to study groups scientifically.

The systems to organizations relies heavily upon achieving negative entropy through openness and feedback. A systemic view on organizations is transdisciplinary and integrative. In other words, it transcends the perspectives of individual disciplines, integrating them on the basis of a common "code", or more exactly, on the basis of the formal apparatus provided by systems theory. The systems approach gives primacy to the interrelationships, not to the elements of the system. It is from these dynamic interrelationships that new properties of the system emerge. In recent years, *systems thinking* has been developed to provide techniques for studying systems in holistic ways to supplement traditional reductionistic methods. In this more recent tradition, systems theory in organizational studies is considered by some as a humanistic extension of the natural sciences.

Software and computing

In the 1960s, systems theory was adopted by the post John Von Neumann computing and information technology field and, in fact, formed the basis of structured analysis and structured design (see also Larry Constantine, Tom DeMarco and Ed Yourdon). It was also the basis for early software engineering and computer-aided software engineering principles.

By the 1970s, General Systems Theory (GST) was the fundamental underpinning of most commercial software design techniques, and by the 1980, W. Vaughn Frick and Albert F. Case, Jr. had used GST to design the "missing link" transformation from system analysis (defining what's needed in a system) to system design (what's actually implemented) using the Yourdon/DeMarco notation. These principles were incorporated into computer-aided software engineering tools delivered by Nastec Corporation, Transform Logic, Inc., KnowledgeWare (see Fran Tarkenton and James Martin), Texas Instruments, Arthur Andersen and ultimately IBM Corporation.

Sociology and Sociocybernetics

Systems theory has also been developed within sociology. An important figure in the sociological systems perspective as developed from GST is Walter Buckley (who from Bertalanffy's theory). Niklas Luhmann (see Luhmann 1994) is also predominant in the literatures for sociology and systems theory. Miller's living systems theory was particularly influential in sociology from the time of the early systems movement. Models for dynamic equilibrium in systems analysis that contrasted classical views from Talcott Parsons and George Homans were influential in integrating concepts with the general movement. With the renewed interest in systems theory on the rise since the 1990s, Bailey (1994) notes the concept of systems in sociology dates back to Auguste Comte in the 19th century, Herbert Spencer and Vilfredo Pareto, and that sociology was readying into its centennial as the new systems theory was emerging following the World Wars. To explore the current inroads of systems theory into sociology (primarily in the form of complexity science) see sociology and complexity science.

In sociology, members of Research Committee 51 of the International Sociological Association (which focuses on sociocybernetics), have sought to identify the sociocybernetic feedback loops which, it is argued, primarily control the operation of society. On the basis of research largely conducted in the area of education, Raven (1995) has, for example, argued that it is these sociocybernetic processes which consistently undermine well intentioned public action and are currently heading our species, at an exponentially increasing rate, toward extinction. See sustainability. He suggests that an understanding of these systems processes will allow us to generate the kind of (non "common-sense") targeted interventions that are required for things to be otherwise - i.e. to halt the destruction of the planet.

Industrial designer, and founder of The Venus Project, Jacque Fresco advocates the utilization of sociocybernetics for the benefits it could bring to society. A major theme of Fresco's is the concept of a resource-based economy that replaces the need for the current monetary economy, which is "scarcity-oriented" or "scarcity-based". Fresco argues that the world is rich in natural resources and energy and that — with modern technology and judicious efficiency — the needs of the global population can be met with abundance, while at the same time removing the current limitations of what is deemed possible due to notions of economic viability.

Systems biology

Systems biology is a term used to describe a number of trends in bioscience research, and a movement which draws on those trends. Proponents describe systems biology as a biology-based inter-disciplinary study field that focuses on complex interactions in biological systems, claiming that it uses a new perspective (holism instead of reduction). Particularly from year 2000 onwards, the term is used widely in the biosciences, and in a variety of contexts. An often stated ambition of systems biology is the modeling and discovery of emergent properties, properties of a system whose theoretical description is only possible using techniques which fall under the remit of systems biology. The term **systems biology** is thought to have been created by Ludwig von Bertalanffy in 1928.^[23]

System dynamics

System Dynamics was founded in the late 1950s by Jay W. Forrester of the MIT Sloan School of Management with the establishment of the MIT System Dynamics Group. At that time, he began applying what he had learned about systems during his work in electrical engineering to everyday kinds of systems. Determining the exact date of the founding of the field of system dynamics is difficult and involves a certain degree of arbitrariness. Jay W. Forrester joined the faculty of the Sloan School at MIT in 1956, where he then developed what is now System Dynamics. The first published article by Jay W. Forrester in the Harvard Business Review on "Industrial Dynamics", was published in 1958. The members of the System Dynamics Society have chosen 1957 to mark the occasion as it is the year in which the work leading to that article, which described the dynamics of a manufacturing supply chain, was done.

As an aspect of systems theory, *system dynamics* is a method for understanding the dynamic behavior of complex systems. The basis of the method is the recognition that the structure of any system — the many circular, interlocking, sometimes time-delayed relationships among its components — is often just as important in determining its behavior as the individual components themselves. Examples are chaos theory and social dynamics. It is also claimed that, because there are often properties-of-the-whole which cannot be found among the properties-of-the-elements, in some cases the behavior of the whole cannot be explained in terms of the behavior of the parts. An example is the properties of these letters which when considered together can give rise to meaning which does not exist in the letters by themselves. This further explains the integration of tools, like language, as a more parsimonious process in the human application of easiest path adaptability through interconnected systems.

Systems engineering

Systems engineering is an interdisciplinary approach and means for enabling the realization and deployment of successful systems. It can be viewed as the application of engineering techniques to the engineering of systems, as well as the application of a systems approach to engineering efforts.^[24] Systems engineering integrates other disciplines and specialty groups into a team effort, forming a structured development process that proceeds from concept to production to operation and disposal. Systems engineering considers both the business and the technical needs of all customers, with the goal of providing a quality product that meets the user needs.^[25]

Systems psychology

Systems psychology is a branch of psychology that studies human behaviour and experience in complex systems. It is inspired by systems theory and systems thinking, and based on the theoretical work of Roger Barker, Gregory Bateson, Humberto Maturana and others. It is an approach in psychology, in which groups and individuals, are considered as systems in homeostasis. Systems psychology "includes the domain of engineering psychology, but in addition is more concerned with societal systems and with the study of motivational, affective, cognitive and group behavior than is engineering psychology."^[26] In systems psychology "characteristics of organizational behaviour for example individual needs, rewards, expectations, and attributes of the people interacting with the systems are considered in the process in order to create an effective system".^[27] The Systems psychology includes an illusion of homeostatic systems, although most of the living systems are in a continuous disequilibrium of various degrees.

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External links

- Systems theory (<http://pespmc1.vub.ac.be/SYSTHEOR.html>) at Principia Cybernetica Web

Organizations

- International Society for the System Sciences (<http://projects.isss.org/Main/Primer>)
- New England Complex Systems Institute (<http://www.necsi.edu/>)
- System Dynamics Society (<http://www.systemdynamics.org/>)

Systems analysis

Systems analysis is the study of sets of interacting entities, including computer systems analysis. This field is closely related to requirements analysis or operations research. It is also "an explicit formal inquiry carried out to help someone (referred to as the decision maker) identify a better course of action and make a better decision than he might otherwise have made."^[1]

Overview

The terms analysis and synthesis come from Greek where they mean respectively "to take apart" and "to put together". These terms are in scientific disciplines from mathematics and logic to economy and psychology to denote similar investigative procedures. Analysis is defined as the procedure by which we break down an intellectual or substantial whole into parts. Synthesis is defined as the procedure by which we combine separate elements or components in order to form a coherent whole.^[2] Systems analysis researchers apply methodology to the analysis of systems involved to form an overall picture. System analysis is used in every field where there is a work of developing something.

Information technology

The development of a computer-based information system includes a systems analysis phase which produces or enhances the data model which itself is a precursor to creating or enhancing a database (see Christopher J. Date "An Introduction to Database Systems"). There are a number of different approaches to system analysis. When a computer-based information system is developed, systems analysis (according to the Waterfall model) would constitute the following steps:

- The development of a feasibility study, involving determining whether a project is economically, socially, technologically and organizationally feasible.
- Conducting fact-finding measures, designed to ascertain the requirements of the system's end-users. These typically span interviews, questionnaires, or visual observations of work on the existing system.
- Gauging how the end-users would operate the system (in terms of general experience in using computer hardware or software), what the system would be used for etc.

Another view outlines a phased approach to the process. This approach breaks systems analysis into 5 phases:

- Scope definition
 - Problem analysis
-

- Requirements analysis
- Logical design
- Decision analysis

Use cases are a widely-used systems analysis modeling tool for identifying and expressing the functional requirements of a system. Each use case is a business scenario or event for which the system must provide a defined response. Use cases evolved out of object-oriented analysis; however, their use as a modeling tool has become common in many other methodologies for system analysis and design.

Practitioners

Practitioners of systems analysis are often called up to dissect systems that have grown haphazardly to determine the current components of the system. This was shown during the year 2000 re-engineering effort as business and manufacturing processes were examined as part of the Y2K automation upgrades. Employment utilizing systems analysis include systems analyst, business analyst, manufacturing engineer, enterprise architect, etc.

While practitioners of systems analysis can be called upon to create new systems, they often modify, expand or document existing systems (processes, procedures and methods). A set of components interact with each other to accomplish some specific purpose. Systems are all around us. Our body is itself a system. A business is also a system. People, money, machine, market and material are the components of business system that work together that achieve the common goal of the organization.

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- [2] Tom Ritchey, [<http://www.swemorph.com/pdf/anaeng-r.pdf> Analysis and .

External links

- Systems Analysis, Modelling and Prediction (SAMP), University of Oxford (<http://www.eng.ox.ac.uk/samp>)
- Software Requirement Analysis using UML (<http://www.slideshare.net/dhirajmusings/software-requirement-analysis-using-uml>) article by Dhiraj Shetty.
- *Introduction to Social Macrodynamics* (<http://urss.ru/cgi-bin/db.pl?cp=&page=Book&id=34250&lang=en&blang=en&list=Found>)
- A useful set of guides and a case study about the practical application of business and systems analysis methods (<http://www.cilco.co.uk/briefing-studies/index.html>)
- Complete online tutorial for system analysis and design (<http://www.systemsanalysis.co.nr>)
- A comprehensive description of the discipline of systems analysis from Simmons College, Boston, MA, USA ([www.simmons.edu](http://web.simmons.edu/~benoit/LIS486/SystemsAnalysis.html)) (<http://web.simmons.edu/~benoit/LIS486/SystemsAnalysis.html>)

Relational theory

In physics and philosophy, a **relational theory** is a framework to understand reality or a physical system in such a way that the positions and other properties of objects are only meaningful relative to other objects. In a relational spacetime theory, space does not exist unless there are objects in it; nor does time exist without events. The relational view proposes that space is contained in objects and that an object represents within itself relationships to other objects. Space can be defined through the relations among the objects that it contains considering their variations through time. The alternative spatial theory is an absolute theory in which the space exists independently of any objects that can be immersed in it.

Someone who has constructed or a relational theory or promotes relational theorising is called a **relationist**.

The relational point of view was advocated by in physics by Gottfried von Leibniz, Ernst Mach (in his Mach's principle), and it was rejected by Isaac Newton in his successful description of classical physics. Although Albert Einstein was impressed by Mach's principle, he did not fully incorporate it into his theory of general relativity. Several attempts have been made to formulate a full Machian theory, but most physicists think that none have so far succeeded. For example, see Brans-Dicke theory

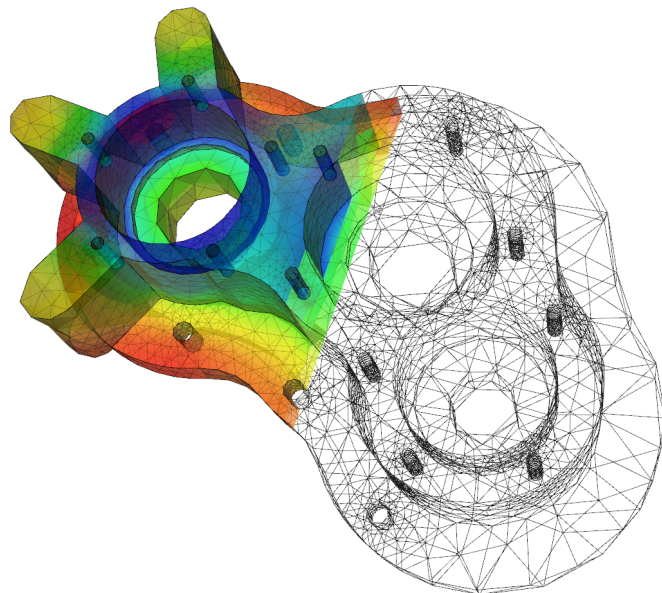
A Relational approach to quantum physics has been developed, in analogy with Einstein's special relativity of space and time.

Relationist physicists such as John Baez and Carlo Rovelli have criticised the leading unified theory of gravity and quantum mechanics, string theory, as retaining absolute space. Some prefer a developing theory of gravity, loop quantum gravity for its 'backgroundlessness'.

Differential Equations

A **differential equation** is a mathematical equation for an unknown function of one or several variables that relates the values of the function itself and its derivatives of various orders. Differential equations play a prominent role in engineering, physics, economics, and other disciplines.

Differential equations arise in many areas of science and technology, specifically, whenever a deterministic relation involving some continuously varying quantities (modeled by functions) and their rates of change in space and/or time (expressed as derivatives) is known or postulated. This is illustrated in classical mechanics, where the motion of a body is described by its position and



Visualization of heat transfer in a pump casing, by solving the heat equation. Heat is being generated internally in the casing and being cooled at the boundary, providing a steady state temperature distribution.

velocity as the time varies. Newton's laws allow one to relate the position, velocity, acceleration and various forces acting on the body and state this relation as a differential equation for the unknown position of the body as a function

of time. In some cases, this differential equation (called an equation of motion) may be solved explicitly.

An example of modelling a real world problem using differential equations is determination of the velocity of a ball falling through the air, considering only gravity and air resistance. The ball's acceleration towards the ground is the acceleration due to gravity minus the deceleration due to air resistance. Gravity is constant but air resistance may be modelled as proportional to the ball's velocity. This means the ball's acceleration, which is the derivative of its velocity, depends on the velocity. Finding the velocity as a function of time involves solving a differential equation.

Differential equations are mathematically studied from several different perspectives, mostly concerned with their solutions—the set of functions that satisfy the equation. Only the simplest differential equations admit solutions given by explicit formulas; however, some properties of solutions of a given differential equation may be determined without finding their exact form. If a self-contained formula for the solution is not available, the solution may be numerically approximated using computers. The theory of dynamical systems puts emphasis on qualitative analysis of systems described by differential equations, while many numerical methods have been developed to determine solutions with a given degree of accuracy.

Directions of study

The study of differential equations is a wide field in pure and applied mathematics, physics, meteorology, and engineering. All of these disciplines are concerned with the properties of differential equations of various types. Pure mathematics focuses on the existence and uniqueness of solutions, while applied mathematics emphasizes the rigorous justification of the methods for approximating solutions. Differential equations play an important role in modelling virtually every physical, technical, or biological process, from celestial motion, to bridge design, to interactions between neurons. Differential equations such as those used to solve real-life problems may not necessarily be directly solvable, i.e. do not have closed form solutions. Instead, solutions can be approximated using numerical methods.

Mathematicians also study weak solutions (relying on weak derivatives), which are types of solutions that do not have to be differentiable everywhere. This extension is often necessary for solutions to exist, and it also results in more physically reasonable properties of solutions, such as possible presence of shocks for equations of hyperbolic type.

The study of the stability of solutions of differential equations is known as stability theory.

Nomenclature

The theory of differential equations is quite developed and the methods used to study them vary significantly with the type of the equation.

- An ordinary differential equation (ODE) is a differential equation in which the unknown function (also known as the **dependent variable**) is a function of a *single* independent variable. In the simplest form, the unknown function is a real or complex valued function, but more generally, it may be vector-valued or matrix-valued: this corresponds to considering a system of ordinary differential equations for a single function. Ordinary differential equations are further classified according to the **order** of the highest derivative of the dependent variable with respect to the independent variable appearing in the equation. The most important cases for applications are first-order and second-order differential equations. In the classical literature also distinction is made between differential equations explicitly solved with respect to the highest derivative and differential equations in an implicit form.
- A partial differential equation (PDE) is a differential equation in which the unknown function is a function of *multiple* independent variables and the equation involves its partial derivatives. The order is defined similarly to the case of ordinary differential equations, but further classification into elliptic, hyperbolic, and parabolic equations, especially for second-order linear equations, is of utmost importance. Some partial differential

equations do not fall into any of these categories over the whole domain of the independent variables and they are said to be of **mixed type**.

Both ordinary and partial differential equations are broadly classified as **linear** and **nonlinear**. A differential equation is **linear** if the unknown function and its derivatives appear to the power 1 (products are not allowed) and **nonlinear** otherwise. The characteristic property of linear equations is that their solutions form an affine subspace of an appropriate function space, which results in much more developed theory of linear differential equations. **Homogeneous** linear differential equations are a further subclass for which the space of solutions is a linear subspace i.e. the sum of any set of solutions or multiples of solutions is also a solution. The coefficients of the unknown function and its derivatives in a linear differential equation are allowed to be (known) functions of the independent variable or variables; if these coefficients are constants then one speaks of a **constant coefficient linear differential equation**.

There are very few methods of explicitly solving nonlinear differential equations; those that are known typically depend on the equation having particular symmetries. Nonlinear differential equations can exhibit very complicated behavior over extended time intervals, characteristic of chaos. Even the fundamental questions of existence, uniqueness, and extendability of solutions for nonlinear differential equations, and well-posedness of initial and boundary value problems for nonlinear PDEs are hard problems and their resolution in special cases is considered to be a significant advance in the mathematical theory (cf. Navier–Stokes existence and smoothness).

Linear differential equations frequently appear as approximations to nonlinear equations. These approximations are only valid under restricted conditions. For example, the harmonic oscillator equation is an approximation to the nonlinear pendulum equation that is valid for small amplitude oscillations (see below).

Examples

In the first group of examples, let u be an unknown function of x , and c and ω are known constants.

- Inhomogeneous first-order linear constant coefficient ordinary differential equation:

$$\frac{du}{dx} = cu + x^2.$$

- Homogeneous second-order linear ordinary differential equation:

$$\frac{d^2u}{dx^2} - x \frac{du}{dx} + u = 0.$$

- Homogeneous second-order linear constant coefficient ordinary differential equation describing the harmonic oscillator:

$$\frac{d^2u}{dx^2} + \omega^2 u = 0.$$

- First-order nonlinear ordinary differential equation:

$$\frac{du}{dx} = u^2 + 1.$$

- Second-order nonlinear ordinary differential equation describing the motion of a pendulum of length L :

$$L \frac{d^2u}{dx^2} + g \sin u = 0.$$

In the next group of examples, the unknown function u depends on two variables x and t or x and y .

- Homogeneous first-order linear partial differential equation:

$$\frac{\partial u}{\partial t} + t \frac{\partial u}{\partial x} = 0.$$

- Homogeneous second-order linear constant coefficient partial differential equation of elliptic type, the Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

- Third-order nonlinear partial differential equation, the Korteweg–de Vries equation:

$$\frac{\partial u}{\partial t} = 6u \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}.$$

Related concepts

- A delay differential equation (DDE) is an equation for a function of a single variable, usually called **time**, in which the derivative of the function at a certain time is given in terms of the values of the function at earlier times.
- A stochastic differential equation (SDE) is an equation in which the unknown quantity is a stochastic process and the equation involves some known stochastic processes, for example, the Wiener process in the case of diffusion equations.
- A differential algebraic equation (DAE) is a differential equation comprising differential and algebraic terms, given in implicit form.

Connection to difference equations

The theory of differential equations is closely related to the theory of difference equations, in which the coordinates assume only discrete values, and the relationship involves values of the unknown function or functions and values at nearby coordinates. Many methods to compute numerical solutions of differential equations or study the properties of differential equations involve approximation of the solution of a differential equation by the solution of a corresponding difference equation.

Universality of mathematical description

Many fundamental laws of physics and chemistry can be formulated as differential equations. In biology and economics differential equations are used to model the behavior of complex systems. The mathematical theory of differential equations first developed, together with the sciences, where the equations had originated and where the results found application. However, diverse problems, sometimes originating in quite distinct scientific fields, may give rise to identical differential equations. Whenever this happens, mathematical theory behind the equations can be viewed as a unifying principle behind diverse phenomena. As an example, consider propagation of light and sound in the atmosphere, and of waves on the surface of a pond. All of them may be described by the same second-order partial differential equation, the wave equation, which allows us to think of light and sound as forms of waves, much like familiar waves in the water. Conduction of heat, the theory of which was developed by Joseph Fourier, is governed by another second-order partial differential equation, the heat equation. It turned out that many diffusion processes, while seemingly different, are described by the same equation; Black-Scholes equation in finance is for instance, related to the heat equation.

Exact solutions

Some differential equations have solutions which can be written in an exact and closed form. These are given here.

In the table below, $H(x)$, $Z(x)$ or $H(y)$ and $Z(y)$ are any integrable functions of x or y , and A, B, C, I, L, N, M are all constants. In general A, B, C, I, L , are real numbers, but N, M, P and Q may be complex. The differential equations are in their equivalent and alternative forms which lead to the solution through integration.

	Differential Equation	General Solution
1	$\frac{dy}{dx} = F(x)$ $dy = F(x)dx$	$y = \int F(x)dx$
2	$\frac{dy}{dx} = F(y)$ $dy = F(y)dx$	$x = \int \frac{dy}{F(y)}$
3	$H(y)\frac{dy}{dx} + Z(x) = 0$ $H(y)dy + Z(x)dx = 0$	$\int H(y)dy + \int Z(x)dx = C$
4	$\frac{dy}{dx} + H(x)y + Z(x) = 0$ $dy + H(x)ydx + Z(x)dx = 0$	$y = -e^{-\int H(x)dx} \int e^{\int H(x)dx} Z(x)dx$
5	$\frac{d^2y}{dx^2} = F(y)$	$x = \pm \int \frac{dy}{\sqrt{2 \int F(y)dy + C_1}} + C_2$
6	$\frac{d^2y}{dx^2} + I\frac{dy}{dx} + Ly = 0$	<p>If $I^2 > 4L$ then $y = Ne^{(-I+\sqrt{I^2-4L})\frac{x}{2}} + Me^{-(I+\sqrt{I^2-4L})\frac{x}{2}}$ If $I^2 = 4L$ then $y = (Ax + B)e^{-Ix/2}$ If $I^2 < 4L$ then $y = e^{-I\frac{x}{2}} \left[P \sin \left(\sqrt{ I^2 - 4L }\frac{x}{2} \right) + Q \cos \left(\sqrt{ I^2 - 4L }\frac{x}{2} \right) \right]$</p>

Notable differential equations

- Newton's Second Law in dynamics (mechanics)
- Hamilton's equations in classical mechanics
- Radioactive decay in nuclear physics
- Newton's law of cooling in thermodynamics
- The wave equation
- Maxwell's equations in electromagnetism
- The heat equation in thermodynamics
- Laplace's equation, which defines harmonic functions
- Poisson's equation
- Einstein's field equation in general relativity
- The Schrödinger equation in quantum mechanics
- The geodesic equation
- The Navier–Stokes equations in fluid dynamics
- The Cauchy–Riemann equations in complex analysis
- The Poisson–Boltzmann equation in molecular dynamics
- The shallow water equations

- Universal differential equation
- The Lorenz equations whose solutions exhibit chaotic flow.

Biology

- Verhulst equation – biological population growth
- von Bertalanffy model – biological individual growth
- Lotka–Volterra equations – biological population dynamics
- Replicator dynamics – may be found in theoretical biology
- Hodgkin-Huxley model - neural action potentials

Economics

- The Black–Scholes PDE
- Exogenous growth model
- Malthusian growth model
- The Vidale-Wolfe advertising model

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External links

- Lectures on Differential Equations ^[3] MIT Open CourseWare Videos
- Online Notes / Differential Equations ^[4] Paul Dawkins, Lamar University
- Differential Equations ^[5], S.O.S. Mathematics
- Introduction to modeling via differential equations ^[6] Introduction to modeling by means of differential equations, with critical remarks.
- Differential Equation Solver ^[7] Java applet tool used to solve differential equations.
- Mathematical Assistant on Web ^[8] Symbolic ODE tool, using Maxima
- Exact Solutions of Ordinary Differential Equations ^[9]
- Collection of ODE and DAE models of physical systems ^[10] MATLAB models
- Notes on Diffy Qs: Differential Equations for Engineers ^[11] An introductory textbook on differential equations by Jiri Lebl of UIUC

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- [4] <http://tutorial.math.lamar.edu/classes/de/de.aspx>
- [5] <http://www.sosmath.com/diffeq/diffeq.html>
- [6] http://www.diptem.unige.it/patrone/differential_equations_intro.htm
- [7] http://publicliterature.org/tools/differential_equation_solver/
- [8] <http://user.mendelu.cz/marik/maw/index.php?lang=en&form=ode>
- [9] <http://eqworld.ipmnet.ru/en/solutions/ode.htm>
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Computational physics

Computational physics is the study and implementation of numerical algorithms to solve problems in physics for which a quantitative theory already exists. It is often regarded as a subdiscipline of theoretical physics but some consider it an intermediate branch between theoretical and experimental physics.

Physicists often have a very precise mathematical theory describing how a system will behave. Unfortunately, it is often the case that solving the theory's equations ab initio in order to produce a useful prediction is not practical. This is especially true with quantum mechanics, where only a handful of simple models admit closed-form, analytic solutions. In cases where the equations can only be solved approximately, computational methods are often used.

Applications of computational physics

Computation now represents an essential component of modern research in accelerator physics, astrophysics, fluid mechanics, lattice field theory/lattice gauge theory (especially lattice quantum chromodynamics), plasma physics (see plasma modeling), solid state physics and soft condensed matter physics. Computational solid state physics, for example, uses density functional theory to calculate properties of solids, a method similar to that used by chemists to study molecules.

As these topics are explored, many more general numerical and mathematical problems are encountered in order to calculate physical properties of the modeled systems. These include, but are not limited to

- Solving differential equations
- Evaluating integrals
- Stochastic methods, especially Monte Carlo methods
- Specialized partial differential equation methods, for example the finite difference method and the finite element method
- The matrix eigenvalue problem – the problem of finding eigenvalues of very large matrices, and their corresponding eigenvectors (eigenstates in quantum physics)
- The pseudo-spectral method

Computational physics also encompasses the tuning of the software/hardware structure to solve problems. Approaches to solving the problems are often very demanding in terms of processing power and/or memory requests.

External links

- C20 IUPAP Commission on Computational Physics ^[1]
- APS DCOMP ^[2]
- IoP CPG (UK) ^[3]
- SciDAC: Scientific Discovery through Advanced Computing ^[4]
- Open Source Physics ^[5]
- SCINET Scientific Software Framework ^[6]

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Dynamical Systems and Symbolic Dynamics

System

System (from Latin *systema*, in turn from Greek *σύστημα* *systema*, "whole compounded of several parts or members, system", literary "composition"^[1]) is a set of interacting or interdependent system components forming an integrated whole.

The concept of an "**integrated whole**" can also be stated in terms of a system embodying a set of relationships which are differentiated from relationships of the set to other elements, and from relationships between an element of the set and elements not a part of the relational regime.

The scientific research field which is engaged in the study of the general properties of systems include systems theory, cybernetics, dynamical systems, thermodynamics and complex systems. They investigate the abstract properties of the matter and organization, searching concepts and principles which are independent of the specific domain, substance, type, or temporal scales of existence.

Most systems share common characteristics, including:

- Systems have structure, defined by components and their composition;
- Systems have behavior, which involves inputs, processing and outputs of material, energy, information, or data;
- Systems have interconnectivity: the various parts of a system have functional as well as structural relationships between each other.
- Systems may have some functions or groups of functions

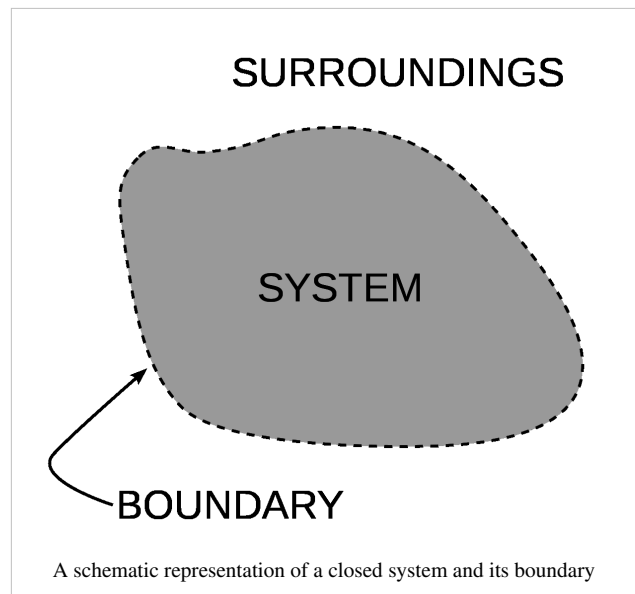
The term *system* may also refer to a set of rules that governs structure and/or behavior.

History

The word *system* in its meaning here, has a long history which can be traced back to Plato (*Philebus*), Aristotle (*Politics*) and Euclid (*Elements*). It had meant "total", "crowd" or "union" in even more ancient times, as it derives from the verb *sunístemi*, uniting, putting together.

In the 19th century the first who developed the concept of a "system" in the natural sciences was the French physicist Nicolas Léonard Sadi Carnot who studied thermodynamics. In 1824 he studied the system which he called the *working substance*, i.e. typically a body of water vapor, in steam engines, in regards to the system's ability to do work when heat is applied to it. The working substance could be put in contact with either a boiler, a cold reservoir (a stream of cold water), or a piston (to which the working body could do work by pushing on it). In 1850, the German physicist Rudolf Clausius generalized this picture to include the concept of the surroundings and began to use the term "working body" when referring to the system.

One of the pioneers of the general systems theory was the biologist Ludwig von Bertalanffy. In 1945 he introduced *models, principles, and laws that apply to generalized systems or their subclasses, irrespective of their particular*



kind, the nature of their component elements, and the relation or 'forces' between them.^[2]

Significant development to the concept of a *system* was done by Norbert Wiener and Ross Ashby who pioneered the use of mathematics to study systems.^{[3] [4]}

In the 1980s the term complex adaptive system was coined at the interdisciplinary Santa Fe Institute by John H. Holland, Murray Gell-Mann and others.

System concepts

Environment and boundaries

Systems theory views the world as a complex system of interconnected parts. We scope a system by defining its boundary; this means choosing which entities are inside the system and which are outside - part of the environment. We then make simplified representations (models) of the system in order to understand it and to predict or impact its future behavior. These models may define the structure and/or the behavior of the system.

Natural and man-made systems

There are natural and man-made (designed) systems. Natural systems may not have an apparent objective but their outputs can be interpreted as purposes. Man-made systems are made with purposes that are achieved by the delivery of outputs. Their parts must be related; they must be “designed to work as a coherent entity” - else they would be two or more distinct systems.

Theoretical Framework

An open system exchanges matter and energy with its surroundings. Most systems are open systems; like a car, coffeemaker, or computer. A closed system exchanges energy, but not matter, with its environment; like Earth or the project Biosphere2 or 3. An isolated system exchanges neither matter nor energy with its environment. A theoretical example of such system is the Universe.

Process and transformation process

A system can also be viewed as a bounded transformation process, that is, a process or collection of processes that transforms inputs into outputs. Inputs are consumed; outputs are produced. The concept of input and output here is very broad. E.g., an output of a passenger ship is the movement of people from departure to destination.

Subsystem

A *subsystem* is a set of elements, which is a system itself, and a component of a larger system.

System Model

A system comprises multiple views. For the man-made systems it may be such views as planning, requirement, design, implementation, deployment, operational, structure, behavior, input data, and output data views. A system model is required to describe and represent all these multiple views.

System Architecture

A system architecture, using one single coalescence model for the description of multiple views such as planning, requirement, design, implementation, deployment, operational, structure, behavior, input data, and output data views, is a kind of system model.

Types of systems

Evidently, there are many types of systems that can be analyzed both quantitatively and qualitatively. For example, with an analysis of urban systems dynamics, [A.W. Steiss]^[5] defines five intersecting systems, including the physical subsystem and behavioral system. For sociological models influenced by systems theory, where Kenneth D. Bailey^[6] defines systems in terms of conceptual, concrete and abstract systems; either isolated, closed, or open, Walter F. Buckley^[7] defines social systems in sociology in terms of mechanical, organic, and process models. Bela H. Banathy^[8] cautions that with any inquiry into a system that understanding the type of system is crucial and defines Natural and Designed systems.

In offering these more global definitions, the author maintains that it is important not to confuse one for the other. The theorist explains that natural systems include sub-atomic systems, living systems, the solar system, the galactic system and the Universe. Designed systems are our creations, our physical structures, hybrid systems which include natural and designed systems, and our conceptual knowledge. The human element of organization and activities are emphasized with their relevant abstract systems and representations. A key consideration in making distinctions among various types of systems is to determine how much freedom the system has to select purpose, goals, methods, tools, etc. and how widely is the freedom to select itself distributed (or concentrated) in the system.

George J. Klir^[9] maintains that no "classification is complete and perfect for all purposes," and defines systems in terms of abstract, real, and conceptual physical systems, bounded and unbounded systems, discrete to continuous, pulse to hybrid systems, et cetera. The interaction between systems and their environments are categorized in terms of relatively closed, and open systems. It seems most unlikely that an absolutely closed system can exist or, if it did, that it could be known by us. Important distinctions have also been made between hard and soft systems.^[10] Hard systems are associated with areas such as systems engineering, operations research and quantitative systems analysis. Soft systems are commonly associated with concepts developed by Peter Checkland and Brian Wilson through Soft Systems Methodology (SSM) involving methods such as action research and emphasizing participatory designs. Where hard systems might be identified as more "scientific," the distinction between them is actually often hard to define.

Cultural system

A cultural system may be defined as the interaction of different elements of culture. While a cultural system is quite different from a social system, sometimes both systems together are referred to as the sociocultural system. A major concern in the social sciences is the problem of order. One way that social order has been theorized is according to the degree of integration of cultural and social factors.

Economic system

An economic system is a mechanism (social institution) which deals with the production, distribution and consumption of goods and services in a particular society. The economic system is composed of people, institutions and their relationships to resources, such as the convention of property. It addresses the problems of economics, like the allocation and scarcity of resources.

Application of the system concept

Systems modeling is generally a basic principle in engineering and in social sciences. The system is the representation of the entities under concern. Hence inclusion to or exclusion from system context is dependent of the intention of the modeler.

No model of a system will include all features of the real system of concern, and no model of a system must include all entities belonging to a real system of concern.

Systems in information and computer science

In computer science and information science, **system** is a software system which has components as its structure and observable Inter-process communications as its behavior. Again, an example will illustrate: There are systems of counting, as with Roman numerals, and various systems for filing papers, or catalogues, and various library systems, of which the Dewey Decimal System is an example. This still fits with the definition of components which are connected together (in this case in order to facilitate the flow of information).

System can also be used referring to a framework, be it software or hardware, designed to allow software programs to run, see platform.

Systems in engineering and physics

In engineering and physics, a physical system is the portion of the universe that is being studied (of which a thermodynamic system is one major example). Engineering also has the concept of a system that refers to all of the parts and interactions between parts of a complex project. Systems engineering refers to the branch of engineering that studies how this type of system should be planned, designed, implemented, built, and maintained.

Systems in social and cognitive sciences and management research

Social and cognitive sciences recognize systems in human person models and in human societies. They include human brain functions and human mental processes as well as normative ethics systems and social/cultural behavioral patterns.

In management science, operations research and organizational development (OD), human organizations are viewed as **systems** (conceptual systems) of interacting components such as subsystems or system aggregates, which are carriers of numerous complex business processes (organizational behaviors) and organizational structures. Organizational development theorist Peter Senge developed the notion of organizations as systems in his book *The Fifth Discipline*.

Systems thinking is a style of thinking/reasoning and problem solving. It starts from the recognition of system properties in a given problem. It can be a leadership competency. Some people can *think globally while acting locally*. Such people consider the potential consequences of their decisions on other parts of larger systems. This is also a basis of systemic coaching in psychology.

Organizational theorists such as Margaret Wheatley have also described the workings of organizational systems in new metaphoric contexts, such as quantum physics, chaos theory, and the self-organization of systems.

Systems applied to strategic thinking

In 1988, military strategist, John A. Warden III introduced his Five Ring System model in his book, *The Air Campaign* contending that any complex system could be broken down into five concentric rings. Each ring—Leadership, Processes, Infrastructure, Population and Action Units—could be used to isolate key elements of any system that needed change. The model was used effectively by Air Force planners in the First Gulf War.^{[11] [12]}

^[13] In the late 1990s, Warden applied this five ring model to business strategy.^[14]

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- *Definitionen von "System" (1572-2002)* (http://www.muellerscience.com/SPEZIALITAETEN/System/System_Definitionen.htm) by Roland Müller, 2001-2007 (most in German).
- *Theory and Practical Exercises of System Dynamics* (<http://www.dinamica-de-sistemas.com/libros/dynamics.htm>) by Juan Martin (also in Spanish)

Dynamics

Dynamics (from Greek *δυναμικός* - *dynamikos* "powerful", from *δύναμις* - *dynamis* "power") may refer to:

Physics and engineering

- Dynamics (mechanics), the time evolution of physical processes
 - Aerodynamics, the study of gases in motion
 - Analytical dynamics refers to the motion of bodies as induced by external forces
 - Flight dynamics, the science of aircraft and spacecraft design
 - Fluid dynamics or *hydrodynamics*, the study of fluid flow
 - Computational fluid dynamics, a way of studying fluid dynamics using numerical methods
 - Molecular dynamics, the study of motion on the molecular level
 - Langevin dynamics, an approach to the mathematical modeling of molecular dynamics
 - Brownian dynamics, a simplified version of Langevin dynamics
 - Quantum chromodynamics, a theory of the strong interaction (color force)
 - Quantum electrodynamics, description of how matter and light interact
 - Relativistic dynamics, a combination of relativistic and quantum concepts
 - Stellar dynamics, in astrophysics, a description of the collective motion of stars
 - System dynamics, the study of the behavior of complex systems
 - Thermodynamics, the study of the relationships between heat and mechanical energy

Sociology and psychology

- Group dynamics, the study of social group processes
- Power dynamics, the dynamics of power, used in sociology
- Psychodynamics, the study of the underlying psychological forces driving human behaviour
- Spiral Dynamics, a social development theory
- Social dynamics, the ability of a society to react to changes

Computer science and mathematics

- Dynamic data structure, a data structure where the data elements may change
 - Dynamical system, a concept describing a point's time dependency
 - Symbolic dynamics, a method to model dynamical systems
 - Dynamic programming, a method of solving complex problems by breaking them down into simpler steps
 - Dynamic program analysis, a set of methods for analyzing computer software
-

Companies

- Arrow Dynamics, roller coaster designer
- Boston Dynamics, robot designer
- Crystal Dynamics, video game developer
- General Dynamics, defence contractor

Other

- Dynamics (music), the softness or loudness of a sound or note
- Force Dynamics, a semantic concept about how entities interact with reference to force
- Microsoft Dynamics, a line of business software
- Neural oscillation in the field of *neurodynamics*, a rhythmic pattern in the brain
- Population dynamics, in life sciences, the changes in the composition of a population

Dynamical systems theory

Dynamical systems theory is an area of applied mathematics used to describe the behavior of complex dynamical systems, usually by employing differential equations or difference equations. When differential equations are employed, the theory is called *continuous dynamical systems*. When difference equations are employed, the theory is called *discrete dynamical systems*. When the time variable runs over a set which is discrete over some intervals and continuous over other intervals or is any arbitrary time-set such as a cantor set then one gets dynamic equations on time scales. Some situations may also be modelled by mixed operators such as differential-difference equations.

This theory deals with the long-term qualitative behavior of dynamical systems, and the studies of the solutions to the equations of motion of systems that are primarily mechanical in nature; although this includes both planetary orbits as well as the behaviour of electronic circuits and the solutions to partial differential equations that arise in biology. Much of modern research is focused on the study of chaotic systems.

This field of study is also called just *Dynamical systems*, *Systems theory* or longer as *Mathematical Dynamical Systems Theory* and the *Mathematical theory of dynamical systems*.

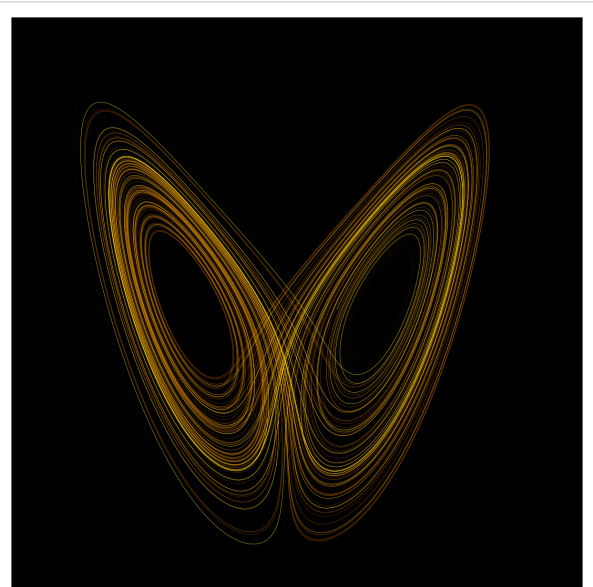
Overview

Dynamical systems theory and chaos theory deal with the long-term qualitative behavior of dynamical systems. Here, the focus is not on finding precise solutions to the equations defining the dynamical system (which is often hopeless), but rather to answer questions like "Will the system settle down to a steady state in the long term, and if so, what are the possible steady states?", or "Does the long-term behavior of the system depend on its initial condition?"

An important goal is to describe the fixed points, or steady states of a given dynamical system; these are values of the variable which won't change over time. Some of these fixed points are *attractive*, meaning that if the system starts out in a nearby state, it will converge towards the fixed point.

Similarly, one is interested in *periodic points*, states of the system which repeat themselves after several timesteps. Periodic points can also be attractive. Sarkovskii's theorem is an interesting statement about the number of periodic points of a one-dimensional discrete dynamical system.

Even simple nonlinear dynamical systems often exhibit almost random, completely unpredictable behavior that has been called *chaos*. The branch of dynamical systems which deals with the clean definition and investigation of chaos is called chaos theory.



The Lorenz attractor is an example of a non-linear dynamical system. Studying this system helped give rise to Chaos theory.

History

The concept of dynamical systems theory has its origins in Newtonian mechanics. There, as in other natural sciences and engineering disciplines, the evolution rule of dynamical systems is given implicitly by a relation that gives the state of the system only a short time into the future.

Before the advent of fast computing machines, solving a dynamical system required sophisticated mathematical techniques and could only be accomplished for a small class of dynamical systems.

Some excellent presentations of mathematical dynamic system theory include Beltrami (1987), Luenberger (1979), Padulo and Arbib (1974), and Strogatz (1994).^[1]

Concepts

Dynamical systems

The dynamical system concept is a mathematical formalization for any fixed "rule" which describes the time dependence of a point's position in its ambient space. Examples include the mathematical models that describe the swinging of a clock pendulum, the flow of water in a pipe, and the number of fish each spring in a lake.

A dynamical system has a *state* determined by a collection of real numbers, or more generally by a set of points in an appropriate *state space*. Small changes in the state of the system correspond to small changes in the numbers. The numbers are also the coordinates of a geometrical space—a manifold. The *evolution rule* of the dynamical system is a fixed rule that describes what future states follow from the current state. The rule is deterministic: for a given time interval only one future state follows from the current state.

Dynamicism

Dynamicism, also termed the *dynamic hypothesis* or the *dynamic hypothesis in cognitive science* or *dynamic cognition*, is a new approach in cognitive science exemplified by the work of philosopher Tim van Gelder. It argues that differential equations are more suited to modelling cognition than more traditional computer models.

Nonlinear system

In mathematics, a nonlinear system is a system which is not linear, i.e. a system which does not satisfy the superposition principle. Less technically, a nonlinear system is any problem where the variable(s) to be solved for cannot be written as a linear sum of independent components. A nonhomogenous system, which is linear apart from the presence of a function of the independent variables, is nonlinear according to a strict definition, but such systems are usually studied alongside linear systems, because they can be transformed to a linear system as long as a particular solution is known.

Related fields

Arithmetic dynamics

Arithmetic dynamics is a field that emerged in the 1990s that amalgamates two areas of mathematics, dynamical systems and number theory. Classically, discrete dynamics refers to the study of the iteration of self-maps of the complex plane or real line. Arithmetic dynamics is the study of the number-theoretic properties of integer, rational, p-adic, and/or algebraic points under repeated application of a polynomial or rational function.

Chaos theory

Chaos theory describes the behavior of certain dynamical systems – that is, systems whose state evolves with time – that may exhibit dynamics that are highly sensitive to initial conditions (popularly referred to as the butterfly effect). As a result of this sensitivity, which manifests itself as an exponential growth of perturbations in the initial conditions, the behavior of chaotic systems appears to be random. This happens even though these systems are deterministic, meaning that their future dynamics are fully defined by their initial conditions, with no random elements involved. This behavior is known as deterministic chaos, or simply *chaos*.

Complex systems

Complex systems is a scientific field, which studies the common properties of systems considered complex in nature, society and science. It is also called *complex systems theory*, *complexity science*, *study of complex systems* and/or *sciences of complexity*. The key problems of such systems are difficulties with their formal modeling and simulation. From such perspective, in different research contexts complex systems are defined on the base of their different attributes.

The study of complex systems is bringing new vitality to many areas of science where a more typical reductionist strategy has fallen short. *Complex systems* is therefore often used as a broad term encompassing a research approach to problems in many diverse disciplines including neurosciences, social sciences, meteorology, chemistry, physics, computer science, psychology, artificial life, evolutionary computation, economics, earthquake prediction, molecular biology and inquiries into the nature of living cells themselves.

Control theory

Control theory is an interdisciplinary branch of engineering and mathematics, that deals with influencing the behavior of dynamical systems.

Ergodic theory

Ergodic theory is a branch of mathematics that studies dynamical systems with an invariant measure and related problems. Its initial development was motivated by problems of statistical physics.

Functional analysis

Functional analysis is the branch of mathematics, and specifically of analysis, concerned with the study of vector spaces and operators acting upon them. It has its historical roots in the study of functional spaces, in particular transformations of functions, such as the Fourier transform, as well as in the study of differential and integral equations. This usage of the word *functional* goes back to the calculus of variations, implying a function whose argument is a function. Its use in general has been attributed to mathematician and physicist Vito Volterra and its founding is largely attributed to mathematician Stefan Banach.

Graph dynamical systems

The concept of graph dynamical systems (GDS) can be used to capture a wide range of processes taking place on graphs or networks. A major theme in the mathematical and computational analysis of GDS is to relate their structural properties (e.g. the network connectivity) and the global dynamics that result.

Projected dynamical systems

Projected dynamical systems is a mathematical theory investigating the behaviour of dynamical systems where solutions are restricted to a constraint set. The discipline shares connections to and applications with both the static world of optimization and equilibrium problems and the dynamical world of ordinary differential equations. A projected dynamical system is given by the flow to the projected differential equation.

Symbolic dynamics

Symbolic dynamics is the practice of modelling a topological or smooth dynamical system by a discrete space consisting of infinite sequences of abstract symbols, each of which corresponds to a state of the system, with the dynamics (evolution) given by the shift operator.

System dynamics

System dynamics is an approach to understanding the behaviour of complex systems over time. It deals with internal feedback loops and time delays that affect the behaviour of the entire system.^[2] What makes using system dynamics different from other approaches to studying complex systems is the use of feedback loops and stocks and flows. These elements help describe how even seemingly simple systems display baffling nonlinearity.

Topological dynamics

Topological dynamics is a branch of the theory of dynamical systems in which qualitative, asymptotic properties of dynamical systems are studied from the viewpoint of general topology.

Applications

In biomechanics

In sports biomechanics, dynamical systems theory has emerged in the movement sciences as a viable framework for modeling athletic performance. From a dynamical systems perspective, the human movement system is a highly intricate network of co-dependent sub-systems (e.g. respiratory, circulatory, nervous, skeletomuscular, perceptual) that are composed of a large number of interacting components (e.g. blood cells, oxygen molecules, muscle tissue, metabolic enzymes, connective tissue and bone). In dynamical systems theory, movement patterns emerge through generic processes of self-organization found in physical and biological systems.^[3]

In cognitive science

Dynamical system theory has been applied in the field of neuroscience and cognitive development, especially in the neo-Piagetian theories of cognitive development. It is the belief that cognitive development is best represented by physical theories rather than theories based on syntax and AI. It also believes that differential equations are the most appropriate tool for modeling human behavior. These equations are interpreted to represent an agent's cognitive trajectory through state space. In other words, dynamicists argue that psychology should be (or is) the description (via differential equations) of the cognitions and behaviors of an agent under certain environmental and internal pressures. The language of chaos theory is also frequently adopted.

In it, the learner's mind reaches a state of disequilibrium where old patterns have broken down. This is the phase transition of cognitive development. Self organization (the spontaneous creation of coherent forms) sets in as activity levels link to each other. Newly formed macroscopic and microscopic structures support each other, speeding up the process. These links form the structure of a new state of order in the mind through a process called *scallop*ing (the repeated building up and collapsing of complex performance.) This new, novel state is progressive, discrete, idiosyncratic and unpredictable.^[4]

Dynamic systems theory has recently been used to explain a long-unanswered problem in child development referred to as the A-not-B error.^[5]

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External links

- Dynamic Systems (<http://www.cogs.indiana.edu/Publications/techreps2000/241/241.html>) Encyclopedia of Cognitive Science entry.
- Definition of dynamical system (<http://mathworld.wolfram.com/DynamicalSystem.html>) in MathWorld.
- DSWeb (<http://www.dynamicalsystems.org/>) Dynamical Systems Magazine

Symbolic dynamics

In mathematics, **symbolic dynamics** is the practice of modeling a topological or smooth dynamical system by a discrete space consisting of infinite sequences of abstract symbols, each of which corresponds to a state of the system, with the dynamics (evolution) given by the shift operator. Formally, a Markov partition is used to provide a finite cover for the smooth system; each set of the cover is associated with a single symbol, and the sequences of symbols result as a trajectory of the system moves from one of the covering sets to another.

History

The idea goes back to Jacques Hadamard's 1898 paper on the geodesics on surfaces of negative curvature. It was applied by Marston Morse in 1921 to the construction of a nonperiodic recurrent geodesic. Related work was done by Emil Artin in 1924 (for the system now called Artin billiard), P. J. Myrberg, Paul Koebe, Jakob Nielsen, G. A. Hedlund.

The first formal treatment was developed by Morse and Hedlund in their 1938 paper. George Birkhoff, Norman Levinson and M. L. Cartwright–J. E. Littlewood have applied similar methods to qualitative analysis of nonautonomous second order differential equations.

Claude Shannon used symbolic sequences and shifts of finite type in his 1948 paper *A mathematical theory of communication* that gave birth to information theory.

The theory was further advanced in the 1960s and 1970s, notably, in the works of Steve Smale and his school, and of Yakov Sinai and the Soviet school of ergodic theory. A spectacular application of the methods of symbolic dynamics is Sharkovskii's theorem about periodic orbits of a continuous map of an interval into itself (1964).

Examples

Concepts such as heteroclinic orbits and homoclinic orbits have a particularly simple representation in symbolic dynamics.

Applications

Symbolic dynamics originated as a method to study general dynamical systems; now its techniques and ideas have found significant applications in data storage and transmission, linear algebra, the motions of the planets and many other areas. The distinct feature in symbolic dynamics is that time is measured in *discrete* intervals. So at each time interval the system is in a particular *state*. Each state is associated with a symbol and the evolution of the system is described by an infinite sequence of symbols — represented effectively as strings. If the system states are not inherently discrete, then the state vector must be discretized, so as to get a coarse-grained description of the system.

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Molecular dynamics

Molecular dynamics (MD) is computer simulation of physical movements by atoms and molecules.

Molecular dynamics simulation is frequently used in the study of proteins and biomolecules, as well as in materials science. It is tempting, though not entirely accurate, to describe the technique as a "virtual microscope" with high temporal and spatial resolution. Whereas it is possible to take "still snapshots" of crystal structures and probe features of the motion of molecules through NMR, no current experimental technique allows access to all the time scales of motion with atomic resolution. Richard Feynman once said that "If we were to name the most powerful assumption of all, which leads one on and on in an attempt to understand life, it is that all things are made of atoms, and that everything that living things do can be understood in terms of the jiggings and wiggings of atoms." Molecular dynamics lets scientists peer into the motion of individual atoms in a way which is not possible in laboratory experiments.

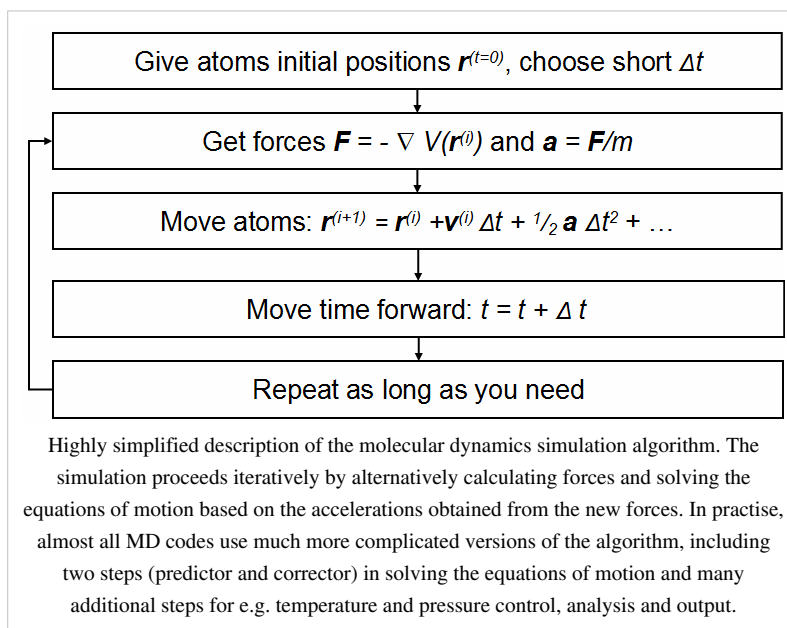
Molecular dynamics is a specialized discipline of molecular modeling and computer simulation based on statistical mechanics; the main justification of the MD method is that statistical ensemble averages are equal to time averages of the system, known as the ergodic hypothesis. MD has also been termed "statistical mechanics by numbers" and "Laplace's vision of Newtonian mechanics" of predicting the future by animating nature's forces^[1] ^[2] and allowing insight into molecular motion on an atomic scale. However, long MD simulations are mathematically ill-conditioned, generating cumulative errors in numerical integration that can be minimized with proper selection of algorithms and parameters, but not eliminated entirely. Furthermore, current potential energy functions (also called force-fields) are, in many cases, not sufficiently accurate to reproduce the dynamics of molecular systems, so the much more computationally demanding Ab Initio Molecular Dynamics method must be used. Nevertheless, molecular dynamics techniques allow detailed time and space resolution into representative behavior in phase space for carefully selected systems.

Before it became possible to simulate molecular dynamics with computers, some undertook the hard work of trying it with physical models such as macroscopic spheres. The idea was to arrange them to replicate the properties of a liquid. J.D. Bernal said, in 1962: "... I took a number of rubber balls and stuck them together with rods of a selection of different lengths ranging from 2.75 to 4 inches. I tried to do this in the first place as casually as possible, working in my own office, being interrupted every five minutes or so and not remembering what I had done before the interruption."^[3] Fortunately, now computers keep track of bonds during a simulation.

Because molecular systems generally consist of a vast number of particles, it is in general impossible to find the properties of such complex systems analytically. When the number of particles interacting is higher than two, the result is chaotic motion (see n-body problem). MD simulation circumvents the analytical intractability by using numerical methods. It represents an interface between laboratory experiments and theory, and can be understood as a "virtual experiment". MD probes the relationship between molecular structure, movement and function. Molecular dynamics is a multidisciplinary method. Its laws and theories stem from mathematics, physics, and chemistry, and it employs algorithms from computer science and information theory. It was originally conceived within theoretical physics in the late 1950s^[4] and early 1960s,^[5] but is applied today mostly in materials science and the modeling of biomolecules.

Areas of Application

There is a significant difference between the focus and methods used by chemists and physicists, and this is reflected in differences in the jargon used by the different fields. In chemistry and biophysics, the interaction between the particles is either described by a "force field" (**classical MD**), a quantum chemical model, or a mix between the two. These terms are not used in physics, where the interactions are usually described by the name of the theory or approximation being used and called the potential energy, or just the "potential".



Beginning in theoretical physics, the method of MD gained popularity in materials science and since the 1970s also in biochemistry and biophysics. In chemistry, MD serves as an important tool in protein structure determination and refinement using experimental tools such as X-ray crystallography and NMR. It has also been applied with limited success as a method of refining protein structure predictions. In physics, MD is used to examine the dynamics of atomic-level phenomena that cannot be observed directly, such as thin film growth and ion-subplantation. It is also used to examine the physical properties of nanotechnological devices that have not or cannot yet be created.

In applied mathematics and theoretical physics, molecular dynamics is a part of the research realm of dynamical systems, ergodic theory and statistical mechanics in general. The concepts of energy conservation and molecular entropy come from thermodynamics. Some techniques to calculate conformational entropy such as principal components analysis come from information theory. Mathematical techniques such as the transfer operator become applicable when MD is seen as a Markov chain. Also, there is a large community of mathematicians working on volume preserving, symplectic integrators for more computationally efficient MD simulations.

MD can also be seen as a special case of the discrete element method (DEM) in which the particles have spherical shape (e.g. with the size of their van der Waals radii.) Some authors in the DEM community employ the term MD rather loosely, even when their simulations do not model actual molecules.

Design Constraints

Design of a molecular dynamics simulation should account for the available computational power. Simulation size (n =number of particles), timestep and total time duration must be selected so that the calculation can finish within a reasonable time period. However, the simulations should be long enough to be relevant to the time scales of the natural processes being studied. To make statistically valid conclusions from the simulations, the time span simulated should match the kinetics of the natural process. Otherwise, it is analogous to making conclusions about how a human walks from less than one footstep. Most scientific publications about the dynamics of proteins and DNA use data from simulations spanning nanoseconds (10^{-9} s) to microseconds (10^{-6} s). To obtain these simulations, several CPU-days to CPU-years are needed. Parallel algorithms allow the load to be distributed among CPUs; an example is the spatial or force decomposition algorithm [6].

During a classical MD simulation, the most CPU intensive task is the evaluation of the potential (force field) as a function of the particles' internal coordinates. Within that energy evaluation, the most expensive one is the

non-bonded or non-covalent part. In Big O notation, common molecular dynamics simulations scale by $O(n^2)$ if all pair-wise electrostatic and van der Waals interactions must be accounted for explicitly. This computational cost can be reduced by employing electrostatics methods such as Particle Mesh Ewald ($O(n \log(n))$), P3M or good spherical cutoff techniques ($O(n)$).

Another factor that impacts total CPU time required by a simulation is the size of the integration timestep. This is the time length between evaluations of the potential. The timestep must be chosen small enough to avoid discretization errors (i.e. smaller than the fastest vibrational frequency in the system). Typical timesteps for classical MD are in the order of 1 femtosecond (10^{-15} s). This value may be extended by using algorithms such as SHAKE, which fix the vibrations of the fastest atoms (e.g. hydrogens) into place. Multiple time scale methods have also been developed, which allow for extended times between updates of slower long-range forces.^{[7] [8] [9]}

For simulating molecules in a solvent, a choice should be made between explicit solvent and implicit solvent. Explicit solvent particles (such as the TIP3P, SPC/E and SPC-f water models) must be calculated expensively by the force field, while implicit solvents use a mean-field approach. Using an explicit solvent is computationally expensive, requiring inclusion of roughly ten times more particles in the simulation. But the granularity and viscosity of explicit solvent is essential to reproduce certain properties of the solute molecules. This is especially important to reproduce kinetics.

In all kinds of molecular dynamics simulations, the simulation box size must be large enough to avoid boundary condition artifacts. Boundary conditions are often treated by choosing fixed values at the edges (which may cause artifacts), or by employing periodic boundary conditions in which one side of the simulation loops back to the opposite side, mimicking a bulk phase.

Microcanonical ensemble (NVE)

In the **microcanonical**, or **NVE** ensemble, the system is isolated from changes in moles (N), volume (V) and energy (E). It corresponds to an adiabatic process with no heat exchange. A microcanonical molecular dynamics trajectory may be seen as an exchange of potential and kinetic energy, with total energy being conserved. For a system of N particles with coordinates \mathbf{X} and velocities \mathbf{V} , the following pair of first order differential equations may be written in Newton's notation as

$$\begin{aligned} \mathbf{F}(\mathbf{X}) &= -\nabla U(\mathbf{X}) = M\dot{\mathbf{V}}(t) \\ \mathbf{V}(t) &= \dot{\mathbf{X}}(t). \end{aligned}$$

The potential energy function $U(\mathbf{X})$ of the system is a function of the particle coordinates \mathbf{X} . It is referred to simply as the "potential" in Physics, or the "force field" in Chemistry. The first equation comes from Newton's laws; the force \mathbf{F} acting on each particle in the system can be calculated as the negative gradient of $U(\mathbf{X})$.

For every timestep, each particle's position \mathbf{X} and velocity \mathbf{V} may be integrated with a symplectic method such as Verlet. The time evolution of \mathbf{X} and \mathbf{V} is called a trajectory. Given the initial positions (e.g. from theoretical knowledge) and velocities (e.g. randomized Gaussian), we can calculate all future (or past) positions and velocities.

One frequent source of confusion is the meaning of temperature in MD. Commonly we have experience with macroscopic temperatures, which involve a huge number of particles. But temperature is a statistical quantity. If there is a large enough number of atoms, statistical temperature can be estimated from the *instantaneous temperature*, which is found by equating the kinetic energy of the system to $nk_B T/2$ where n is the number of degrees of freedom of the system.

A temperature-related phenomenon arises due to the small number of atoms that are used in MD simulations. For example, consider simulating the growth of a copper film starting with a substrate containing 500 atoms and a deposition energy of 100 eV. In the real world, the 100 eV from the deposited atom would rapidly be transported through and shared among a large number of atoms ($\sim 10^{10}$ or more) with no big change in temperature. When there are only 500 atoms, however, the substrate is almost immediately vaporized by the deposition. Something similar

happens in biophysical simulations. The temperature of the system in NVE is naturally raised when macromolecules such as proteins undergo exothermic conformational changes and binding.

Canonical ensemble (NVT)

In the canonical ensemble, moles (N), volume (V) and temperature (T) are conserved. It is also sometimes called constant temperature molecular dynamics (CTMD). In NVT, the energy of endothermic and exothermic processes is exchanged with a thermostat.

A variety of thermostat methods is available to add and remove energy from the boundaries of an MD system in a more or less realistic way, approximating the canonical ensemble. Popular techniques to control temperature include velocity rescaling, the Nosé-Hoover thermostat, Nosé-Hoover chains, the Berendsen thermostat and Langevin dynamics. Note that the Berendsen thermostat might introduce the flying ice cube effect, which leads to unphysical translations and rotations of the simulated system.

It is not trivial to obtain a canonical distribution of conformations and velocities using these algorithms. How this depends on system size, thermostat choice, thermostat parameters, time step and integrator is the subject of many articles in the field.

Isothermal-Isobaric (NPT) ensemble

In the isothermal-isobaric ensemble, moles (N), pressure (P) and temperature (T) are conserved. In addition to a thermostat, a barostat is needed. It corresponds most closely to laboratory conditions with a flask open to ambient temperature and pressure.

In the simulation of biological membranes, isotropic pressure control is not appropriate. For lipid bilayers, pressure control occurs under constant membrane area (NPAT) or constant surface tension " γ " (NP γ T).

Generalized ensembles

The replica exchange method is a generalized ensemble. It was originally created to deal with the slow dynamics of disordered spin systems. It is also called parallel tempering. The replica exchange MD (REMD) formulation^[10] tries to overcome the multiple-minima problem by exchanging the temperature of non-interacting replicas of the system running at several temperatures.

Potentials in MD simulations

A molecular dynamics simulation requires the definition of a potential function, or a description of the terms by which the particles in the simulation will interact. In chemistry and biology this is usually referred to as a force field. Potentials may be defined at many levels of physical accuracy; those most commonly used in chemistry are based on molecular mechanics and embody a classical treatment of particle-particle interactions that can reproduce structural and conformational changes but usually cannot reproduce chemical reactions.

The reduction from a fully quantum description to a classical potential entails two main approximations. The first one is the Born-Oppenheimer approximation, which states that the dynamics of electrons is so fast that they can be considered to react instantaneously to the motion of their nuclei. As a consequence, they may be treated separately. The second one treats the nuclei, which are much heavier than electrons, as point particles that follow classical Newtonian dynamics. In classical molecular dynamics the effect of the electrons is approximated as a single potential energy surface, usually representing the ground state.

When finer levels of detail are required, potentials based on quantum mechanics are used; some techniques attempt to create hybrid classical/quantum potentials where the bulk of the system is treated classically but a small region is treated as a quantum system, usually undergoing a chemical transformation.

Empirical potentials

Empirical potentials used in chemistry are frequently called force fields, while those used in materials physics are called just empirical or analytical potentials.

Most force fields in chemistry are empirical and consist of a summation of bonded forces associated with chemical bonds, bond angles, and bond dihedrals, and non-bonded forces associated with van der Waals forces and electrostatic charge. Empirical potentials represent quantum-mechanical effects in a limited way through ad-hoc functional approximations. These potentials contain free parameters such as atomic charge, van der Waals parameters reflecting estimates of atomic radius, and equilibrium bond length, angle, and dihedral; these are obtained by fitting against detailed electronic calculations (quantum chemical simulations) or experimental physical properties such as elastic constants, lattice parameters and spectroscopic measurements.

Because of the non-local nature of non-bonded interactions, they involve at least weak interactions between all particles in the system. Its calculation is normally the bottleneck in the speed of MD simulations. To lower the computational cost, force fields employ numerical approximations such as shifted cutoff radii, reaction field algorithms, particle mesh Ewald summation, or the newer Particle-Particle Particle Mesh (P3M).

Chemistry force fields commonly employ preset bonding arrangements (an exception being *ab-initio* dynamics), and thus are unable to model the process of chemical bond breaking and reactions explicitly. On the other hand, many of the potentials used in physics, such as those based on the bond order formalism can describe several different coordinations of a system and bond breaking. Examples of such potentials include the Brenner potential^[11] for hydrocarbons and its further developments for the C-Si-H and C-O-H systems. The ReaxFF potential^[12] can be considered a fully reactive hybrid between bond order potentials and chemistry force fields.

Pair potentials vs. many-body potentials

The potential functions representing the non-bonded energy are formulated as a sum over interactions between the particles of the system. The simplest choice, employed in many popular force fields, is the "pair potential", in which the total potential energy can be calculated from the sum of energy contributions between pairs of atoms. An example of such a pair potential is the non-bonded Lennard-Jones potential (also known as the 6-12 potential), used for calculating van der Waals forces.

$$U(r) = 4\epsilon \left[\left(\frac{\sigma}{r} \right)^{12} - \left(\frac{\sigma}{r} \right)^6 \right]$$

Another example is the Born (ionic) model of the ionic lattice. The first term in the next equation is Coulomb's law for a pair of ions, the second term is the short-range repulsion explained by Pauli's exclusion principle and the final term is the dispersion interaction term. Usually, a simulation only includes the dipolar term, although sometimes the quadrupolar term is included as well.

$$U_{ij}(r_{ij}) = \sum \frac{z_i z_j}{4\pi\epsilon_0 r_{ij}} + \sum A_l \exp \frac{-r_{ij}}{p_l} + \sum C_l r_{ij}^{-n_l} + \dots$$

In many-body potentials, the potential energy includes the effects of three or more particles interacting with each other. In simulations with pairwise potentials, global interactions in the system also exist, but they occur only through pairwise terms. In many-body potentials, the potential energy cannot be found by a sum over pairs of atoms, as these interactions are calculated explicitly as a combination of higher-order terms. In the statistical view, the dependency between the variables cannot in general be expressed using only pairwise products of the degrees of freedom. For example, the Tersoff potential,^[13] which was originally used to simulate carbon, silicon and germanium and has since been used for a wide range of other materials, involves a sum over groups of three atoms, with the angles between the atoms being an important factor in the potential. Other examples are the embedded-atom method (EAM)^[14] and the Tight-Binding Second Moment Approximation (TBSMA) potentials,^[15] where the electron density of states in the region of an atom is calculated from a sum of contributions from surrounding atoms,

and the potential energy contribution is then a function of this sum.

Semi-empirical potentials

Semi-empirical potentials make use of the matrix representation from quantum mechanics. However, the values of the matrix elements are found through empirical formulae that estimate the degree of overlap of specific atomic orbitals. The matrix is then diagonalized to determine the occupancy of the different atomic orbitals, and empirical formulae are used once again to determine the energy contributions of the orbitals.

There are a wide variety of semi-empirical potentials, known as tight-binding potentials, which vary according to the atoms being modeled.

Polarizable potentials

Most classical force fields implicitly include the effect of polarizability, e.g. by scaling up the partial charges obtained from quantum chemical calculations. These partial charges are stationary with respect to the mass of the atom. But molecular dynamics simulations can explicitly model polarizability with the introduction of induced dipoles through different methods, such as Drude particles or fluctuating charges. This allows for a dynamic redistribution of charge between atoms which responds to the local chemical environment.

For many years, polarizable MD simulations have been touted as the next generation. For homogenous liquids such as water, increased accuracy has been achieved through the inclusion of polarizability.^[16] Some promising results have also been achieved for proteins.^[17] However, it is still uncertain how to best approximate polarizability in a simulation.

Ab-initio methods

In classical molecular dynamics, a single potential energy surface (usually the ground state) is represented in the force field. This is a consequence of the Born-Oppenheimer approximation. In excited states, chemical reactions or a more accurate representation is needed, electronic behavior can be obtained from first principles by using a quantum mechanical method, such as Density Functional Theory. This is known as *Ab Initio* Molecular Dynamics (AIMD). Due to the cost of treating the electronic degrees of freedom, the computational cost of this simulations is much higher than classical molecular dynamics. This implies that AIMD is limited to smaller systems and shorter periods of time.

Ab-initio quantum-mechanical methods may be used to calculate the potential energy of a system on the fly, as needed for conformations in a trajectory. This calculation is usually made in the close neighborhood of the reaction coordinate. Although various approximations may be used, these are based on theoretical considerations, not on empirical fitting. *Ab-Initio* calculations produce a vast amount of information that is not available from empirical methods, such as density of electronic states or other electronic properties. A significant advantage of using *ab-initio* methods is the ability to study reactions that involve breaking or formation of covalent bonds, which correspond to multiple electronic states.

A popular software for *ab-initio* molecular dynamics is the Car-Parrinello Molecular Dynamics (CPMD) package based on the density functional theory.

Hybrid QM/MM

QM (quantum-mechanical) methods are very powerful. However, they are computationally expensive, while the MM (classical or molecular mechanics) methods are fast but suffer from several limitations (require extensive parameterization; energy estimates obtained are not very accurate; cannot be used to simulate reactions where covalent bonds are broken/formed; and are limited in their abilities for providing accurate details regarding the chemical environment). A new class of method has emerged that combines the good points of QM (accuracy) and

MM (speed) calculations. These methods are known as mixed or hybrid quantum-mechanical and molecular mechanics methods (hybrid QM/MM). The methodology for such techniques was introduced by Warshel and coworkers. In the recent years have been pioneered by several groups including: Arieh Warshel (University of Southern California), Weitao Yang (Duke University), Sharon Hammes-Schiffer (The Pennsylvania State University), Donald Truhlar and Jiali Gao (University of Minnesota) and Kenneth Merz (University of Florida).

The most important advantage of hybrid QM/MM methods is the speed. The cost of doing classical molecular dynamics (MM) in the most straightforward case scales $O(n^2)$, where n is the number of atoms in the system. This is mainly due to electrostatic interactions term (every particle interacts with every other particle). However, use of cutoff radius, periodic pair-list updates and more recently the variations of the particle-mesh Ewald's (PME) method has reduced this between $O(n)$ to $O(n^2)$. In other words, if a system with twice as many atoms is simulated then it would take between two to four times as much computing power. On the other hand the simplest *ab-initio* calculations typically scale $O(n^3)$ or worse (Restricted Hartree-Fock calculations have been suggested to scale $\sim O(n^{2.7})$). To overcome the limitation, a small part of the system is treated quantum-mechanically (typically active-site of an enzyme) and the remaining system is treated classically.

In more sophisticated implementations, QM/MM methods exist to treat both light nuclei susceptible to quantum effects (such as hydrogens) and electronic states. This allows generation of hydrogen wave-functions (similar to electronic wave-functions). This methodology has been useful in investigating phenomena such as hydrogen tunneling. One example where QM/MM methods have provided new discoveries is the calculation of hydride transfer in the enzyme liver alcohol dehydrogenase. In this case, tunneling is important for the hydrogen, as it determines the reaction rate.^[18]

Coarse-graining and reduced representations

At the other end of the detail scale are coarse-grained and lattice models. Instead of explicitly representing every atom of the system, one uses "pseudo-atoms" to represent groups of atoms. MD simulations on very large systems may require such large computer resources that they cannot easily be studied by traditional all-atom methods. Similarly, simulations of processes on long timescales (beyond about 1 microsecond) are prohibitively expensive, because they require so many timesteps. In these cases, one can sometimes tackle the problem by using reduced representations, which are also called coarse-grained models.

Examples for coarse graining (CG) methods are discontinuous molecular dynamics (CG-DMD)^[19] ^[20] and Go-models.^[21] Coarse-graining is done sometimes taking larger pseudo-atoms. Such united atom approximations have been used in MD simulations of biological membranes. The aliphatic tails of lipids are represented by a few pseudo-atoms by gathering 2 to 4 methylene groups into each pseudo-atom.

The parameterization of these very coarse-grained models must be done empirically, by matching the behavior of the model to appropriate experimental data or all-atom simulations. Ideally, these parameters should account for both enthalpic and entropic contributions to free energy in an implicit way. When coarse-graining is done at higher levels, the accuracy of the dynamic description may be less reliable. But very coarse-grained models have been used successfully to examine a wide range of questions in structural biology.

Examples of applications of coarse-graining in biophysics:

- protein folding studies are often carried out using a single (or a few) pseudo-atoms per amino acid;
- DNA supercoiling has been investigated using 1-3 pseudo-atoms per basepair, and at even lower resolution;
- Packaging of double-helical DNA into bacteriophage has been investigated with models where one pseudo-atom represents one turn (about 10 basepairs) of the double helix;
- RNA structure in the ribosome and other large systems has been modeled with one pseudo-atom per nucleotide.

The simplest form of coarse-graining is the "united atom" (sometimes called "extended atom") and was used in most early MD simulations of proteins, lipids and nucleic acids. For example, instead of treating all four atoms of a CH_3 methyl group explicitly (or all three atoms of CH_2 methylene group), one represents the whole group with a single

pseudo-atom. This pseudo-atom must, of course, be properly parameterized so that its van der Waals interactions with other groups have the proper distance-dependence. Similar considerations apply to the bonds, angles, and torsions in which the pseudo-atom participates. In this kind of united atom representation, one typically eliminates all explicit hydrogen atoms except those that have the capability to participate in hydrogen bonds ("polar hydrogens"). An example of this is the Charmm 19 force-field.

The polar hydrogens are usually retained in the model, because proper treatment of hydrogen bonds requires a reasonably accurate description of the directionality and the electrostatic interactions between the donor and acceptor groups. A hydroxyl group, for example, can be both a hydrogen bond donor and a hydrogen bond acceptor, and it would be impossible to treat this with a single OH pseudo-atom. Note that about half the atoms in a protein or nucleic acid are nonpolar hydrogens, so the use of united atoms can provide a substantial savings in computer time.

Examples of applications

Molecular dynamics is used in many fields of science.

- First macromolecular MD simulation published (1977, Size: 500 atoms, Simulation Time: 9.2 ps=0.0092 ns, Program: CHARMM precursor) Protein: Bovine Pancreatic Trypsin Inhibitor. This is one of the best studied proteins in terms of folding and kinetics. Its simulation published in Nature magazine paved the way for understanding protein motion as essential in function and not just accessory.^[22]
- MD is the standard method to treat collision cascades in the heat spike regime, i.e. the effects that energetic neutron and ion irradiation have on solids and solid surfaces.^{[23] [24]}

The following two biophysical examples are not run-of-the-mill MD simulations. They illustrate notable efforts to produce simulations of a system of very large size (a complete virus) and very long simulation times (500 microseconds):

- MD simulation of the complete satellite tobacco mosaic virus (**STMV**) (2006, Size: 1 million atoms, Simulation time: 50 ns, program: NAMD) This virus is a small, icosahedral plant virus which worsens the symptoms of infection by Tobacco Mosaic Virus (TMV). Molecular dynamics simulations were used to probe the mechanisms of viral assembly. The entire STMV particle consists of 60 identical copies of a single protein that make up the viral capsid (coating), and a 1063 nucleotide single stranded RNA genome. One key finding is that the capsid is very unstable when there is no RNA inside. The simulation would take a single 2006 desktop computer around 35 years to complete. It was thus done in many processors in parallel with continuous communication between them.^[25]
- Folding Simulations of the Villin Headpiece in All-Atom Detail (2006, Size: 20,000 atoms; Simulation time: 500 μ s = 500,000 ns, Program: folding@home) This simulation was run in 200,000 CPU's of participating personal computers around the world. These computers had the folding@home program installed, a large-scale distributed computing effort coordinated by Vijay Pande at Stanford University. The kinetic properties of the Villin Headpiece protein were probed by using many independent, short trajectories run by CPU's without continuous real-time communication. One technique employed was the Pfold value analysis, which measures the probability of folding before unfolding of a specific starting conformation. Pfold gives information about transition state structures and an ordering of conformations along the folding pathway. Each trajectory in a Pfold calculation can be relatively short, but many independent trajectories are needed.^[26]

Molecular dynamics algorithms

Integrators

- Verlet-Stoermer integration
- Runge-Kutta integration
- Beeman's algorithm
- Gear predictor - corrector
- Constraint algorithms (for constrained systems)
- Symplectic integrator

Short-range interaction algorithms

- Cell lists
- Verlet list
- Bonded interactions

Long-range interaction algorithms

- Ewald summation
- Particle Mesh Ewald (PME)
- Particle-Particle Particle Mesh P3M
- Reaction Field Method

Parallelization strategies

- Domain decomposition method (Distribution of system data for parallel computing)
- Molecular Dynamics - Parallel Algorithms ^[27]

Major software for MD simulations

- AutoDock suite of automated docking tools,
 - Autodock Vina improved local search algorithm, suite of automated docking tools,
 - Abalone (classical, implicit water)
 - ABINIT (DFT)
 - ACEMD ^[28] (running on NVIDIA GPUs: heavily optimized with CUDA)
 - ADUN ^[29] (classical, P2P database for simulations)
 - AMBER (classical)
 - Ascalaph ^[30] (classical, GPU accelerated)
 - CASTEP (DFT)
 - CPMD (DFT)
 - CP2K ^[31] (DFT)
 - CHARMM (classical, the pioneer in MD simulation, extensive analysis tools)
 - COSMOS ^[32] (classical and hybrid QM/MM, quantum-mechanical atomic charges with BPT)
 - Desmond (classical, parallelization with up to thousands of CPU's)
 - Culgi ^[33] (classical, OPLS-AA, Dreiding, Nerd, and TraPPE-UA force fields)
 - DL_POLY ^[34] (classical)
 - ESPResSo (classical, coarse-grained, parallel, extensible)
 - Fireball ^[35] (tight-binding DFT)
 - GROMACS (classical)
 - GROMOS (classical)
-

- GULP (classical)
- Hippo ^[36] (classical)
- HOOMD-Blue ^[37] (classical, accelerated by NVIDIA GPUs, heavily optimized with CUDA)
- Kalypso ^[38] MD simulation of atomic collisions in solids
- LAMMPS (classical, large-scale with spatial-decomposition of simulation domain for parallelism)
- LPMD ^[39] Las Palmeras Molecular Dynamics: flexible and modular MD.
- MacroModel (classical)
- MACSIMUS ^[40] (classical, polarizability, thread-based parallelization)
- MDynaMix (classical, parallel)
- MOLDY ^[41] (classical, parallel) latest release ^[42]
- Materials Studio ^[43] (Forcite MD using COMPASS, Dreiding, Universal, cvff and pcff forcefields in serial or parallel, QMERA (QM+MD), ONESTEP (DFT), etc.)
- MOSCITO (classical)
- NAMD (classical, parallelization with up to thousands of CPU's)
- nano-Material Simulation Toolkit ^[44]
- NEWTON-X ^[45] (ab initio, surface-hopping dynamics)
- ORAC (classical)
- ProtoMol ^[46] (classical, extensible, includes multigrid electrostatics)
- PWscf (DFT)
- RedMD ^[47] (coarse-grained simulations package on GNU licence)
- S/PHI/nX ^[48] (DFT)
- SIESTA (DFT)
- Tremolo-X
- VASP (DFT)
- TINKER (classical)
- YASARA ^[49] (classical)
- XMD (classical)

Related software

- Avizo - 3d visualization and analysis software.
 - BOSS - MC in OPLS
 - Chimera - Molecular visualization and analysis package, including trajectory support.
 - esra ^[50] - Lightweight molecular modeling and analysis library (Java/Jython/Mathematica).
 - Molecular Workbench ^[51] - Interactive molecular simulations on your desktop.
 - Packmol ^[52] Package for building starting configurations for MD in an automated fashion.
 - Punto ^[53] is a freely available visualisation tool for particle simulations.
 - PyMol - Molecular Visualization software written in python.
 - Sirius - Molecular modeling, analysis and visualization of MD trajectories.
 - VMD - MD simulation trajectories can be visualized and analyzed.
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Specialized hardware for MD simulations

- Anton - A specialized, massively parallel supercomputer designed to execute MD simulations.
- MDGRAPE - A special purpose system built for molecular dynamics simulations, especially protein structure prediction.

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External links

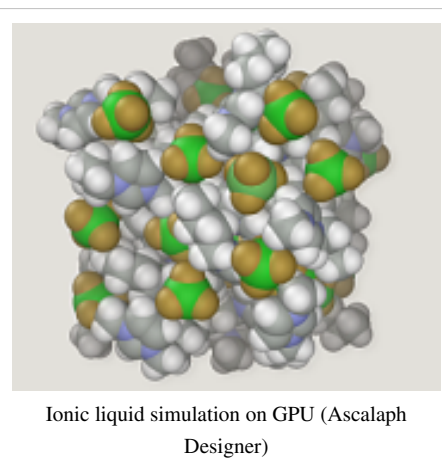
- The Blue Gene Project (<http://researchweb.watson.ibm.com/bluegene/>) (IBM)JawBreakers.org
- D. E. Shaw Research (<http://deshawresearch.com/>) (D. E. Shaw Research)
- Molecular Physics (<http://www.tandf.co.uk/journals/titles/00268976.asp>)
- Statistical mechanics of Nonequilibrium Liquids (<http://www.phys.unsw.edu.au/~gary/book.html>) Lecture Notes on non-equilibrium MD
- Introductory Lecture on Classical Molecular Dynamics (<http://www.fz-juelich.de/nic-series/volume10/sutmann.pdf>) by Dr. Godehard Sutmann, NIC, Forschungszentrum Jülich, Germany
- Introductory Lecture on Ab Initio Molecular Dynamics and Ab Initio Path Integrals (<http://www.fz-juelich.de/nic-series/volume10/tuckerman2.pdf>) by Mark E. Tuckerman, New York University, USA
- Introductory Lecture on Ab initio molecular dynamics: Theory and Implementation (<http://www.fz-juelich.de/nic-series/Volume1/marx.pdf>) by Dominik Marx, Ruhr-Universität Bochum and Jürg Hutter, Universität Zürich
- Online course on (MSE 597G) An Introduction to Molecular Dynamics (<http://nanohub.org/resources/5838>) by Alejandro Strachan
- Lecture Notes on *Short Course on Molecular Dynamics Simulation* Ashlie Martini (2009) (<http://nanohub.org/resources/7570>)

Molecular modeling on GPU

Molecular modeling on GPU is the technique of using a graphics processing unit (GPU) for molecular simulations. ^[1]

In 2007, NVIDIA introduced video cards that could be used not only to show graphics but also for scientific calculations. These cards include many arithmetic units (currently up to 512) working in parallel. Long before this event, the computational power of video cards was used to accelerate calculations. What was new is that stream processing made it possible to develop parallel programs in a high-level language. This technology substantially simplified programming by enabling programs to be written in C/C++.

Quantum chemistry calculations ^{[2] [3] [4] [5] [6]} and molecular mechanics simulations ^{[7] [8] [9]} (molecular modeling in terms of classical mechanics) are among beneficial applications of this technology. The video cards can accelerate the calculations tens of times. Thus, a PC with such a card has the power similar to that of a cluster of workstations based on the common processors.



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External links

- More links for MD on GPUs (http://www.nvidia.com/object/molecular_dynamics.html)

GPU accelerated software

Programs

- Abalone
- AceMD (<http://www.acellera.com/index.php?arg=acemd>) the biomolecular MD package used by GPUGRID
- AMBER on GPUs version (<http://ambermd.org/gpus/>)
- Ascalaph (<http://www.biomolecular-modeling.com/Products.html>) on GPUs version – Ascalaph Liquid GPU (<http://www.biomolecular-modeling.com/Ascalaph/Ascalaph-Liquid.html>)
- BigDFT *Ab initio* program based on wavelet
- GROMACS on GPUs version (https://simtk.org/project/xml/downloads.xml?group_id=161#package_id600)
- HALMD (<http://colberg.org/research/halmd/>) – Highly Accelerated Large-scale MD package
- HOOMD (<http://codeblue.umich.edu/hoomd-blue/index.html>) – Highly Optimized Object Oriented Molecular Dynamics
- LAMMPS on GPUs version – gpulammps (<http://code.google.com/p/gpulammps/>)
- TeraChem - Quantum chemistry and *ab initio* Molecular Dynamics
- VMD & NAMD on GPUs versions (<http://www.ks.uiuc.edu/Research/gpu/>)

API

- OpenMM (<http://simtk.org/home/openmm/>) – an API for accelerating molecular dynamics on GPUs, v1.0 provides GPU-accelerated version of GROMACS

Distributed computing projects

- GPUGRID (<http://www.gpugrid.net/>) distributed supercomputing infrastructure
- Folding@Home (<http://folding.stanford.edu/>) distributed computing project

Monte Carlo Methods

Monte Carlo methods (or **Monte Carlo experiments**) are a class of computational algorithms that rely on repeated random sampling to compute their results. Monte Carlo methods are often used in simulating physical and mathematical systems. These methods are most suited to calculation by a computer and tend to be used when it is infeasible to compute an exact result with a deterministic algorithm.^[1] This method is also used to complement the theoretical derivations.

Monte Carlo methods are especially useful for simulating systems with many coupled degrees of freedom, such as fluids, disordered materials, strongly coupled solids, and cellular structures (see cellular Potts model). They are used to model phenomena with significant uncertainty in inputs, such as the calculation of risk in business. They are widely used in mathematics, for example to evaluate multidimensional definite integrals with complicated boundary conditions. When Monte Carlo simulations have been applied in space exploration and oil exploration, their predictions of failures, cost overruns and schedule overruns are routinely better than human intuition or alternative "soft" methods.^[2]

The Monte Carlo method was coined in the 1940s by John von Neumann and Stanislaw Ulam, while they were working on nuclear weapon projects in the Los Alamos National Laboratory. It was named in homage to Monte Carlo casino, a famous casino, where Ulam's uncle would often gamble away his money.^[3]

Introduction

Monte Carlo methods vary, but tend to follow a particular pattern:

1. Define a domain of possible inputs.
2. Generate inputs randomly from a probability distribution over the domain.
3. Perform a deterministic computation on the inputs.
4. Aggregate the results.

For example, given that a circle inscribed in a square and the square itself have a ratio of areas that is $\pi/4$, the value of π can be approximated using a Monte Carlo method:^[4]

1. Draw a square on the ground, then inscribe a circle within it.
2. Uniformly scatter some objects of uniform size (grains of rice or sand) over the square.
3. Count the number of objects inside the circle and the total number of objects.
4. The ratio of the two counts is an estimate of the ratio of the two areas, which is $\pi/4$. Multiply the result by 4 to estimate π .

In this procedure the domain of inputs is the square that circumscribes our circle. We generate random inputs by scattering grains over the square then perform a computation on each input (test whether it falls within the circle). Finally, we aggregate the results to obtain our final result, the approximation of π .

To get an accurate approximation for π this procedure should have two other common properties of Monte Carlo methods. First, the inputs should truly be random. If grains are purposefully dropped into only the center of the

circle, they will not be uniformly distributed, and so our approximation will be poor. Second, there should be a large number of inputs. The approximation will generally be poor if only a few grains are randomly dropped into the whole square. On average, the approximation improves as more grains are dropped.

History

Before the Monte Carlo method was developed, simulations tested a previously understood deterministic problem and statistical sampling was used to estimate uncertainties in the simulations. Monte Carlo simulations invert this approach, solving deterministic problems using a probabilistic analog (see Simulated annealing).

An early variant of the Monte Carlo method can be seen in the Buffon's needle experiment, in which π can be estimated by dropping needles on a floor made of parallel strips of wood. In the 1930s, Enrico Fermi first experimented with the Monte Carlo method while studying neutron diffusion, but did not publish anything on it.^[3]

In 1946, physicists at Los Alamos Scientific Laboratory were investigating radiation shielding and the distance that neutrons would likely travel through various materials. Despite having most of the necessary data, such as the average distance a neutron would travel in a substance before it collided with an atomic nucleus or how much energy the neutron was likely to give off following a collision, the problem could not be solved with analytical calculations. Stanisław Ulam had the idea of using random experiments. He recounts his inspiration as follows:

The first thoughts and attempts I made to practice [the Monte Carlo Method] were suggested by a question which occurred to me in 1946 as I was convalescing from an illness and playing solitaires. The question was what are the chances that a Canfield solitaire laid out with 52 cards will come out successfully? After spending a lot of time trying to estimate them by pure combinatorial calculations, I wondered whether a more practical method than "abstract thinking" might not be to lay it out say one hundred times and simply observe and count the number of successful plays. This was already possible to envisage with the beginning of the new era of fast computers, and I immediately thought of problems of neutron diffusion and other questions of mathematical physics, and more generally how to change processes described by certain differential equations into an equivalent form interpretable as a succession of random operations. Later [in 1946, I] described the idea to John von Neumann, and we began to plan actual calculations.

—Stanisław Ulam^[5]

Being secret, the work of von Neumann and Ulam required a code name. Von Neumann chose the name "Monte Carlo". The name is a reference to the Monte Carlo Casino in Monaco where Ulam's uncle would borrow money to gamble.^{[1] [6] [7]} Using lists of "truly" random numbers was extremely slow, von Neumann developed a form of making pseudorandom numbers, using the middle-square method. Though this method has been criticized as crude, von Neumann was aware of this: he justified it as being faster than any other method at his disposal, and also noted that when it went awry it did so obviously, unlike methods which could be subtly incorrect.

Monte Carlo methods were central to the simulations required for the Manhattan Project, though severely limited by the computational tools at the time. In the 1950s they were used at Los Alamos for early work relating to the development of the hydrogen bomb, and became popularized in the fields of physics, physical chemistry, and operations research. The Rand Corporation and the U.S. Air Force were two of the major organizations responsible for funding and disseminating information on Monte Carlo methods during this time, and they began to find a wide application in many different fields.

Uses of Monte Carlo methods require large amounts of random numbers, and it was their use that spurred the development of pseudorandom number generators, which were far quicker to use than the tables of random numbers that had been previously used for statistical sampling.

Definitions

There is no consensus on how *Monte Carlo* should be defined. For example, Ripley^[8] defines most probabilistic modeling as *stochastic simulation*, with *Monte Carlo* being reserved for Monte Carlo integration and Monte Carlo statistical tests. Sawilowsky^[9] distinguishes between a simulation, Monte Carlo method, and a Monte Carlo simulation. A simulation is a fictitious representation of reality. A Monte Carlo method is a technique that can be used to solve a mathematical or statistical problem. A Monte Carlo simulation uses repeated sampling to determine the properties of some phenomenon. Examples:

- Drawing a pseudo-random uniform variable from the interval $[0,1]$ can be used to simulate the tossing of a coin: If the value is less than or equal to 0.50 designate the outcome as heads, but if the value is greater than 0.50 designate the outcome as tails. This is a simulation, but not a Monte Carlo simulation.
- The area of an irregular figure inscribed in a unit square can be determined by throwing darts at the square and computing the ratio of hits within the irregular figure to the total number of darts thrown. This is a Monte Carlo method of determining area, but not a simulation.
- Drawing a large number of pseudo-random uniform variables from the interval $[0,1]$, and assigning values less than or equal to 0.50 as heads and greater than 0.50 as tails, is a Monte Carlo simulation of the behavior of repeatedly tossing a coin.

Kalos and Whitlock^[4] point out that such distinctions are not always easy to maintain. For example, the emission of radiation from atoms is a natural stochastic process. It can be simulated directly, or its average behavior can be described by stochastic equations that can themselves be solved using Monte Carlo methods. "Indeed, the same computer code can be viewed simultaneously as a 'natural simulation' or as a solution of the equations by natural sampling."

Monte Carlo and random numbers

Interestingly, Monte Carlo simulation methods do not always require truly random numbers to be useful — while for some applications, such as primality testing, unpredictability is vital.^[10] Many of the most useful techniques use deterministic, pseudorandom sequences, making it easy to test and re-run simulations. The only quality usually necessary to make good simulations is for the pseudo-random sequence to appear "random enough" in a certain sense.

What this means depends on the application, but typically they should pass a series of statistical tests. Testing that the numbers are uniformly distributed or follow another desired distribution when a large enough number of elements of the sequence are considered is one of the simplest, and most common ones.

Sawilowsky lists the characteristics of a high quality Monte Carlo simulation:^[9]

- the (pseudo-random) number generator has certain characteristics (*e. g.*, a long "period" before the sequence repeats)
 - the (pseudo-random) number generator produces values that pass tests for randomness
 - there are enough samples to ensure accurate results
 - the proper sampling technique is used
 - the algorithm used is valid for what is being modeled
 - it simulates the phenomenon in question.
-

Monte Carlo simulation versus “what if” scenarios

There are ways of using probabilities that are definitely not Monte Carlo simulations—for example, deterministic modeling using single-point estimates. Each uncertain variable within a model is assigned a “best guess” estimate. Scenarios (such as best, worst, or most likely case) for each input variable are chosen and the results recorded.^[11]

By contrast, Monte Carlo simulations sample probability distribution for each variable to produce hundreds or thousands of possible outcomes. The results are analyzed to get probabilities of different outcomes occurring.^[12] For example, a comparison of a spreadsheet cost construction model run using traditional “what if” scenarios, and then run again with Monte Carlo simulation and Triangular probability distributions shows that the Monte Carlo analysis has a narrower range than the “what if” analysis. This is because the “what if” analysis gives equal weight to all scenarios (see quantifying uncertainty in corporate finance).

Applications

Monte Carlo methods are especially useful for simulating phenomena with significant uncertainty in inputs and systems with a large number of coupled degrees of freedom. Areas of application include:

Physical sciences

Monte Carlo methods are very important in computational physics, physical chemistry, and related applied fields, and have diverse applications from complicated quantum chromodynamics calculations to designing heat shields and aerodynamic forms. In statistical physics Monte Carlo molecular modeling is an alternative to computational molecular dynamics, and Monte Carlo methods are used to compute statistical field theories of simple particle and polymer systems.^[13] Quantum Monte Carlo methods solve the many-body problem for quantum systems. In experimental particle physics, Monte Carlo methods are used for designing detectors, understanding their behavior and comparing experimental data to theory. In astrophysics, they are used to model the evolution of galaxies.^[14]

Monte Carlo methods are also used in the ensemble models that form the basis of modern weather forecasting.

Engineering

Monte Carlo methods are widely used in engineering for sensitivity analysis and quantitative probabilistic analysis in process design. The need arises from the interactive, co-linear and non-linear behavior of typical process simulations. For example,

- in microelectronics engineering, Monte Carlo methods are applied to analyze correlated and uncorrelated variations in analog and digital integrated circuits. This enables designers to estimate realistic 3-sigma corners and effectively optimize circuit yields.
- in geostatistics and geometallurgy, Monte Carlo methods underpin the design of mineral processing flowsheets and contribute to quantitative risk analysis.
- impacts of pollution are simulated^[15] and diesel compared with petrol.^[16]
- In autonomous robotics, Monte Carlo localization can be used to determine the position of a robot, it is often applied to stochastic filters such as the Kalman filter or Particle filter which form the heart of the SLAM (simultaneous Localisation and Mapping) algorithm.

Computational Biology

Monte Carlo methods are used in computational biology, such for as Bayesian inference in phylogeny.

Biological systems such as proteins^[17] membranes,^[18] images of cancer,^[19] are being studied by means of computer simulations.

The systems can be studied in the coarse-grained or *ab initio* frameworks depending on the desired accuracy. Computer simulations allow us to monitor the local environment of a particular molecule to see if some chemical reaction is happening for instance. We can also conduct thought experiments when the physical experiments are not feasible, for instance breaking bonds, introducing impurities at specific sites, changing the local/global structure, or introducing external fields.

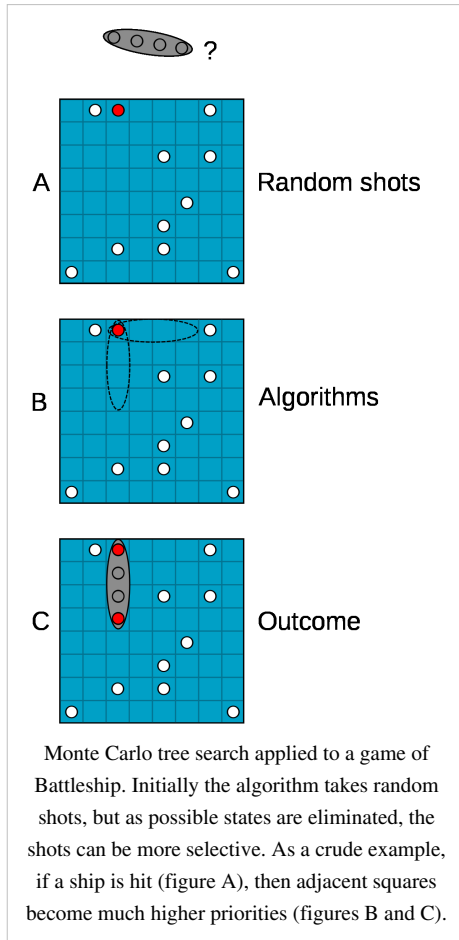
Applied statistics

In applied statistics, Monte Carlo methods are generally used for two purposes:

1. To compare competing statistics for small samples under realistic data conditions. Although Type I error and power properties of statistics can be calculated for data drawn from classical theoretical distributions (*e.g.*, normal curve, Cauchy distribution) for asymptotic conditions (*i. e.*, infinite sample size and infinitesimally small treatment effect), real data often do not have such distributions.^[20]
2. To provide implementations of hypothesis tests that are more efficient than exact tests such as permutation tests (which are often impossible to compute) while being more accurate than critical values for asymptotic distributions.

Monte Carlo methods are also a compromise between approximate randomization and permutation tests. An approximate randomization test is based on a specified subset of all permutations (which entails potentially enormous housekeeping of which permutations have been considered). The Monte Carlo approach is based on a specified number of randomly drawn permutations (exchanging a minor loss in precision if a permutation is drawn twice – or more frequently – for the efficiency of not having to track which permutations have already been selected).

Games



Monte Carlo methods have recently been incorporated in algorithms for playing games that have outperformed previous algorithms in games like Go and Battleship. These algorithms employ *Monte Carlo tree search*. Possible algorithms are organized in a tree and a large number of random simulations are used to estimate the long-term potential of each move. A black box simulator represents the opponent's moves. In games like Battleship, where there is only limited knowledge of the state of the system (*i.e.*, the positions of the ships), a belief state is constructed consisting of probabilities for each state and then initial states are sampled for running simulations. The belief state is updated as the game proceeds, as in the figure. On a 10 x 10 grid, in which the total possible number of moves is 100, one algorithm sank all the ships 50 moves faster, on average, than random play.^[21]

One of the main problems that this approach has in game playing is that it sometimes misses an isolated, very good move. These approaches are often strong strategically but weak tactically, as tactical decisions tend to rely on a small number of crucial moves which are easily missed by the randomly searching Monte Carlo algorithm.

Design and visuals

Monte Carlo methods have also proven efficient in solving coupled integral differential equations of radiation fields and energy transport, and thus these methods have been used in global illumination

computations which produce photo-realistic images of virtual 3D models, with applications in video games, architecture, design, computer generated films, and cinematic special effects.^[22]

Finance and business

Monte Carlo methods in finance are often used to calculate the value of companies, to evaluate investments in projects at a business unit or corporate level, or to evaluate financial derivatives. They can be used to model project schedules, where simulations aggregate estimates for worst-case, best-case, and most likely durations for each task to determine outcomes for the overall project.

Telecommunications

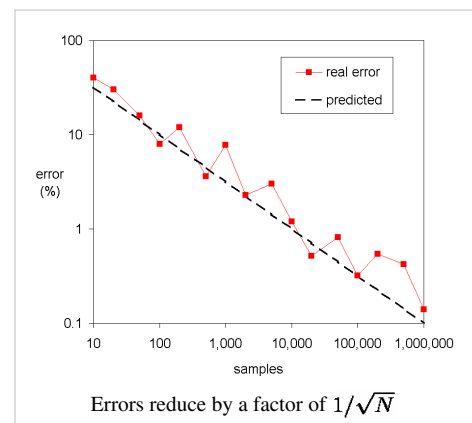
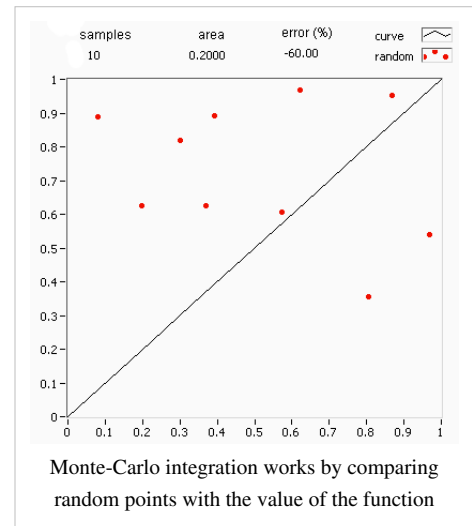
When planning a wireless network, design must be proved to work for a wide variety of scenarios that depend mainly on the number of users, their locations and the services they want to use. Monte Carlo methods are typically used to generate these users and their states. The network performance is then evaluated and, if results are not satisfactory, the network design goes through an optimization process.

Use in mathematics

In general, Monte Carlo methods are used in mathematics to solve various problems by generating suitable random numbers and observing that fraction of the numbers which obeys some property or properties. The method is useful for obtaining numerical solutions to problems which are too complicated to solve analytically. The most common application of the Monte Carlo method is Monte Carlo integration.

Integration

Deterministic numerical integration algorithms work well in a small number of dimensions, but encounter two problems when the functions have many variables. First, the number of function evaluations needed increase rapidly with the number of dimensions. For example, if 10 evaluations provide adequate accuracy in one dimension, then 10^{100} points are needed for 100 dimensions—far too many to be computed. This is called the curse of dimensionality. Second, the boundary of a multidimensional region may be very complicated, so it may not be feasible to reduce the problem to a series of nested one-dimensional integrals.^[23] 100 dimensions is by no means unusual, since in many physical problems, a "dimension" is equivalent to a degree of freedom.



Monte Carlo methods provide a way out of this exponential increase in computation time. As long as the function in question is reasonably well-behaved, it can be estimated by randomly selecting points in 100-dimensional space, and taking some kind of average of the function values at these points. By the law of large numbers, this method will display $1/\sqrt{N}$ convergence—*i.e.*, quadrupling the number of sampled points will halve the error, regardless of the number of dimensions.^[23]

A refinement of this method, known as importance sampling in statistics, involves sampling the points randomly, but more frequently where the integrand is large. To do this precisely one would have to already know the integral, but one can approximate the integral by an integral of a similar function or use adaptive routines such as Stratified sampling, recursive stratified sampling, adaptive umbrella sampling^[24] ^[25] or the VEGAS algorithm.

A similar approach, the quasi-Monte Carlo method, uses low-discrepancy sequences. These sequences "fill" the area better and sample the most important points more frequently, so quasi-Monte Carlo methods can often converge on the integral more quickly.

Another class of methods for sampling points in a volume is to simulate random walks over it (Markov chain Monte Carlo). Such methods include the Metropolis-Hastings algorithm, Gibbs sampling and the Wang and Landau algorithm.

Optimization

Another powerful and very popular application for random numbers in numerical simulation is in numerical optimization. The problem is to minimize (or maximize) functions of some vector that often has a large number of dimensions. Many problems can be phrased in this way: for example, a computer chess program could be seen as trying to find the set of, say, 10 moves that produces the best evaluation function at the end. In the traveling salesman problem the goal is to minimize distance traveled. There are also applications to engineering design, such as multidisciplinary design optimization.

Most Monte Carlo optimization methods are based on random walks. Essentially, the program moves randomly on a multi-dimensional surface, preferring moves that reduce the function, but sometimes moving "uphill".

Inverse problems

Probabilistic formulation of inverse problems leads to the definition of a probability distribution in the model space. This probability distribution combines *a priori* information with new information obtained by measuring some observable parameters (data). As, in the general case, the theory linking data with model parameters is nonlinear, the *a posteriori* probability in the model space may not be easy to describe (it may be multimodal, some moments may not be defined, etc.).

When analyzing an inverse problem, obtaining a maximum likelihood model is usually not sufficient, as we normally also wish to have information on the resolution power of the data. In the general case we may have a large number of model parameters, and an inspection of the marginal probability densities of interest may be impractical, or even useless. But it is possible to pseudorandomly generate a large collection of models according to the posterior probability distribution and to analyze and display the models in such a way that information on the relative likelihoods of model properties is conveyed to the spectator. This can be accomplished by means of an efficient Monte Carlo method, even in cases where no explicit formula for the *a priori* distribution is available.

The best-known importance sampling method, the Metropolis algorithm, can be generalized, and this gives a method that allows analysis of (possibly highly nonlinear) inverse problems with complex *a priori* information and data with an arbitrary noise distribution.^{[26] [27]}

Computational mathematics

Monte Carlo methods are useful in many areas of computational mathematics, where a "lucky choice" can find the correct result. A classic example is Rabin's algorithm for primality testing: for any n which is not prime, a random x has at least a 75% chance of proving that n is not prime. Hence, if n is not prime, but x says that it might be, we have observed at most a 1-in-4 event. If 10 different random x say that " n is probably prime" when it is not, we have observed a one-in-a-million event. In general a Monte Carlo algorithm of this kind produces one correct answer with a guarantee **n is composite, and x proves it so**, but another one without, but with a guarantee of not getting this answer when it is wrong **too often**—in this case at most 25% of the time. See also Las Vegas algorithm for a related, but different, idea.

Notes

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- [2] Hubbard 2009
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- [5] Eckardt 1987
- [6] Grinstead & Snell 1997
- [7] Anderson 1986
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- [9] Sawilowsky 2003
- [10] Davenport 1992
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External links

- Overview and reference list (<http://mathworld.wolfram.com/MonteCarloMethod.html>), Mathworld
- Introduction to Monte Carlo Methods (<http://www.phy.ornl.gov/csep/CSEP/MC/MC.html>), Computational Science Education Project
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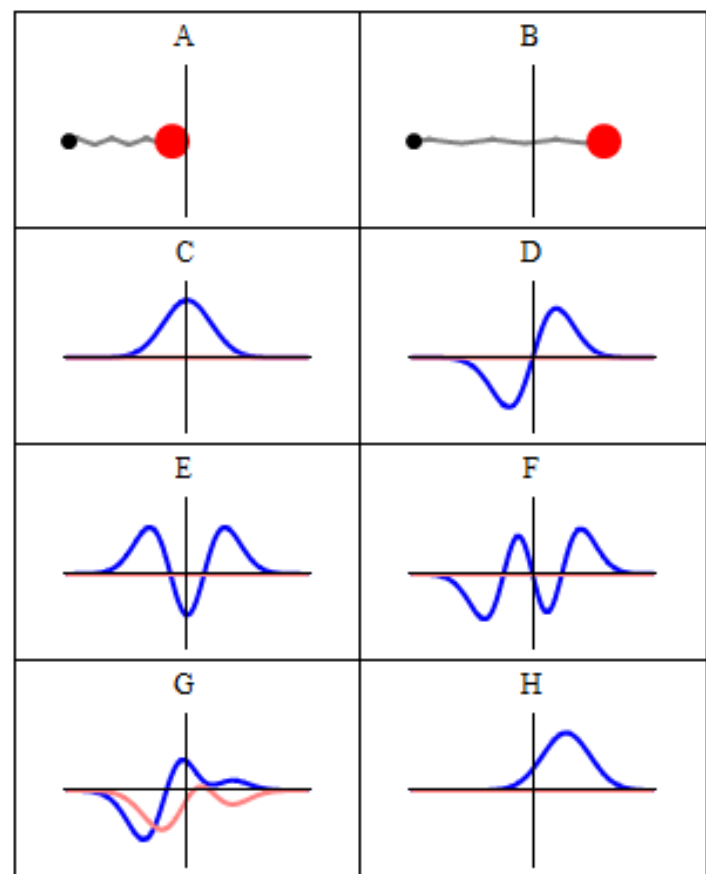
Quantum Dynamics

Quantum mechanics, also known as **quantum physics** or **quantum theory**, is a branch of physics providing a mathematical description of the dual particle-like and wave-like behavior and interaction of matter and energy. Quantum mechanics describes the time evolution of physical systems via a mathematical structure called the wave function. The wave function encapsulates the probability that the system is to be found in a given state at a given time. Quantum mechanics also allows one to calculate the effect on the system of making measurements of properties of the system by defining the effect of those measurements on the wave function. This leads to the well-known uncertainty principle as well as enduring debate over the role of the experimenter, epitomised in the Schrödinger's Cat thought experiment.

Quantum mechanics differs significantly from classical mechanics in its predictions when the scale of observations becomes comparable to the atomic and sub-atomic scale, the so-called quantum realm. However, many macroscopic properties of systems can only be fully understood and explained with the use of quantum mechanics. Phenomena such as superconductivity, the properties of materials such as semiconductors and nuclear and chemical reaction mechanisms observed as macroscopic behaviour, cannot be explained using classical mechanics.

The term was coined by Max Planck, and derives from the observation that some physical quantities can be changed only by discrete amounts, or quanta, as multiples of the Planck constant, rather than being capable of varying continuously or by any arbitrary amount. For example, the angular momentum, or more generally the action, of an electron bound into an atom or molecule is quantized. Although an unbound electron does not exhibit quantized energy levels, one which is bound in an atomic orbital has quantized values of angular momentum. In the context of quantum mechanics, the wave–particle duality of energy and matter and the uncertainty principle provide a unified view of the behavior of photons, electrons and other atomic-scale objects.

The mathematical formulations of quantum mechanics are abstract. Similarly, the implications are often counter-intuitive in terms of classical physics. The centerpiece of the mathematical formulation is the wavefunction (defined by Schrödinger's wave equation), which describes the probability amplitude of the position and momentum of a particle. Mathematical manipulations of the wavefunction usually involve the bra-ket notation, which requires



Some trajectories of a harmonic oscillator (a ball attached to a spring) in classical mechanics (A-B) and quantum mechanics (C-H). In quantum mechanics, the position of the ball is represented by a wave (called the wavefunction), with real part shown in blue and imaginary part in red. Some of the trajectories, such as C,D,E,F, are standing waves (or "stationary states"). Each standing-wave frequency is proportional to a possible energy level of the oscillator. This "energy quantization" does not occur in classical physics, where the oscillator can have *any* energy.

an understanding of complex numbers and linear functionals. The wavefunction treats the object as a quantum harmonic oscillator and the mathematics is akin to that of acoustic resonance.

Many of the results of quantum mechanics do not have models that are easily visualized in terms of classical mechanics; for instance, the ground state in the quantum mechanical model is a non-zero energy state that is the lowest permitted energy state of a system, rather than a traditional classical system that is thought of as simply being at rest with zero kinetic energy.

Fundamentally, it attempts to explain the peculiar behaviour of matter and energy at the subatomic level—an attempt which has produced more accurate results than classical physics in predicting how individual particles behave. But many unexplained anomalies remain.

Historically, the earliest versions of quantum mechanics were formulated in the first decade of the 20th Century, around the time that atomic theory and the corpuscular theory of light as interpreted by Einstein first came to be widely accepted as scientific fact; these later theories can be viewed as quantum theories of matter and electromagnetic radiation.

Following Schrödinger's breakthrough in deriving his wave equation in the mid-1920s, quantum theory was significantly reformulated away from the old quantum theory, towards the quantum mechanics of Werner Heisenberg, Max Born, Wolfgang Pauli and their associates, becoming a science of probabilities based upon the Copenhagen interpretation of Niels Bohr. By 1930, the reformulated theory had been further unified and formalized by the work of Paul Dirac and John von Neumann, with a greater emphasis placed on measurement, the statistical nature of our knowledge of reality, and philosophical speculations about the role of the observer.

The Copenhagen interpretation quickly became (and remains) the orthodox interpretation. However, due to the absence of conclusive experimental evidence there are also many competing interpretations.

Quantum mechanics has since branched out into almost every aspect of physics, and into other disciplines such as quantum chemistry, quantum electronics, quantum optics and quantum information science. Much 19th Century physics has been re-evaluated as the classical limit of quantum mechanics and its more advanced developments in terms of quantum field theory, string theory, and speculative quantum gravity theories.

History

The history of quantum mechanics dates back to the 1838 discovery of cathode rays by Michael Faraday. This was followed by the 1859 statement of the black body radiation problem by Gustav Kirchhoff, the 1877 suggestion by Ludwig Boltzmann that the energy states of a physical system can be discrete, and the 1900 quantum hypothesis of Max Planck.^[1] Planck's hypothesis that energy is radiated and absorbed in discrete "quanta", or "energy elements", precisely matched the observed patterns of black body radiation. According to Planck, each energy element E is proportional to its frequency ν :

$$E = h\nu$$

where h is Planck's constant. Planck cautiously insisted that this was simply an aspect of the processes of absorption and emission of radiation and had nothing to do with the physical reality of the radiation itself.^[2] However, in 1905 Albert Einstein interpreted Planck's quantum hypothesis realistically and used it to explain the photoelectric effect, in which shining light on certain materials can eject electrons from the material. Einstein postulated that light itself consists of individual quanta of energy, later called photons.^[3]

The foundations of quantum mechanics were established during the first half of the twentieth century by Niels Bohr, Werner Heisenberg, Max Planck, Louis de Broglie, Albert Einstein, Erwin Schrödinger, Max Born, John von Neumann, Paul Dirac, Wolfgang Pauli, David Hilbert, and others. In the mid-1920s, developments in quantum mechanics led to its becoming the standard formulation for atomic physics. In the summer of 1925, Bohr and Heisenberg published results that closed the "Old Quantum Theory". Out of deference to their dual state as particles, light quanta came to be called photons (1926). From Einstein's simple postulation was born a flurry of debating,

theorizing and testing. Thus the entire field of quantum physics emerged, leading to its wider acceptance at the Fifth Solvay Conference in 1927.

The other exemplar that led to quantum mechanics was the study of electromagnetic waves such as light. When it was found in 1900 by Max Planck that the energy of waves could be described as consisting of small packets or quanta, Albert Einstein further developed this idea to show that an electromagnetic wave such as light could be described as a particle - later called the photon - with a discrete energy that was dependent on its frequency. This led to a theory of unity between subatomic particles and electromagnetic waves called wave-particle duality in which particles and waves were neither one nor the other, but had certain properties of both.

While quantum mechanics traditionally described the world of the very small, it is also needed to explain certain recently investigated macroscopic systems such as superconductors and superfluids.

The word *quantum* derives from Latin, meaning "how great" or "how much".^[4] In quantum mechanics, it refers to a discrete unit that quantum theory assigns to certain physical quantities, such as the energy of an atom at rest (see Figure 1). The discovery that particles are discrete packets of energy with wave-like properties led to the branch of physics dealing with atomic and sub-atomic systems which is today called quantum mechanics. It is the underlying mathematical framework of many fields of physics and chemistry, including condensed matter physics, solid-state physics, atomic physics, molecular physics, computational physics, computational chemistry, quantum chemistry, particle physics, nuclear chemistry, and nuclear physics.^[5] Some fundamental aspects of the theory are still actively studied.^[6]

Quantum mechanics is essential to understand the behavior of systems at atomic length scales and smaller. For example, if classical mechanics governed the workings of an atom, electrons would rapidly travel towards and collide with the nucleus, making stable atoms impossible. However, in the natural world the electrons normally remain in an uncertain, non-deterministic "smeared" (wave-particle wave function) orbital path around or through the nucleus, defying classical electromagnetism.^[7]

Quantum mechanics was initially developed to provide a better explanation of the atom, especially the differences in the spectra of light emitted by different isotopes of the same element. The quantum theory of the atom was developed as an explanation for the electron remaining in its orbit, which could not be explained by Newton's laws of motion and Maxwell's laws of classical electromagnetism.

Broadly speaking, quantum mechanics incorporates four classes of phenomena for which classical physics cannot account:

- The quantization of certain physical properties
- Wave-particle duality
- The uncertainty principle
- Quantum entanglement

Mathematical formulations

In the mathematically rigorous formulation of quantum mechanics developed by Paul Dirac^[8] and John von Neumann,^[9] the possible states of a quantum mechanical system are represented by unit vectors (called "state vectors"). Formally, these reside in a complex separable Hilbert space (variously called the "state space" or the "associated Hilbert space" of the system) well defined up to a complex number of norm 1 (the phase factor). In other words, the possible states are points in the projective space of a Hilbert space, usually called the complex projective space. The exact nature of this Hilbert space is dependent on the system; for example, the state space for position and momentum states is the space of square-integrable functions, while the state space for the spin of a single proton is just the product of two complex planes. Each observable is represented by a maximally Hermitian (precisely: by a self-adjoint) linear operator acting on the state space. Each eigenstate of an observable corresponds to an eigenvector of the operator, and the associated eigenvalue corresponds to the value of the observable in that eigenstate. If the

operator's spectrum is discrete, the observable can only attain those discrete eigenvalues.

In the formalism of quantum mechanics, the state of a system at a given time is described by a complex wave function, also referred to as state vector in a complex vector space.^[10] This abstract mathematical object allows for the calculation of probabilities of outcomes of concrete experiments. For example, it allows one to compute the probability of finding an electron in a particular region around the nucleus at a particular time. Contrary to classical mechanics, one can never make simultaneous predictions of conjugate variables, such as position and momentum, with accuracy. For instance, electrons may be considered to be located somewhere within a region of space, but with their exact positions being unknown. Contours of constant probability, often referred to as "clouds", may be drawn around the nucleus of an atom to conceptualize where the electron might be located with the most probability. Heisenberg's uncertainty principle quantifies the inability to precisely locate the particle given its conjugate momentum.^[11]

According to one interpretation, as the result of a measurement the wave function containing the probability information for a system collapses from a given initial state to a particular eigenstate. The possible results of a measurement are the eigenvalues of the operator representing the observable — which explains the choice of *Hermitian* operators, for which all the eigenvalues are real. We can find the probability distribution of an observable in a given state by computing the spectral decomposition of the corresponding operator. Heisenberg's uncertainty principle is represented by the statement that the operators corresponding to certain observables do not commute.

The probabilistic nature of quantum mechanics thus stems from the act of measurement. This is one of the most difficult aspects of quantum systems to understand. It was the central topic in the famous Bohr-Einstein debates, in which the two scientists attempted to clarify these fundamental principles by way of thought experiments. In the decades after the formulation of quantum mechanics, the question of what constitutes a "measurement" has been extensively studied. Newer interpretations of quantum mechanics have been formulated that do away with the concept of "wavefunction collapse"; see, for example, the relative state interpretation. The basic idea is that when a quantum system interacts with a measuring apparatus, their respective wavefunctions become entangled, so that the original quantum system ceases to exist as an independent entity. For details, see the article on measurement in quantum mechanics.^[12] Generally, quantum mechanics does not assign definite values. Instead, it makes predictions using probability distributions; that is, it describes the probability of obtaining possible outcomes from measuring an observable. Often these results are skewed by many causes, such as dense probability clouds^[13] or quantum state nuclear attraction.^{[14] [15]} Naturally, these probabilities will depend on the quantum state at the "instant" of the measurement. Hence, uncertainty is involved in the value. There are, however, certain states that are associated with a definite value of a particular observable. These are known as eigenstates of the observable ("eigen" can be translated from German as meaning inherent or characteristic).^[16]

In the everyday world, it is natural and intuitive to think of everything (every observable) as being in an eigenstate. Everything appears to have a definite position, a definite momentum, a definite energy, and a definite time of occurrence. However, quantum mechanics does not pinpoint the exact values of a particle's position and momentum (since they are conjugate pairs) or its energy and time (since they too are conjugate pairs); rather, it only provides a range of probabilities of where that particle might be given its momentum and momentum probability. Therefore, it is helpful to use different words to describe states having *uncertain* values and states having *definite* values (eigenstate). Usually, a system will not be in an eigenstate of the observable (particle) we are interested in. However, if one measures the observable, the wavefunction will instantaneously be an eigenstate (or generalised eigenstate) of that observable. This process is known as wavefunction collapse, a controversial and much debated process.^[17] It involves expanding the system under study to include the measurement device. If one knows the corresponding wave function at the instant before the measurement, one will be able to compute the probability of collapsing into each of the possible eigenstates. For example, the free particle in the previous example will usually have a wavefunction that is a wave packet centered around some mean position x_0 , neither an eigenstate of position nor of momentum. When one measures the position of the particle, it is impossible to predict with certainty the result.^[12] It is probable, but not

certain, that it will be near x_0 , where the amplitude of the wave function is large. After the measurement is performed, having obtained some result x , the wave function collapses into a position eigenstate centered at x .^[18]

The time evolution of a quantum state is described by the Schrödinger equation, in which the Hamiltonian (the operator corresponding to the total energy of the system) generates time evolution. The time evolution of wave functions is deterministic in the sense that, given a wavefunction at an initial time, it makes a definite prediction of what the wavefunction will be at any later time.^[19]

During a measurement, on the other hand, the change of the wavefunction into another one is not deterministic; it is unpredictable, i.e. random. A time-evolution simulation can be seen here.^{[20] [21]} Wave functions can change as time progresses. An equation known as the Schrödinger equation describes how wavefunctions change in time, a role similar to Newton's second law in classical mechanics. The Schrödinger equation, applied to the aforementioned example of the free particle, predicts that the center of a wave packet will move through space at a constant velocity, like a classical particle with no forces acting on it. However, the wave packet will also spread out as time progresses, which means that the position becomes more uncertain. This also has the effect of turning position eigenstates (which can be thought of as infinitely sharp wave packets) into broadened wave packets that are no longer position eigenstates.^[22]

Some wave functions produce probability distributions that are constant, or independent of time, such as when in a stationary state of constant energy, time drops out of the absolute square of the wave function. Many systems that are treated dynamically in classical mechanics are described by such "static" wave functions. For example, a single electron in an unexcited atom is pictured classically as a particle moving in a circular trajectory around the atomic nucleus, whereas in quantum mechanics it is described by a static, spherically symmetric wavefunction surrounding the nucleus (Fig. 1). (Note that only the lowest angular momentum states, labeled s , are spherically symmetric).^[23]

The Schrödinger equation acts on the entire probability amplitude, not merely its absolute value. Whereas the absolute value of the probability amplitude encodes information about probabilities, its phase encodes information about the interference between quantum states. This gives rise to the wave-like behavior of quantum states. It turns out that analytic solutions of Schrödinger's equation are only available for a small number of model Hamiltonians, of which the quantum harmonic oscillator, the particle in a box, the hydrogen molecular ion and the hydrogen atom are the most important representatives. Even the helium atom, which contains just one more electron than hydrogen, defies all attempts at a fully analytic treatment. There exist several techniques for generating approximate solutions. For instance, in the method known as perturbation theory one uses the analytic results for a simple quantum mechanical model to generate results for a more complicated model related to the simple model by,

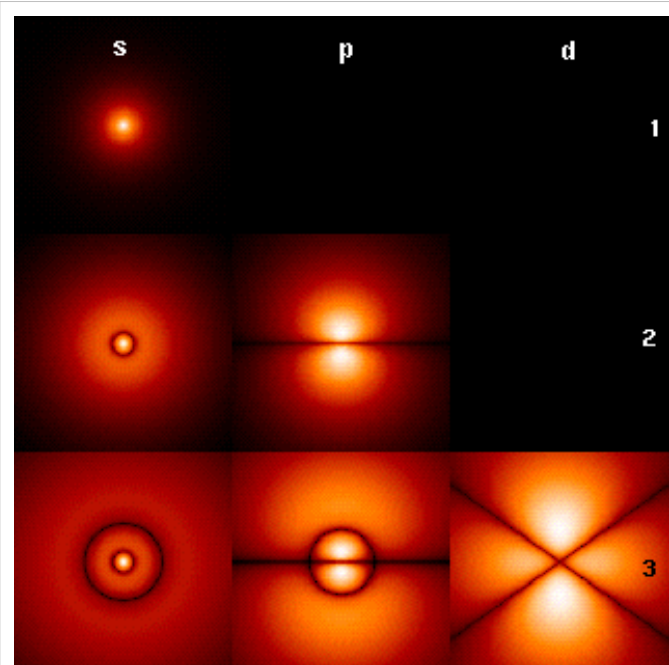


Fig. 1: Probability densities corresponding to the wavefunctions of an electron in a hydrogen atom possessing definite energy levels (increasing from the top of the image to the bottom: $n = 1, 2, 3, \dots$) and angular momentum (increasing across from left to right: s, p, d, \dots). Brighter areas correspond to higher probability density in a position measurement. Wavefunctions like these are directly comparable to Chladni's figures of acoustic modes of vibration in classical physics and are indeed modes of oscillation as well: they possess a sharp energy and thus a keen frequency. The angular momentum and energy are quantized, and only take on discrete values like those shown (as is the case for resonant frequencies in acoustics).

for example, the addition of a weak potential energy. Another method is the "semi-classical equation of motion" approach, which applies to systems for which quantum mechanics produces weak deviations from classical behavior. The deviations can be calculated based on the classical motion. This approach is important for the field of quantum chaos.

There are numerous mathematically equivalent formulations of quantum mechanics. One of the oldest and most commonly used formulations is the transformation theory proposed by Cambridge theoretical physicist Paul Dirac, which unifies and generalizes the two earliest formulations of quantum mechanics, matrix mechanics (invented by Werner Heisenberg)^{[24] [25]} and wave mechanics (invented by Erwin Schrödinger).^[26] In this formulation, the instantaneous state of a quantum system encodes the probabilities of its measurable properties, or "observables". Examples of observables include energy, position, momentum, and angular momentum. Observables can be either continuous (e.g., the position of a particle) or discrete (e.g., the energy of an electron bound to a hydrogen atom).^[27] An alternative formulation of quantum mechanics is Feynman's path integral formulation, in which a quantum-mechanical amplitude is considered as a sum over histories between initial and final states; this is the quantum-mechanical counterpart of action principles in classical mechanics.

Interactions with other scientific theories

The rules of quantum mechanics are fundamental; they assert that the state space of a system is a Hilbert space and that observables of that system are Hermitian operators acting on that space; they do not tell us which Hilbert space or which operators. These can be chosen appropriately in order to obtain a quantitative description of a quantum system. An important guide for making these choices is the correspondence principle, which states that the predictions of quantum mechanics reduce to those of classical physics when a system moves to higher energies or, equivalently, larger quantum numbers (i.e. whereas a single particle exhibits a degree of randomness, in systems incorporating millions of particles averaging takes over and, at the high energy limit, the statistical probability of random behaviour approaches zero). In other words, classical mechanics is simply a quantum mechanics of large systems. This "high energy" limit is known as the *classical* or *correspondence limit*. One can even start from an established classical model of a particular system, and attempt to guess the underlying quantum model that would give rise to the classical model in the correspondence limit.

When quantum mechanics was originally formulated, it was applied to models whose correspondence limit was non-relativistic classical mechanics. For instance, the well-known model of the quantum harmonic oscillator uses an explicitly non-relativistic expression for the kinetic energy of the oscillator, and is thus a quantum version of the classical harmonic oscillator.

Early attempts to merge quantum mechanics with special relativity involved the replacement of the Schrödinger equation with a covariant equation such as the Klein-Gordon equation or the Dirac equation. While these theories were successful in explaining many experimental results, they had certain unsatisfactory qualities stemming from their neglect of the relativistic creation and annihilation of particles. A fully relativistic quantum theory required the development of quantum field theory, which applies quantization to a field rather than a fixed set of particles. The first complete quantum field theory, quantum electrodynamics, provides a fully quantum description of the electromagnetic interaction. The full apparatus of quantum field theory is often unnecessary for describing electrodynamic systems. A simpler approach, one employed since the inception of quantum mechanics, is to treat charged particles as quantum mechanical objects being acted on by a classical electromagnetic field. For example, the elementary quantum model of the hydrogen atom describes the electric field of the hydrogen atom using a classical $-e^2/(4\pi \epsilon_0 r)$ Coulomb potential. This "semi-classical" approach fails if quantum fluctuations in the electromagnetic field play an important role, such as in the emission of photons by charged particles.

Quantum field theories for the strong nuclear force and the weak nuclear force have been developed. The quantum field theory of the strong nuclear force is called quantum chromodynamics, and describes the interactions of subnuclear particles: quarks and gluons. The weak nuclear force and the electromagnetic force were unified, in their

quantized forms, into a single quantum field theory known as electroweak theory, by the physicists Abdus Salam, Sheldon Glashow and Steven Weinberg. These three men shared the Nobel Prize in Physics in 1979 for this work.^[28]

It has proven difficult to construct quantum models of gravity, the remaining fundamental force. Semi-classical approximations are workable, and have led to predictions such as Hawking radiation. However, the formulation of a complete theory of quantum gravity is hindered by apparent incompatibilities between general relativity, the most accurate theory of gravity currently known, and some of the fundamental assumptions of quantum theory. The resolution of these incompatibilities is an area of active research, and theories such as string theory are among the possible candidates for a future theory of quantum gravity.

Classical mechanics has been extended into the complex domain, and complex classical mechanics exhibits behaviours similar to quantum mechanics.^[29]

Quantum mechanics and classical physics

Predictions of quantum mechanics have been verified experimentally to a extremely high degree of accuracy. According to the correspondence principle between classical and quantum mechanics, all objects obey the laws of quantum mechanics, and classical mechanics is just an approximation for large systems (or a statistical quantum mechanics of a large collection of particles). The laws of classical mechanics thus follow from the laws of quantum mechanics as a statistical average at the limit of large systems or large quantum numbers.^[30] However, chaotic systems do not have good quantum numbers, and quantum chaos studies the relationship between classical and quantum descriptions in these systems.

Quantum coherence is an essential difference between classical and quantum theories, and is illustrated by the Einstein-Podolsky-Rosen paradox. Quantum interference involves adding together *probability amplitudes*, whereas when classical waves interfere there is an adding together of *intensities*. For microscopic bodies, the extension of the system is much smaller than the coherence length, which gives rise to long-range entanglement and other nonlocal phenomena characteristic of quantum systems.^[31] Quantum coherence is not typically evident at macroscopic scales, although an exception to this rule can occur at extremely low temperatures, when quantum behavior can manifest itself on more macroscopic scales (see Bose-Einstein condensate and Quantum machine). This is in accordance with the following observations:

- Many macroscopic properties of a classical system are a direct consequences of the quantum behavior of its parts. For example, the stability of bulk matter (which consists of atoms and molecules which would quickly collapse under electric forces alone), the rigidity of solids, and the mechanical, thermal, chemical, optical and magnetic properties of matter are all results of the interaction of electric charges under the rules of quantum mechanics.^[32]
- While the seemingly exotic behavior of matter posited by quantum mechanics and relativity theory become more apparent when dealing with extremely fast-moving or extremely tiny particles, the laws of classical Newtonian physics remain accurate in predicting the behavior of the vast majority of large objects—of the order of the size of large molecules and bigger—at velocities much smaller than the velocity of light.^[33]

Relativity and quantum mechanics

Main articles: Quantum gravity and Theory of everything

Even with the defining postulates of both Einstein's theory of general relativity and quantum theory being indisputably supported by rigorous and repeated empirical evidence and while they do not directly contradict each other theoretically (at least with regard to primary claims), they are resistant to being incorporated within one cohesive model.^[34]

Einstein himself is well known for rejecting some of the claims of quantum mechanics. While clearly contributing to the field, he did not accept the more philosophical consequences and interpretations of quantum mechanics, such as the lack of deterministic causality and the assertion that a single subatomic particle can occupy numerous areas of space at one time. He also was the first to notice some of the apparently exotic consequences of entanglement and used them to formulate the Einstein-Podolsky-Rosen paradox, in the hope of showing that quantum mechanics had unacceptable implications. This was 1935, but in 1964 it was shown by John Bell (see Bell inequality) that, although Einstein was correct in identifying seemingly paradoxical implications of quantum mechanical nonlocality, these implications could be experimentally tested. Alain Aspect's initial experiments in 1982, and many subsequent experiments since, have verified quantum entanglement.

According to the paper of J. Bell and the Copenhagen interpretation (the common interpretation of quantum mechanics by physicists since 1927), and contrary to Einstein's ideas, quantum mechanics was not at the same time

- a "realistic" theory
- and a *local* theory.

The Einstein-Podolsky-Rosen paradox shows in any case that there exist experiments by which one can measure the state of one particle and instantaneously change the state of its entangled partner, although the two particles can be an arbitrary distance apart; however, this effect does not violate causality, since no transfer of information happens. Quantum entanglement is at the basis of quantum cryptography, with high-security commercial applications in banking and government.

Gravity is negligible in many areas of particle physics, so that unification between general relativity and quantum mechanics is not an urgent issue in those applications. However, the lack of a correct theory of quantum gravity is an important issue in cosmology and physicists' search for an elegant "theory of everything". Thus, resolving the inconsistencies between both theories has been a major goal of twentieth- and twenty-first-century physics. Many prominent physicists, including Stephen Hawking, have labored in the attempt to discover a theory underlying *everything*, combining not only different models of subatomic physics, but also deriving the universe's four forces—the strong force, electromagnetism, weak force, and gravity—from a single force or phenomenon. While Stephen Hawking was initially a believer in the Theory of Everything, after considering Gödel's Incompleteness Theorem, concluded that one was not obtainable, and stated such publicly in his lecture, "Gödel and the end of physics" in 2002.^[35] One of the leaders in this field is Edward Witten, a theoretical physicist who formulated the groundbreaking M-theory, which is an attempt at describing the supersymmetrical based string theory.

Attempts at a unified field theory

As of 2011 the quest for unifying the fundamental forces through quantum mechanics is still ongoing. Quantum electrodynamics (or "quantum electromagnetism"), which is currently (in the perturbative regime at least) the most accurately tested physical theory,^[36] has been successfully merged with the weak nuclear force into the electroweak force and work is currently being done to merge the electroweak and strong force into the electrostrong force. Current predictions state that at around 10^{14} GeV the three aforementioned forces are fused into a single unified field,^[37] Beyond this "grand unification," it is speculated that it may be possible to merge gravity with the other three gauge symmetries, expected to occur at roughly 10^{19} GeV. However— and while special relativity is parsimoniously incorporated into quantum electrodynamics—the expanded general relativity, currently the best

theory describing the gravitation force, has not been fully incorporated into quantum theory.

Philosophical implications

Since its inception, the many counter-intuitive results of quantum mechanics have provoked strong philosophical debate and many interpretations. Even fundamental issues such as Max Born's basic rules concerning probability amplitudes and probability distributions took decades to be appreciated.

Richard Feynman said, "I think I can safely say that nobody understands quantum mechanics."^[38]

The Copenhagen interpretation, due largely to the Danish theoretical physicist Niels Bohr, is the interpretation of the quantum mechanical formalism most widely accepted amongst physicists. According to it, the probabilistic nature of quantum mechanics is not a temporary feature which will eventually be replaced by a deterministic theory, but instead must be considered to be a final renunciation of the classical ideal of causality. In this interpretation, it is believed that any well-defined application of the quantum mechanical formalism must always make reference to the experimental arrangement, due to the complementarity nature of evidence obtained under different experimental situations.

Albert Einstein, himself one of the founders of quantum theory, disliked this loss of determinism in measurement. (A view paraphrased as "God does not play dice with the universe.") Einstein held that there should be a local hidden variable theory underlying quantum mechanics and that, consequently, the present theory was incomplete. He produced a series of objections to the theory, the most famous of which has become known as the Einstein-Podolsky-Rosen paradox. John Bell showed that the EPR paradox led to experimentally testable differences between quantum mechanics and local realistic theories. Experiments have been performed confirming the accuracy of quantum mechanics, thus demonstrating that the physical world cannot be described by local realistic theories.^[39] The *Bohr-Einstein debates* provide a vibrant critique of the Copenhagen Interpretation from an epistemological point of view.

The Everett many-worlds interpretation, formulated in 1956, holds that all the possibilities described by quantum theory simultaneously occur in a multiverse composed of mostly independent parallel universes.^[40] This is not accomplished by introducing some new axiom to quantum mechanics, but on the contrary by *removing* the axiom of the collapse of the wave packet: All the possible consistent states of the measured system and the measuring apparatus (including the observer) are present in a *real* physical (not just formally mathematical, as in other interpretations) quantum superposition. Such a superposition of consistent state combinations of different systems is called an entangled state. While the multiverse is deterministic, we perceive non-deterministic behavior governed by probabilities, because we can observe only the universe, i.e. the consistent state contribution to the mentioned superposition, we inhabit. Everett's interpretation is perfectly consistent with John Bell's experiments and makes them intuitively understandable. However, according to the theory of quantum decoherence, the parallel universes will never be accessible to us. This inaccessibility can be understood as follows: Once a measurement is done, the measured system becomes entangled with both the physicist who measured it and a huge number of other particles, some of which are photons flying away towards the other end of the universe; in order to prove that the wave function did not collapse one would have to bring all these particles back and measure them again, together with the system that was measured originally. This is completely impractical, but even if one could theoretically do this, it would destroy any evidence that the original measurement took place (including the physicist's memory).

Applications

Quantum mechanics had enormous success in explaining many of the features of our world. The individual behaviour of the subatomic particles that make up all forms of matter—electrons, protons, neutrons, photons and others—can often only be satisfactorily described using quantum mechanics. Quantum mechanics has strongly influenced string theory, a candidate for a theory of everything (see reductionism) and the multiverse hypothesis.

Quantum mechanics is important for understanding how individual atoms combine covalently to form chemicals or molecules. The application of quantum mechanics to chemistry is known as quantum chemistry. (Relativistic) quantum mechanics can in principle mathematically describe most of chemistry. Quantum mechanics can provide quantitative insight into ionic and covalent bonding processes by explicitly showing which molecules are energetically favorable to which others, and by approximately how much.^[41] Most of the calculations performed in computational chemistry rely on quantum mechanics.^[42]

Much of modern technology operates at a scale where quantum effects are significant. Examples include the laser, the transistor (and thus the microchip), the electron microscope, and magnetic resonance imaging. The study of semiconductors led to the invention of the diode and the transistor, which are indispensable for modern electronics.

Researchers are currently seeking robust methods of directly manipulating quantum states. Efforts are being made to develop quantum cryptography, which will allow guaranteed secure transmission of information. A more distant goal is the development of quantum computers, which are expected to perform certain computational tasks exponentially faster than classical computers. Another active research topic is quantum teleportation, which deals with techniques to transmit quantum information over arbitrary distances.

Quantum tunneling is vital in many devices, even in the simple light switch, as otherwise the electrons in the electric current could not penetrate the potential barrier made up of a layer of oxide. Flash memory chips found in USB drives use quantum tunneling to erase their memory cells.

Quantum mechanics primarily applies to the atomic regimes of matter and energy, but some systems exhibit quantum mechanical effects on a large scale; superfluidity (the frictionless flow of a liquid at temperatures near absolute zero) is one well-known example. Quantum theory also provides accurate descriptions for many previously unexplained phenomena such as black body radiation and the stability of electron orbitals. It has also given insight into the workings of many different biological systems, including smell receptors and protein structures.^[43] Recent work on photosynthesis has provided evidence that quantum correlations play an essential role in this most fundamental process of the plant kingdom.^[44] Even so, classical physics often can be a good approximation to results otherwise obtained by **quantum physics**, typically in circumstances with large numbers of particles or large quantum numbers. (However, some open questions remain in the field of quantum chaos.)

Examples

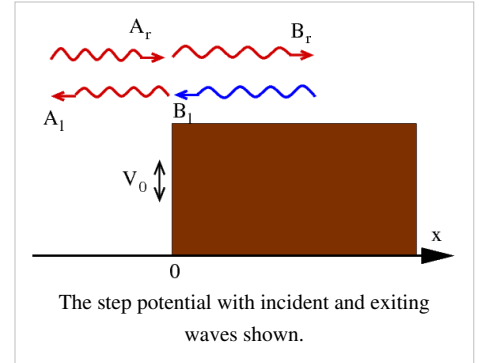
Free particle

For example, consider a free particle. In quantum mechanics, there is wave-particle duality so the properties of the particle can be described as the properties of a wave. Therefore, its quantum state can be represented as a wave of arbitrary shape and extending over space as a wave function. The position and momentum of the particle are observables. The Uncertainty Principle states that both the position and the momentum cannot simultaneously be measured with full precision at the same time. However, one can measure the position alone of a moving free particle creating an eigenstate of position with a wavefunction that is very large (a Dirac delta) at a particular position x and zero everywhere else. If one performs a position measurement on such a wavefunction, the result x will be obtained with 100% probability (full certainty). This is called an eigenstate of position (mathematically more precise: a *generalized position eigenstate (eigendistribution)*). If the particle is in an eigenstate of position then its momentum is completely unknown. On the other hand, if the particle is in an eigenstate of momentum then its

position is completely unknown.^[45] In an eigenstate of momentum having a plane wave form, it can be shown that the wavelength is equal to h/p , where h is Planck's constant and p is the momentum of the eigenstate.^[46]

Step potential

The potential in this case is given by:



$$V(x) = \begin{cases} 0, & x < 0, \\ V_0, & x \geq 0, \end{cases}$$

The solutions are superpositions of left and right moving waves:

$$\psi_L(x) = \frac{1}{\sqrt{k_0}} (A_r e^{ik_0 x} + A_l e^{-ik_0 x}) \quad x < 0,$$

$$\psi_R(x) = \frac{1}{\sqrt{k_1}} (B_r e^{ik_1 x} + B_l e^{-ik_1 x}) \quad x > 0$$

where the wave vectors are related to the energy via

$$k_0 = \sqrt{2mE/\hbar^2}, \text{ and}$$

$$k_1 = \sqrt{2m(E - V_0)/\hbar^2}$$

and the coefficients A and B are determined from the boundary conditions and by imposing a continuous derivative to the solution.

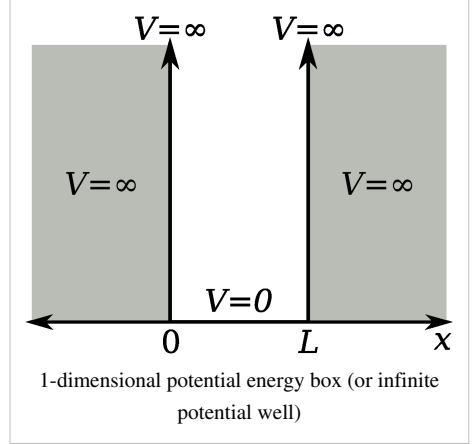
Each term of the solution can be interpreted as an incident, reflected or transmitted component of the wave, allowing the calculation of transmission and reflection coefficients. In contrast to classical mechanics, incident particles with energies higher than the size of the potential step are still partially reflected.

Rectangular potential barrier

This is a model for the quantum tunneling effect, which has important applications to modern devices such as flash memory and the scanning tunneling microscope.

Particle in a box

The particle in a 1-dimensional potential energy box is the most simple example where restraints lead to the quantization of energy levels. The box is defined as having zero potential energy inside a certain region and infinite potential energy everywhere outside that region. For the 1-dimensional case in the x direction, the time-independent Schrödinger equation can be written as:^[47]



$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} = E\psi.$$

Writing the differential operator

$$\hat{p}_x = -i\hbar \frac{d}{dx}$$

the previous equation can be seen to be evocative of the classic analogue

$$\frac{1}{2m} \hat{p}_x^2 = E$$

with E as the energy for the state ψ , in this case coinciding with the kinetic energy of the particle.

The general solutions of the Schrödinger equation for the particle in a box are:

$$\psi(x) = Ae^{ikx} + Be^{-ikx} \quad E = \frac{\hbar^2 k^2}{2m}$$

or, from Euler's formula,

$$\psi(x) = C \sin kx + D \cos kx.$$

The presence of the walls of the box determines the values of C , D , and k . At each wall ($x = 0$ and $x = L$), $\psi = 0$.

Thus when $x = 0$,

$$\psi(0) = 0 = C \sin 0 + D \cos 0 = D$$

and so $D = 0$. When $x = L$,

$$\psi(L) = 0 = C \sin kL.$$

C cannot be zero, since this would conflict with the Born interpretation. Therefore $\sin kL = 0$, and so it must be that kL is an integer multiple of π . Therefore,

$$k = \frac{n\pi}{L} \quad n = 1, 2, 3, \dots$$

The quantization of energy levels follows from this constraint on k , since

$$E = \frac{\hbar^2 \pi^2 n^2}{2mL^2} = \frac{n^2 \hbar^2}{8mL^2}.$$

Finite potential well

This is generalization of the infinite potential well problem to potential wells of finite depth.

Harmonic oscillator

As in the classical case, the potential for the quantum harmonic oscillator is given by:

$$V(x) = \frac{1}{2}m\omega^2 x^2$$

This problem can be solved either by directly solving the Schrödinger equation directly, which is not trivial, or by using the more elegant ladder method, first proposed by Paul Dirac. The eigenstates are given by:

$$\psi_n(x) = \sqrt{\frac{1}{2^n n!}} \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \cdot e^{-\frac{m\omega x^2}{2\hbar}} \cdot H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right), \quad n = 0, 1, 2, \dots$$

where H_n are the Hermite polynomials:

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2})$$

and the corresponding energy levels are

$$E_n = \hbar\omega \left(n + \frac{1}{2}\right).$$

This is another example which illustrates the quantification of energy for bound states.

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- Introduction to Quantum Theory at Quantiki. (http://www.quantiki.org/wiki/index.php/Introduction_to_Quantum_Theory)
- Quantum Physics Made Relatively Simple (<http://bethe.cornell.edu/>): three video lectures by Hans Bethe
- H is for h-bar. (<http://www.nonlocal.com/hbar/>)
- Quantum Mechanics Books Collection (<http://www.freebookcentre.net/Physics/Quantum-Mechanics-Books.html>): Collection of free books

Course material

- Doron Cohen: Lecture notes in Quantum Mechanics (comprehensive, with advanced topics). (<http://arxiv.org/abs/quant-ph/0605180>)
- MIT OpenCourseWare: Chemistry (<http://ocw.mit.edu/OcwWeb/Chemistry/index.htm>).
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Mathematical Formulations of Quantum Dynamics

The **mathematical formulations of quantum mechanics** are those mathematical formalisms that permit a rigorous description of quantum mechanics. Such are distinguished from mathematical formalisms for theories developed prior to the early 1900s by the use of abstract mathematical structures, such as infinite-dimensional Hilbert spaces and operators on these spaces. Many of these structures are drawn from functional analysis, a research area within pure mathematics that was influenced in part by the needs of quantum mechanics. In brief, values of physical observables such as energy and momentum were no longer considered as values of functions on phase space, but as eigenvalues; more precisely: as spectral values (point spectrum plus absolute continuous plus singular continuous spectrum) of linear operators in Hilbert space.^[1]

These formulations of quantum mechanics continue to be used today. At the heart of the description are ideas of *quantum state* and *quantum observable* which are radically different from those used in previous models of physical reality. While the mathematics permits calculation of many quantities that can be measured experimentally, there is a definite theoretical limit to values that can be simultaneously measured. This limitation was first elucidated by Heisenberg through a thought experiment, and is represented mathematically in the new formalism by the non-commutativity of quantum observables.

Prior to the emergence of quantum mechanics as a separate theory, the mathematics used in physics consisted mainly of differential geometry and partial differential equations; probability theory was used in statistical mechanics. Geometric intuition clearly played a strong role in the first two and, accordingly, theories of relativity were formulated entirely in terms of geometric concepts. The phenomenology of quantum physics arose roughly between 1895 and 1915, and for the 10 to 15 years before the emergence of quantum theory (around 1925) physicists continued to think of quantum theory within the confines of what is now called classical physics, and in particular within the same mathematical structures. The most sophisticated example of this is the Sommerfeld–Wilson–Ishiwara quantization rule, which was formulated entirely on the classical phase space.

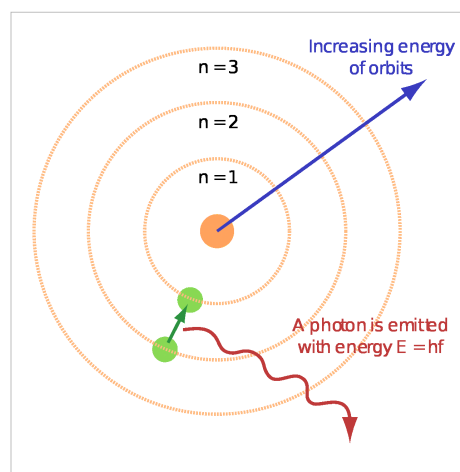
History of the formalism

The "old quantum theory" and the need for new mathematics

In the 1890s, Planck was able to derive the blackbody spectrum which was later used to avoid the classical ultraviolet catastrophe by making the unorthodox assumption that, in the interaction of radiation with matter, energy could only be exchanged in discrete units which he called quanta. Planck postulated a direct proportionality between the frequency of radiation and the quantum of energy at that frequency. The proportionality constant, h , is now called Planck's constant in his honor.

In 1905, Einstein explained certain features of the photoelectric effect by assuming that Planck's energy quanta were actual particles, which were later dubbed photons.

All of these developments were phenomenological and flew in the face of the theoretical physics of the time. Bohr and Sommerfeld went on to modify classical mechanics in an attempt to deduce the Bohr model from first principles. They proposed that, of all closed classical orbits traced by a mechanical system in its phase space, only the ones that enclosed an area which was a multiple of Planck's constant were actually allowed. The most sophisticated version of this formalism was the so-called Sommerfeld–Wilson–Ishiwara quantization. Although the Bohr model of the hydrogen atom could be explained in this way, the spectrum of the helium atom (classically an unsolvable 3-body problem) could not be predicted. The mathematical status of quantum theory remained uncertain for some time.



In 1923 de Broglie proposed that wave-particle duality applied not only to photons but to electrons and every other physical system.

The situation changed rapidly in the years 1925–1930, when working mathematical foundations were found through the groundbreaking work of Erwin Schrödinger, Werner Heisenberg, Max Born, Pascual Jordan, and the foundational work of John von Neumann, Hermann Weyl and Paul Dirac, and it became possible to unify several different approaches in terms of a fresh set of ideas. The physical interpretation of the theory was also clarified in these years after Werner Heisenberg discovered the uncertainty relations and Niels Bohr introduced the idea of complementarity.

The "new quantum theory"

Erwin Schrödinger's wave mechanics originally was the first successful attempt at replicating the observed quantization of atomic spectra with the help of a precise mathematical realization of de Broglie's wave-particle duality. Schrödinger's wave mechanics were created independently, uniquely based on de Broglie's concepts, less formal and easier to understand, visualize and exploit. Within a year, it was shown that the two theories were equivalent. Schrödinger himself initially did not understand the fundamental probabilistic nature of quantum mechanics, as he thought that the absolute square of the wave function of an electron should be interpreted as the charge density of an object smeared out over an extended, possibly infinite, volume of space, but Max Born introduced the interpretation of the absolute square of the wave function as the probability distribution of the position of a *pointlike* object. Born's idea was soon taken over by Niels Bohr in Copenhagen, who then became the "father" of the Copenhagen interpretation of quantum mechanics. Schrödinger's wave function can be seen to be closely related to the classical Hamilton–Jacobi equation. The correspondence to classical mechanics was even more explicit, although somewhat more formal, in Heisenberg's matrix mechanics. I.e., the equation for the operators in the Heisenberg representation, as it is now called, closely translates to classical equations for the dynamics of certain quantities in the Hamiltonian formalism of classical mechanics, where one uses Poisson brackets.

To be more precise: already before Schrödinger the young student Werner Heisenberg invented his matrix mechanics, which was the first correct quantum mechanics, i.e. the essential breakthrough. Heisenberg's matrix mechanics formulation was based on algebras of infinite matrices, being certainly very radical in light of the mathematics of classical physics, although he started from the index-terminology of the experimentalists of that time, not even knowing that his "index-schemes" were matrices. In fact, in these early years linear algebra was not generally known to physicists in its present form.

Although Schrödinger himself after a year proved the equivalence of his wave-mechanics and Heisenberg's matrix mechanics, the reconciliation of the two approaches is generally attributed to Paul Dirac, who wrote a lucid account in his 1930 classic *Principles of Quantum Mechanics*, being the third, and perhaps most important, person working independently in that field (he soon was the only one, who found a relativistic generalization of the theory). In his

above-mentioned account, he introduced the bra-ket notation, together with an abstract formulation in terms of the Hilbert space used in functional analysis; he showed that Schrödinger's and Heisenberg's approaches were two different representations of the same theory and found a third, most general one, which represented the dynamics of the system. His work was particularly fruitful in all kind of generalizations of the field. Concerning quantum mechanics, Dirac's method is now called canonical quantization.

The first complete mathematical formulation of this approach is generally credited to John von Neumann's 1932 book *Mathematical Foundations of Quantum Mechanics*, although Hermann Weyl had already referred to Hilbert spaces (which he called *unitary spaces*) in his 1927 classic book. It was developed in parallel with a new approach to the mathematical spectral theory based on linear operators rather than the quadratic forms that were David Hilbert's approach a generation earlier. Though theories of quantum mechanics continue to evolve to this day, there is a basic framework for the mathematical formulation of quantum mechanics which underlies most approaches and can be traced back to the mathematical work of John von Neumann. In other words, discussions about *interpretation* of the theory, and extensions to it, are now mostly conducted on the basis of shared assumptions about the mathematical foundations.

Later developments

The application of the new quantum theory to electromagnetism resulted in quantum field theory, which was developed starting around 1930. Quantum field theory has driven the development of more sophisticated formulations of quantum mechanics, of which the one presented here is a simple special case. In fact, the difficulties involved in implementing any of the following formulations cannot be said yet to have been solved in a satisfactory fashion except for ordinary quantum mechanics.

- Feynman path integrals
- axiomatic, algebraic and constructive quantum field theory
- geometric quantization
- quantum field theory in curved spacetime
- C^* algebra formalism
- Generalized Statistical Model of Quantum Mechanics

On a different front, von Neumann originally dispatched quantum measurement with his infamous postulate on the collapse of the wavefunction, raising a host of philosophical problems. Over the intervening 70 years, the *problem of measurement* became an active research area and itself spawned some new formulations of quantum mechanics.

- Relative state/Many-worlds interpretation of quantum mechanics
- Decoherence
- Consistent histories formulation of quantum mechanics
- Quantum logic formulation of quantum mechanics

A related topic is the relationship to classical mechanics. Any new physical theory is supposed to reduce to successful old theories in some approximation. For quantum mechanics, this translates into the need to study the so-called classical limit of quantum mechanics. Also, as Bohr emphasized, human cognitive abilities and language are inextricably linked to the classical realm, and so classical descriptions are intuitively more accessible than quantum ones. In particular, quantization, namely the construction of a quantum theory whose classical limit is a given and known classical theory, becomes an important area of quantum physics in itself.

Finally, some of the originators of quantum theory (notably Einstein and Schrödinger) were unhappy with what they thought were the philosophical implications of quantum mechanics. In particular, Einstein took the position that quantum mechanics must be incomplete, which motivated research into so-called hidden-variable theories. The issue of hidden variables has become in part an experimental issue with the help of quantum optics.

- de Broglie–Bohm–Bell pilot wave formulation of quantum mechanics
- Bell's inequalities

- Kochen–Specker theorem

Mathematical structure of quantum mechanics

A physical system is generally described by three basic ingredients: states; observables; and dynamics (or law of time evolution) or, more generally, a group of physical symmetries. A classical description can be given in a fairly direct way by a phase space model of mechanics: states are points in a symplectic phase space, observables are real-valued functions on it, time evolution is given by a one-parameter group of symplectic transformations of the phase space, and physical symmetries are realized by symplectic transformations. A quantum description consists of a Hilbert space of states, observables are self adjoint operators on the space of states, time evolution is given by a one-parameter group of unitary transformations on the Hilbert space of states, and physical symmetries are realized by unitary transformations.

Postulates of quantum mechanics

The following summary of the mathematical framework of quantum mechanics can be partly traced back to von Neumann's postulates.

- Each physical system is associated with a (topologically) separable complex Hilbert space H with inner product $\langle \phi | \psi \rangle$. Rays (one-dimensional subspaces) in H are associated with states of the system. In other words, physical states can be identified with equivalence classes of vectors of length 1 in H , where two vectors represent the same state if they differ only by a phase factor. *Separability* is a mathematically convenient hypothesis, with the physical interpretation that countably many observations are enough to uniquely determine the state.
- The Hilbert space of a composite system is the Hilbert space tensor product of the state spaces associated with the component systems (for instance, J.M. Jauch, *Foundations of quantum mechanics*, section 11-7). For a non-relativistic system consisting of a finite number of distinguishable particles, the component systems are the individual particles.
- Physical symmetries act on the Hilbert space of quantum states unitarily or antiunitarily due to Wigner's theorem (supersymmetry is another matter entirely).
- Physical observables are represented by densely-defined self-adjoint operators on H .

The expected value (in the sense of probability theory) of the observable A for the system in state represented by the unit vector $|\psi\rangle \in H$ is

$$\langle \psi | A | \psi \rangle$$

By spectral theory, we can associate a probability measure to the values of A in any state ψ . We can also show that the possible values of the observable A in any state must belong to the spectrum of A . In the special case A has only discrete spectrum, the possible outcomes of measuring A are its eigenvalues.

More generally, a state can be represented by a so-called density operator, which is a trace class, nonnegative self-adjoint operator ρ normalized to be of trace 1. The expected value of A in the state ρ is

$$\text{tr}(A\rho)$$

If ρ_ψ is the orthogonal projector onto the one-dimensional subspace of H spanned by $|\psi\rangle$, then

$$\text{tr}(A\rho_\psi) = \langle \psi | A | \psi \rangle$$

Density operators are those that are in the closure of the convex hull of the one-dimensional orthogonal projectors. Conversely, one-dimensional orthogonal projectors are extreme points of the set of density operators. Physicists also call one-dimensional orthogonal projectors *pure states* and other density operators *mixed states*.

One can in this formalism state Heisenberg's uncertainty principle and prove it as a theorem, although the exact historical sequence of events, concerning who derived what and under which framework, is the subject of historical

investigations outside the scope of this article.

Furthermore, to the postulates of quantum mechanics one should also add basic statements on the properties of spin and Pauli's exclusion principle, see below.

Superselection sectors. The correspondence between states and rays needs to be refined somewhat to take into account so-called superselection sectors. States in different superselection sectors cannot influence each other, and the relative phases between them are unobservable.

Pictures of dynamics

- In the so-called Schrödinger picture of quantum mechanics, the dynamics is given as follows:

The time evolution of the state is given by a differentiable function from the real numbers \mathbf{R} , representing instants of time, to the Hilbert space of system states. This map is characterized by a differential equation as follows: If $|\psi(t)\rangle$ denotes the state of the system at any one time t , the following Schrödinger equation holds:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle$$

where H is a densely-defined self-adjoint operator, called the system Hamiltonian, i is the imaginary unit and \hbar is the reduced Planck constant. As an observable, H corresponds to the total energy of the system.

Alternatively, by Stone's theorem one can state that there is a strongly continuous one-parameter unitary group $U(t): H \rightarrow H$ such that

$$|\psi(t+s)\rangle = U(t) |\psi(s)\rangle$$

for all times s, t . The existence of a self-adjoint Hamiltonian H such that

$$U(t) = e^{-(i/\hbar)tH}$$

is a consequence of Stone's theorem on one-parameter unitary groups. (It is assumed that H does not depend on time and that the perturbation starts at $t_0 = 0$; otherwise one must use the Dyson series, formally written as

$$U(t) = \mathcal{T} \left\{ \exp \left(-\frac{i}{\hbar} \int_{t_0}^t dt' H(t') \right) \right\},$$

where \mathcal{T} is Dyson's time-ordering symbol.

(This symbol permutes a product of noncommuting operators of the form

$$B_1(t_1) \cdot B_2(t_2) \cdot \dots \cdot B_n(t_n)$$

into the uniquely determined re-ordered expression

$$B_{i_1}(t_{i_1}) \cdot B_{i_2}(t_{i_2}) \cdot \dots \cdot B_{i_n}(t_{i_n}) \text{ with } t_{i_1} \geq t_{i_2} \geq \dots \geq t_{i_n}.$$

The result is a causal chain, the primary *cause* in the past on the utmost r.h.s., and finally the present *effect* on the utmost l.h.s. .)

- The Heisenberg picture of quantum mechanics focuses on observables and instead of considering states as varying in time, it regards the states as fixed and the observables as changing. To go from the Schrödinger to the Heisenberg picture one needs to define time-independent states and time-dependent operators thus:

$$|\psi\rangle = |\psi(0)\rangle$$

$$A(t) = U(-t)AU(t).$$

It is then easily checked that the expected values of all observables are the same in both pictures

$$\langle \psi | A(t) | \psi \rangle = \langle \psi(t) | A | \psi(t) \rangle$$

and that the time-dependent Heisenberg operators satisfy

$$i\hbar \frac{d}{dt} A(t) = [A(t), H].$$

This assumes A is not time dependent in the Schrödinger picture. Notice the commutator expression is purely formal when one of the operators is unbounded. One would specify a representation for the expression to make sense of it.

- The so-called Dirac picture or interaction picture has time-dependent *states* and observables, evolving with respect to different Hamiltonians. This picture is most useful when the evolution of the observables can be solved exactly, confining any complications to the evolution of the states. For this reason, the Hamiltonian for the observables is called "free Hamiltonian" and the Hamiltonian for the states is called "interaction Hamiltonian". In symbols:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H_{\text{int}}(t) |\psi(t)\rangle$$

$$i\hbar \frac{d}{dt} A(t) = [A(t), H_0].$$

The interaction picture does not always exist, though. In interacting quantum field theories, Haag's theorem states that the interaction picture does not exist. This is because the Hamiltonian cannot be split into a free and an interacting part within a superselection sector. Moreover, even if in the Schrödinger picture the Hamiltonian does not depend on time, e.g. $H = H_0 + V$, in the interaction picture it does, at least, if V does not commute with H_0 , since

$$H_{\text{int}}(t) \equiv e^{(i/\hbar)tH_0} V e^{(-i/\hbar)tH_0}.$$

So the above-mentioned Dyson-series has to be used anyhow.

The Heisenberg picture is the closest to classical Hamiltonian mechanics (for example, the commutators appearing in the above equations directly translate into the classical Poisson brackets); but this is already rather "high-browed", and the Schrödinger picture is considered easiest to visualize and understand by most people, to judge from pedagogical accounts of quantum mechanics. The Dirac picture is the one used in perturbation theory, and is specially associated to quantum field theory and many-body physics.

Similar equations can be written for any one-parameter unitary group of symmetries of the physical system. Time would be replaced by a suitable coordinate parameterizing the unitary group (for instance, a rotation angle, or a translation distance) and the Hamiltonian would be replaced by the conserved quantity associated to the symmetry (for instance, angular or linear momentum).

Representations

The original form of the Schrödinger equation depends on choosing a particular representation of Heisenberg's canonical commutation relations. The Stone–von Neumann theorem states all irreducible representations of the finite-dimensional Heisenberg commutation relations are unitarily equivalent. This is related to quantization and the correspondence between classical and quantum mechanics, and is therefore not strictly part of the general mathematical framework.

The quantum harmonic oscillator is an exactly-solvable system where the possibility of choosing among more than one representation can be seen in all its glory. There, apart from the Schrödinger (position or momentum) representation one encounters the Fock (number) representation and the Bargmann-Segal (phase space or coherent state) representation. All three are unitarily equivalent.

Time as an operator

The framework presented so far singles out time as *the* parameter that everything depends on. It is possible to formulate mechanics in such a way that time becomes itself an observable associated to a self-adjoint operator. At the classical level, it is possible to arbitrarily parameterize the trajectories of particles in terms of an unphysical parameter s , and in that case the time t becomes an additional generalized coordinate of the physical system. At the quantum level, translations in s would be generated by a "Hamiltonian" $H - E$, where E is the energy operator and H

is the "ordinary" Hamiltonian. However, since s is an unphysical parameter, *physical* states must be left invariant by " s -evolution", and so the physical state space is the kernel of $H - E$ (this requires the use of a rigged Hilbert space and a renormalization of the norm).

This is related to quantization of constrained systems and quantization of gauge theories. It is also possible to formulate a quantum theory of "events" where time becomes an observable (see D. Edwards).

Spin

In addition to their other properties all particles possess a quantity, which has no correspondence at all in conventional physics, namely the spin, which is some kind of *intrinsic angular momentum* (therefore the name). In the position representation, instead of a wavefunction without spin, $\psi = \psi(\mathbf{r})$, one has with spin: $\psi = \psi(\mathbf{r}, \sigma)$, where σ belongs to the following discrete set of values

$$\sigma \in \{-S \cdot \hbar, -(S-1) \cdot \hbar, \dots, +(S-1) \cdot \hbar, +S \cdot \hbar\}.$$

One distinguishes bosons ($S = 0$ or 1 or 2 or ...) and fermions ($S = 1/2$ or $3/2$ or $5/2$ or ...)

Pauli's principle

The property of spin relates to another basic property concerning systems of N identical particles: Pauli's exclusion principle, which is a consequence of the following permutation behaviour of an N -particle wave function; again in the position representation one must postulate that for the transposition of any two of the N particles one always should have

$$\psi(\dots; \mathbf{r}_i, \sigma_i; \dots; \mathbf{r}_j, \sigma_j; \dots) \stackrel{!}{=} (-1)^{2S} \cdot \psi(\dots; \mathbf{r}_j, \sigma_j; \dots; \mathbf{r}_i, \sigma_i; \dots)$$

i.e., on transposition of the arguments of any two particles the wavefunction should *reproduce*, apart from a prefactor $(-1)^{2S}$ which is $+1$ for bosons, but (-1) for fermions. Electrons are fermions with $S = 1/2$; quanta of light are bosons with $S = 1$. In nonrelativistic quantum mechanics all particles are either bosons or fermions; in relativistic quantum theories also "supersymmetric" theories exist, where a particle is a linear combination of a bosonic and a fermionic part. Only in dimension $d=2$ one can construct entities where $(-1)^{2S}$ is replaced by an arbitrary complex number with magnitude 1 (\rightarrow anyons).

Although *spin* and the *Pauli principle* can only be derived from relativistic generalizations of quantum mechanics the properties mentioned in the last two paragraphs belong to the basic postulates already in the non-relativistic limit. Especially, many important properties in natural science, e.g. the periodic system of chemistry, are consequences of the two properties.

The problem of measurement

The picture given in the preceding paragraphs is sufficient for description of a completely isolated system. However, it fails to account for one of the main differences between quantum mechanics and classical mechanics, that is the effects of measurement.^[2] The von Neumann description of quantum measurement of an observable A , when the system is prepared in a pure state ψ is the following (note, however, that von Neumann's description dates back to the 1930s and is based on experiments as performed during that time – more specifically the Compton–Simon experiment; it is not applicable to most present-day measurements within the quantum domain):

- Let A have spectral resolution

$$A = \int \lambda dE_A(\lambda),$$

where E_A is the resolution of the identity (also called projection-valued measure) associated to A . Then the probability of the measurement outcome lying in an interval B of \mathbf{R} is $|\mathbb{E}_A(B) \psi|^2$. In other words, the probability is obtained by integrating the characteristic function of B against the countably additive measure

$$\langle \psi | E_A \psi \rangle.$$

- If the measured value is contained in B , then immediately after the measurement, the system will be in the (generally non-normalized) state $E_A(B) \psi$. If the measured value does not lie in B , replace B by its complement for the above state.

For example, suppose the state space is the n -dimensional complex Hilbert space \mathbf{C}^n and A is a Hermitian matrix with eigenvalues λ_i , with corresponding eigenvectors ψ_i . The projection-valued measure associated with A , E_A , is then

$$E_A(B) = |\psi_i\rangle\langle\psi_i|,$$

where B is a Borel set containing only the single eigenvalue λ_i . If the system is prepared in state

$$|\psi\rangle$$

Then the probability of a measurement returning the value λ_i can be calculated by integrating the spectral measure

$$\langle \psi | E_A \psi \rangle$$

over B_i . This gives trivially

$$\langle \psi | \psi_i \rangle \langle \psi_i | \psi \rangle = |\langle \psi | \psi_i \rangle|^2.$$

The characteristic property of the von Neumann measurement scheme is that repeating the same measurement will give the same results. This is also called the *projection postulate*.

A more general formulation replaces the projection-valued measure with a positive-operator valued measure (POVM). To illustrate, take again the finite-dimensional case. Here we would replace the rank-1 projections

$$|\psi_i\rangle\langle\psi_i|$$

by a finite set of positive operators

$$F_i F_i^*$$

whose sum is still the identity operator as before (the resolution of identity). Just as a set of possible outcomes $\{\lambda_1 \dots \lambda_n\}$ is associated to a projection-valued measure, the same can be said for a POVM. Suppose the measurement outcome is λ_i . Instead of collapsing to the (unnormalized) state

$$|\psi_i\rangle\langle\psi_i|\psi\rangle$$

after the measurement, the system now will be in the state

$$F_i |\psi\rangle.$$

Since the $F_i F_i^*$'s need not be mutually orthogonal projections, the projection postulate of von Neumann no longer holds.

The same formulation applies to general mixed states.

In von Neumann's approach, the state transformation due to measurement is distinct from that due to time evolution in several ways. For example, time evolution is deterministic and unitary whereas measurement is non-deterministic and non-unitary. However, since both types of state transformation take one quantum state to another, this difference was viewed by many as unsatisfactory. The POVM formalism views measurement as one among many other quantum operations, which are described by completely positive maps which do not increase the trace.

In any case it seems that the above-mentioned problems can only be resolved if the time evolution included not only the quantum system, but also, and essentially, the classical measurement apparatus (see above).

The *relative state* interpretation

An alternative interpretation of measurement is Everett's relative state interpretation, which was later dubbed the "many-worlds interpretation" of quantum mechanics.

List of mathematical tools

Part of the folklore of the subject concerns the mathematical physics textbook *Methods of Mathematical Physics* put together by Richard Courant from David Hilbert's Göttingen University courses. The story is told (by mathematicians) that physicists had dismissed the material as not interesting in the current research areas, until the advent of Schrödinger's equation. At that point it was realised that the mathematics of the new quantum mechanics was already laid out in it. It is also said that Heisenberg had consulted Hilbert about his matrix mechanics, and Hilbert observed that his own experience with infinite-dimensional matrices had derived from differential equations, advice which Heisenberg ignored, missing the opportunity to unify the theory as Weyl and Dirac did a few years later. Whatever the basis of the anecdotes, the mathematics of the theory was conventional at the time, whereas the physics was radically new.

The main tools include:

- linear algebra: complex numbers, eigenvectors, eigenvalues
- functional analysis: Hilbert spaces, linear operators, spectral theory
- differential equations: partial differential equations, separation of variables, ordinary differential equations, Sturm–Liouville theory, eigenfunctions
- harmonic analysis: Fourier transforms

See also: list of mathematical topics in quantum theory.

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Quantum Chemistry and Biochemical Dynamics

Quantum chemistry applies quantum theory to the explanation and prediction of chemical behaviour. The majority of quantum chemical investigations, at the time of writing, determine the energies of molecules, using heavy computations based on approximate solutions of the Schrödinger equation. Other quantum chemical studies use semi-empirical and other methods that are also based on quantum mechanical principles, and deal with time dependent problems. Quantum chemical studies relate to the ground state of individual atoms and molecules, to excited states, and to the transition states that occur during chemical reactions. Quantum chemical results include molecular geometry, the strengths and other characteristics of chemical bonds, optical and other spectra, intermolecular forces, chemical reactivity and many other chemical properties and features of chemical behaviour. Many quantum chemical studies assume the nuclei are at rest (Born-Oppenheimer approximation). Many calculations involve iterative methods that include self-consistent field methods. Major goals of quantum chemistry include increasing the accuracy of the results for small molecular systems, and increasing the size of large molecules that can be processed, which is limited by scaling considerations—the computation time increases as a power of the number of atoms.

An alternative approach

Quantum chemistry is a branch of theoretical chemistry which applies quantum mechanics and quantum field theory to address problems in chemistry. One application of quantum chemistry is the electronic behavior of atoms and molecules relative to their chemical reactivity. Quantum chemistry lies on the border between chemistry and physics. Thus, significant contributions have been made by scientists from both fields. It has a strong and active overlap with the field of atomic physics and molecular physics, as well as physical chemistry.

Quantum chemistry mathematically describes the fundamental behavior of matter at the molecular scale,^[1] but can span from elementary particles such as electrons (fermions) and photons (bosons) to the cosmos such as star formation.^[2] It is, in principle, possible to describe all chemical systems using this theory. In practice, only the simplest chemical systems may realistically be investigated in purely quantum mechanical terms, and approximations must be made for most practical purposes (e.g., Hartree-Fock, post Hartree-Fock or Density functional theory, see computational chemistry for more details). Hence a detailed understanding of quantum mechanics is not necessary for most chemistry, as the important implications of the theory (principally the orbital approximation) can be understood and applied in simpler terms.

In quantum mechanics the Hamiltonian, or the physical state, of a particle can be expressed as the sum of two operators, one corresponding to kinetic energy and the other to potential energy. The Hamiltonian in the Schrödinger wave equation used in quantum chemistry does not contain terms for the spin of the electron.

Solutions of the Schrödinger equation for the hydrogen atom gives the form of the wave function for atomic orbitals, and the relative energy of the various orbitals. The orbital approximation can be used to understand the other atoms e.g. helium, lithium and carbon.

History

The history of quantum chemistry essentially began with the 1838 discovery of cathode rays by Michael Faraday, the 1859 statement of the black body radiation problem by Gustav Kirchhoff, the 1877 suggestion by Ludwig Boltzmann that the energy states of a physical system could be discrete, and the 1900 quantum hypothesis by Max Planck that any energy radiating atomic system can theoretically be divided into a number of discrete energy elements ϵ such that each of these energy elements is proportional to the frequency ν with which they each individually radiate energy, as defined by the following formula:

$$\epsilon = h\nu$$

where h is a numerical value called Planck's Constant. Then, in 1905, to explain the photoelectric effect (1839), i.e., that shining light on certain materials can function to eject electrons from the material, Albert Einstein postulated, based on Planck's quantum hypothesis, that light itself consists of individual quantum particles, which later came to be called photons (1926). In the years to follow, this theoretical basis slowly began to be applied to chemical structure, reactivity, and bonding.

Electronic structure

The first step in solving a quantum chemical problem is usually solving the Schrödinger equation (or Dirac equation in relativistic quantum chemistry) with the electronic molecular Hamiltonian. This is called determining the **electronic structure** of the molecule. It can be said that the electronic structure of a molecule or crystal implies essentially its chemical properties. An exact solution for the Schrödinger equation can only be obtained for the hydrogen atom. Since all other atomic, or molecular systems, involve the motions of three or more "particles", their Schrödinger equations cannot be solved exactly and so approximate solutions must be sought.

Wave model

The foundation of quantum mechanics and quantum chemistry is the **wave model**, in which the atom is a small, dense, positively charged nucleus surrounded by electrons. Unlike the earlier Bohr model of the atom, however, the wave model describes electrons as "clouds" moving in orbitals, and their positions are represented by probability distributions rather than discrete points. The strength of this model lies in its predictive power. Specifically, it predicts the pattern of chemically similar elements found in the periodic table. The wave model is so named because electrons exhibit properties (such as interference) traditionally associated with waves. See wave-particle duality.

Valence bond

Although the mathematical basis of quantum chemistry had been laid by Schrödinger in 1926, it is generally accepted that the first true calculation in quantum chemistry was that of the German physicists Walter Heitler and Fritz London on the hydrogen (H_2) molecule in 1927. Heitler and London's method was extended by the American theoretical physicist John C. Slater and the American theoretical chemist Linus Pauling to become the **Valence-Bond (VB)** [or **Heitler-London-Slater-Pauling (HLSP)**] method. In this method, attention is primarily devoted to the pairwise interactions between atoms, and this method therefore correlates closely with classical chemists' drawings of bonds.

Molecular orbital

An alternative approach was developed in 1929 by Friedrich Hund and Robert S. Mulliken, in which electrons are described by mathematical functions delocalized over an entire molecule. The **Hund-Mulliken** approach or **molecular orbital (MO) method** is less intuitive to chemists, but has turned out capable of predicting spectroscopic properties better than the VB method. This approach is the conceptional basis of the **Hartree-Fock method** and further post Hartree-Fock methods.

Density functional theory

The **Thomas-Fermi model** was developed independently by Thomas and Fermi in 1927. This was the first attempt to describe many-electron systems on the basis of electronic density instead of wave functions, although it was not very successful in the treatment of entire molecules. The method did provide the basis for what is now known as **density functional theory**. Though this method is less developed than post Hartree-Fock methods, its significantly lower computational requirements (scaling typically no worse than n^3 with respect to n basis functions) allow it to tackle larger polyatomic molecules and even macromolecules. This computational affordability and often comparable accuracy to MP2 and CCSD (post-Hartree-Fock methods) has made it one of the most popular methods in computational chemistry at present.

Chemical dynamics

A further step can consist of solving the Schrödinger equation with the total molecular Hamiltonian in order to study the motion of molecules. Direct solution of the Schrödinger equation is called *quantum molecular dynamics*, within the semiclassical approximation *semiclassical molecular dynamics*, and within the classical mechanics framework *molecular dynamics (MD)*. Statistical approaches, using for example Monte Carlo methods, are also possible.

Adiabatic chemical dynamics

In **adiabatic dynamics**, interatomic interactions are represented by single scalar potentials called potential energy surfaces. This is the Born-Oppenheimer approximation introduced by Born and Oppenheimer in 1927. Pioneering applications of this in chemistry were performed by Rice and Ramsperger in 1927 and Kassel in 1928, and generalized into the RRKM theory in 1952 by Marcus who took the transition state theory developed by Eyring in 1935 into account. These methods enable simple estimates of unimolecular reaction rates from a few characteristics of the potential surface.

Non-adiabatic chemical dynamics

Non-adiabatic dynamics consists of taking the interaction between several coupled potential energy surface (corresponding to different electronic quantum states of the molecule). The coupling terms are called **vibronic couplings**. The pioneering work in this field was done by Stueckelberg, Landau, and Zener in the 1930s, in their work on what is now known as the Landau-Zener transition. Their formula allows the transition probability between two diabatic potential curves in the neighborhood of an avoided crossing to be calculated.

Quantum chemistry and quantum field theory

The application of quantum field theory (QFT) to chemical systems and theories has become increasingly common in the modern physical sciences. One of the first and most fundamentally explicit appearances of this is seen in the theory of the photomagnetron. In this system, plasmas, which are ubiquitous in both physics and chemistry, are studied in order to determine the basic quantization of the underlying bosonic field. However, quantum field theory is of interest in many fields of chemistry, including: nuclear chemistry, astrochemistry, sonochemistry, and quantum hydrodynamics. Field theoretic methods have also been critical in developing the ab initio Effective Hamiltonian theory of semi-empirical pi-electron methods.

Further reading

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- [2] "Astrophysics and Astrochemistry". *Astrophysics and Astrochemistry*. (<http://www.astrochemistry.eu/>).

External links

- The Sherrill Group - Notes (<http://vergil.chemistry.gatech.edu/notes/index.html>)
- ChemViz Curriculum Support Resources (<http://www.shodor.org/chemviz/>)
- Early ideas in the history of quantum chemistry (<http://www.quantum-chemistry-history.com/>)
- The Particle Adventure (<http://particleadventure.org/>)

Nobel lectures by quantum chemists

- Walter Kohn's Nobel lecture (<http://nobelprize.org/chemistry/laureates/1998/kohn-lecture.html>)
- Rudolph Marcus' Nobel lecture (<http://nobelprize.org/chemistry/laureates/1992/marcus-lecture.html>)
- Robert Mulliken's Nobel lecture (<http://nobelprize.org/chemistry/laureates/1966/mulliken-lecture.html>)
- Linus Pauling's Nobel lecture (<http://nobelprize.org/chemistry/laureates/1954/pauling-lecture.html>)
- John Pople's Nobel lecture (<http://nobelprize.org/chemistry/laureates/1998/pople-lecture.html>)

List of quantum chemistry and solid state physics software

Quantum chemistry computer programs are used in computational chemistry to implement the methods of quantum chemistry. Most include the Hartree–Fock (HF) and some post-Hartree–Fock methods. They may also include density functional theory (DFT), molecular mechanics or semi-empirical quantum chemistry methods. The programs include both open source and commercial software. Most of them are large, often containing several separate programs, and have developed over many years.

The following table illustrates the capabilities of the most versatile software packages that show an entry in two or more columns of the table.

Package	License [†]	Lang.	Basis	Periodic [‡]	Mol. mech.	Semi-emp.	HF	Post-HF	DFT
ABINIT	GPL	Fortran	PW	3d	Yes	No	No	No	Yes
ACES II	Academic	Fortran	GTO	No	No	No	Yes	Yes	Yes
ACES II MAB	Academic	Fortran	GTO	No	No	No	Yes	Yes	No
ADF	Commercial	Fortran	STO	Any	Yes	Yes ⁴	Yes	No	Yes
Atomistix ToolKit (ATK)	Commercial	C++/Python	NAO/EHT	3d ⁹	Yes	Yes	No	No	Yes
BigDFT	GPL	Fortran	Wavelet	Any	Yes	No	Yes	No	Yes
CADPAC	Academic	Fortran	GTO	No	No	No	Yes	Yes	Yes
CASINO (QMC)	Academic	Fortran	GTO / PW / Spline / Grid / STO	Any	No	No	Yes	Yes	No
CASTEP	Academic (UK) / Commercial	Fortran	PW	3d	Yes	No	Yes ⁵	Yes	Yes
CFOUR	Academic	Fortran	GTO	No	No	No	Yes	Yes	No
COLUMBUS	Academic	Fortran	GTO	No	No	No	Yes	Yes	No
CONQUEST	Academic (UK)	Unknown	Unknown	Unknown	Unknown	Unknown	Unknown	Unknown	Unknown
COSMOS	Commercial	Unknown	Unknown	Unknown	Yes	Yes	No	No	No
CP2K	GPL	Fortran 95	Hybrid GTO / PW	Any	Yes	Yes	Yes	No	Yes
CPMD	Academic	Fortran	PW	Any	Yes	No	Yes	No	Yes
CRYSTAL	Academic (UK) / Commercial	Fortran	GTO	Any	Yes	No	Yes	Yes ¹⁰	Yes
DACAPO	GPL ? ¹	Fortran	PW	3d	Yes	No	No	No	Yes
DALTON	Academic	Fortran	GTO	No	No	No	Yes	Yes	Yes
DFTB+ ^[1]	Academic / Commercial	Fortran 95	NAO	Any	Yes	Yes	No	No	No

DFT++ ^[2]	GPL	C++	PW / Wavelet	3d	Yes	No	No	No	Yes
DIRAC	Academic	Fortran 77, Fortran 90, C	GTO	No	No	No	Yes	Yes	Yes
DMol3 ^[3]	Commercial	Unknown	Numeric AOs	3d	No	No	No	No	Yes
ErgoSCF ^[4]	GPL	C++	GTO	No	No	No	Yes	Yes	Yes
EXCITING	GPL	Fortran 95	FP-LAPW	Unknown	Unknown	Yes	Unknown	Unknown	Yes
FLEUR ^[5]	Academic	Unknown	FP-(L)APW+lo	3d, 2d, 1d	No	No	Yes	Yes	Yes
FHI-aims ^[6]	Commercial	Fortran	NAO	Any	Yes	No	Yes	Yes	Yes
FreeON	GPL	Fortran 95	GTO	Any	Yes	No	Yes	Yes	Yes
Firefly / PC GAMESS	Academic	Unknown	GTO	No	Yes ³	Yes	Yes	Yes	Yes
GAMESS (UK)	Academic (UK) / Commercial	Fortran	GTO	No	No	Yes	Yes	Yes	Yes
GAMESS (US)	Academic	Fortran	GTO	No	Yes ²	Yes	Yes	Yes	Yes
GAUSSIAN	Commercial	Fortran	GTO	Any	Yes	Yes	Yes	Yes	Yes
GPAW ^[7]	GPL	Python / C	Grid / NAO	Any	Yes	Unknown	Yes ⁵	No	Yes
hBar Lab ⁷	Commercial	Unknown	GTO	No	No	No	Yes	Yes	Yes
HiLAPW ^[8]	Unknown	Unknown	FLAPW	3d	No	No	No	No	Yes
JAGUAR	Commercial	Unknown	GTO	Unknown	Yes	No	Yes	Yes	Yes
MADNESS	GPL	C++	Wavelet	Unknown	No	No	Yes	No	Yes
MOLCAS	Commercial	Fortran	GTO	No	Yes	Yes	Yes	Yes	Yes
MOLPRO	Commercial	Unknown	GTO	Unknown	No	No	Yes	Yes	Yes
MOPAC	Academic / Commercial	Fortran	Unknown	Unknown	Unknown	Yes	No	No	No
MPQC	LGPL	C++	GTO	No	No	No	Yes	Yes	Yes
NWChem	Academic	Fortran 77 / C	Unknown	Unknown	Yes	No	Yes	Yes	Yes
OCTOPUS	GPL	Unknown	Grid	Any	Yes	No	No	No	Yes
ONETEP	Academic (UK) / Commercial	Fortran	PW	Any	Yes	No	Yes ⁵	No	Yes
OpenAtom	Academic	Charm++ (C++)	DVR	Unknown	Yes	No	No	No	Yes
OpenMX ^[9]	GPL	C	NAO	3d	Yes	No	No	No	Yes
ORCA ^[10]	Academic	C++	GTO	Any	Yes	Yes	Yes	Yes	Yes

PLATO	Academic	Unknown	NAO	Any	Yes	No	No	No	Yes
PQS	Commercial	Unknown	Unknown	Unknown	Yes	Yes	Yes	Yes	Yes
Priroda-06 ^[11]	Academic	Unknown	GTO	No	No	No	Yes	Yes	Yes
PSI	GPL	C / C++	GTO	No	No	No	Yes	Yes	No
PWscf ⁶	GPL	Fortran	PW	3d	No	No	Yes	No	Yes
PyQuante	BSD	Python	GTO	No	No	Yes	Yes	Yes	Yes
Q-Chem	Commercial	Fortran / C++	GTO	No	Yes	Yes	Yes	Yes	Yes
Quantemol-N	Academic / Commercial	Fortran	GTO	No	Yes	Yes	Yes	Yes	No
Quantum ESPRESSO ^[12]	GPL	Fortran	PW	3d	No	No	Yes	No	Yes
RSPT ^[13]	Academic	Fortran / C	FP-LMTO	3d	No	No	No	No	Yes
SPARTAN	Commercial	Unknown	GTO	No	Yes	Yes	Yes	Yes	Yes
SIESTA	Academic	Fortran	NAO	3d	Yes	No	No	No	Yes
TB-LMTO ^[14]	Academic	Fortran	LMTO	3d	No	No	No	No	Yes
TERACHEM ⁸	Commercial	C/CUDA	GTO	No	Yes	No	Yes	No	Yes
TURBOMOLE	Commercial	Fortran	GTO	No	Yes	No	Yes	Yes	Yes
VASP	Academic(AT)/ Commercial	Fortran	PW	Any	No	No	Yes	Yes	Yes
WIEN2k	Commercial	Unknown	FP-(L)APW+lo	3d	Yes	No	No	No	Yes

[†] “Academic”: academic (no cost) license possible upon request; “Commercial”: commercially distributed.

[‡] Support for periodic systems (3d-crystals, 2d-slabs, 1d-rods and isolated molecules): 3d-periodic codes always allow the simulation of systems with lower dimensionality within a supercell. Specified here is the capability for actual simulation within lower periodicity.

¹ The CAMPOS project^[15] (which includes Dacapo) states that all code is GPL. The Dacapo distribution itself does not contain any license information.

² Through interface to TINKER

³ Through Ascalaph^[30]

⁴ Through interface to MOPAC

⁵ Using exact exchange DFT

⁶ Distributed with Quantum ESPRESSO^[12]

⁷ Web service integrating MPQC.

⁸ TeraChem is the first fully GPU-accelerated quantum chemistry software.

⁹ Atomistix ToolKit also contains finite-bias NEGF electron transport calculations with open boundary conditions.

¹⁰ Through CRYSCOR^[16] program.

Further programs

- AIMPRO^[17]
- Ascalaph Designer
- PWPAW / Atompaw
- deMon2K^[22]
- DFTB+^[24]
- Fireball^[26]
- FSatom^[18]
- MAPS^[19]
- NRLMOL^[20]
- ORCA^[10]
- ParaGauss^[25]
- PARATEC
- PARSEC
- Petot^[21]
- Socorro^[23]
- S/PHI/nX^[48]
- SPR-KKR^[27]

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- [5] <http://www.flapw.de>
- [6] <http://www.fhi-berlin.mpg.de/aims/>
- [7] <https://wiki.fysik.dtu.dk/gpaw/>
- [8] <http://home.hiroshima-u.ac.jp/fpc/manuals/HiLAPW/HiLAPW.html>
- [9] <http://www.openmx-square.org/>
- [10] <http://www.thch.uni-bonn.de/tc/orca/>
- [11] <http://www.physto.se/~laikov/p/>
- [12] <http://www.quantum-espresso.org>
- [13] <http://www.rspt.net>
- [14] <http://www.fkf.mpg.de/andersen/>
- [15] <http://www.camd.dtu.dk/software.aspx>
- [16] <http://www.cryscor.unito.it>
- [17] <http://aimpro.ncl.ac.uk/>
- [18] <http://www.tddft.org/fsatom>
- [19] <http://www.scienomics.com/Products/maps/>
- [20] <http://quantum.utep.edu/nrlmol/nrlmol.html>
- [21] <http://hpcrd.lbl.gov/~linwang/PEtot/PEtot.html>
- [22] http://www.demon-software.com/public_html/program.html
- [23] <http://dft.sandia.gov/Socorro/mainpage.html>
- [24] <http://www.dftb-plus.info/>
- [25] <http://qcl.theochem.tu-muenchen.de/ParaGauss.html>
- [26] <http://www.fireball-dft.org>
- [27] <http://olymp.cup.uni-muenchen.de/ak/ebert/SPRKKR>

Basic Concepts in Symbolic Dynamics

Sequential dynamical system

Sequential dynamical systems (SDSs) are a class of graph dynamical systems. They are discrete dynamical systems which generalize many aspects of for example classical cellular automata, and they provide a framework for studying asynchronous processes over graphs. The analysis of SDSs uses techniques from combinatorics, abstract algebra, graph theory, dynamical systems and probability theory.

Definition

An SDS is constructed from the following components:

- A finite *graph* Y with vertex set $v[Y] = \{1, 2, \dots, n\}$. Depending on the context the graph can be directed or undirected.
- A state x_v for each vertex i of Y taken from a finite set K . The *system state* is the n -tuple $x = (x_1, x_2, \dots, x_n)$, and $x[i]$ is the tuple consisting of the states associated to the vertices in the 1-neighborhood of i in Y (in some fixed order).
- A *vertex function* f_i for each vertex i . The vertex function maps the state of vertex i at time t to the vertex state at time $t + 1$ based on the states associated to the 1-neighborhood of i in Y .
- A word $w = (w_1, w_2, \dots, w_m)$ over $v[Y]$.

It is convenient to introduce the Y -local maps F_i constructed from the vertex functions by

$$F_i(x) = (x_1, x_2, \dots, x_{i-1}, f_i(x[i]), x_{i+1}, \dots, x_n) .$$

The word w specifies the sequence in which the Y -local maps are composed to derive the sequential dynamical system map $F: K^n \rightarrow K^n$ as

$$[F_Y, w] = F_{w(m)} \circ F_{w(m-1)} \circ \dots \circ F_{w(2)} \circ F_{w(1)} .$$

If the update sequence is a permutation one frequently speaks of a *permutation SDS* to emphasize this point. The *phase space* associated to a sequential dynamical system with map $F: K^n \rightarrow K^n$ is the finite directed graph with vertex set K^n and directed edges $(x, F(x))$. The structure of the phase space is governed by the properties of the graph Y , the vertex functions (f_i) , and the update sequence w . A large part of SDS research seeks to infer phase space properties based on the structure of the system constituents.

Example

Consider the case where Y is the graph with vertex set $\{1, 2, 3\}$ and undirected edges $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$ (a triangle or 3-circle) with vertex states from $K = \{0, 1\}$. For vertex functions use the symmetric, boolean function $\text{nor} : K^3 \rightarrow K$ defined by $\text{nor}(x, y, z) = (1+x)(1+y)(1+z)$ with boolean arithmetic. Thus, the only case in which the function nor returns the value 1 is when all the arguments are 0. Pick $w = (1, 2, 3)$ as update sequence. Starting from the initial system state $(0, 0, 0)$ at time $t = 0$ one computes the state of vertex 1 at time $t=1$ as $\text{nor}(0, 0, 0) = 1$. The state of vertex 2 at time $t=1$ is $\text{nor}(1, 0, 0) = 0$. Note that the state of vertex 1 at time $t=1$ is used immediately. Next one obtains the state of vertex 3 at time $t=1$ as $\text{nor}(1, 0, 0) = 0$. This completes the update sequence, and one concludes that the Nor-SDS map sends the system state $(0, 0, 0)$ to $(1, 0, 0)$. The system state $(1, 0, 0)$ is in turned mapped to $(0, 1, 0)$ by an application of the SDS map.

References

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- Predecessor and Permutation Existence Problems for Sequential Dynamical Systems ^[1]
- Genetic Sequential Dynamical Systems ^[2]

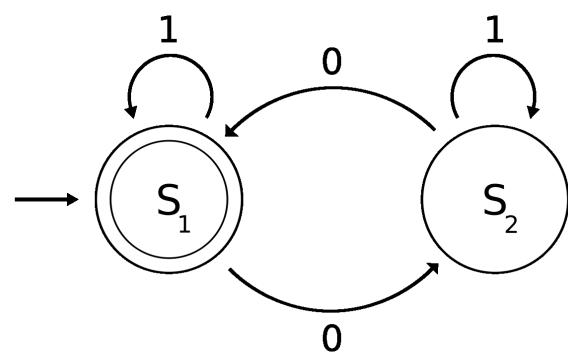
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- [1] <http://www.emis.de/journals/DMTCS/pdfpapers/dmAB0106.pdf>
 [2] <http://arxiv.org/pdf/math.DS/0603370>

Automata theory

In theoretical computer science, **automata theory** is the study of abstract machines (or more appropriately, abstract 'mathematical' machines or systems) and the computational problems that can be solved using these machines. These abstract machines are called automata.

The figure at right illustrates a finite state machine, which is one well-known variety of automaton. This automaton consists of states (represented in the figure by circles), and transitions (represented by arrows). As the automaton sees a symbol of input, it makes a *transition* (or *jump*) to another state, according to its *transition function* (which takes the current state and the recent symbol as its inputs).



An example of automata and study of mathematical properties of such automata is automata theory

Automata theory is also closely related to formal language theory, as the automata are often classified by the class of formal languages they are able to recognize. An automaton can be a finite representation of a formal language that may be an infinite set.

Automata play a major role in compiler design and parsing.

Automata

Following is an introductory definition of one type of automata, which attempts to help one grasp the essential concepts involved in automata theory.

Informal description

An automaton is supposed to *run* on some given sequence or string of *inputs* in discrete time steps. At each time step, an automaton gets one input that is picked up from a set of *symbols* or *letters*, which is called an *alphabet*. At any time, the symbols so far fed to the automaton as input form a finite sequence of symbols, which is called a *word*. An automaton contains a finite set of states. At each instance in time of some run, automaton is *in* one of its states. At each time step when the automaton reads a symbol, it *jumps* or *transits* to next state depending on its current state and on the symbol currently read. This function in terms of the current state and input symbol is called *transition function*. The automaton *reads* the input word one symbol after another in the sequence and transits from state to state according to the transition function, until the word is read completely. Once the input word has been read, the automaton is said to have been *stopped* and the state at which automaton has stopped is called *final state*. Depending

on the final state, it's said that the automaton either *accepts* or *rejects* an input word. There is a subset of states of the automaton, which is defined as the set of *accepting states*. If the final state is an accepting state, then the automaton *accepts* the word. Otherwise, the word is *rejected*. The set of all the words accepted by an automaton is called the *language recognized by the automaton*.

Formal definition

Automaton

An **automaton** is represented formally by the 5-tuple $\langle Q, \Sigma, \delta, q_0, A \rangle$, where:

- Q is a finite set of *states*.
- Σ is a finite set of *symbols*, called the *alphabet* of the automaton.
- δ is the **transition function**, that is, $\delta: Q \times \Sigma \rightarrow Q$.
- q_0 is the *start state*, that is, the state which the automaton is *in* when no input has been processed yet, where $q_0 \in Q$.
- A is a set of states of Q (i.e. $A \subseteq Q$) called **accept states**.

Input word

An automaton reads a finite string of symbols a_1, a_2, \dots, a_n , where $a_i \in \Sigma$, which is called a *input word*. Set of all words is denoted by Σ^* .

Run

A *run* of the automaton on an input word $w = a_1, a_2, \dots, a_n \in \Sigma^*$, is a sequence of states $q_0, q_1, q_2, \dots, q_n$, where $q_i \in Q$ such that q_0 is the start state and $q_i = \delta(q_{i-1}, a_i)$ for $0 < i \leq n$. In words, at first the automaton is at the start state q_0 and then automaton reads symbols of the input word in sequence. When automaton reads symbol a_i then it jumps to state $q_i = \delta(q_{i-1}, a_i)$. q_n said to be the *final state* of the run.

Accepting word

A word $w \in \Sigma^*$ is accepted by the automaton if $q_n \in A$.

Recognized language

An automaton can recognize a formal language. The recognized language $L \subseteq \Sigma^*$ by an automaton is the set of all the words that are accepted by the automaton.

Recognizable languages

The recognizable languages is the set of languages that are recognized by some automaton. For above definition of automata the recognizable languages are regular languages. For different definitions of automata, the recognizable languages are different.

Variations in definition of automata

Automata are defined to study useful machines under mathematical formalism. So, the definition of an automaton is open to variations according to the "real world machine", which we want to model using the automaton. People have studied many variations of automata. Above, the most standard variant is described, which is called deterministic finite automaton. The following are some popular variations in the definition of different components of automata.

Input

- *Finite input*: An automaton that accepts only finite sequence of symbols. The above introductory definition only accepts finite words.
- *Infinite input*: An automaton that accepts infinite words (ω -words). Such automata are called *ω -automata*.
- *Tree word input*: The input may be a *tree of symbols* instead of sequence of symbols. In this case after reading each symbol, the automaton *reads* all the successor symbols in the input tree. It is said that the automaton *makes one copy* of itself for each successor and each such copy starts running on one of the successor symbol from the

state according to the transition relation of the automaton. Such an automaton is called tree automaton.

States

- *Finite states*: An automaton that contains only a finite number of states. The above introductory definition describes automata with finite numbers of states.
- *Infinite states*: An automaton that may not have a finite number of states, or even a countable number of states. For example, the quantum finite automaton or topological automaton has uncountable infinity of states.
- *Stack memory*: An automaton may also contain some extra memory in the form of a stack in which symbols can be pushed and popped. This kind of automaton is called a *pushdown automaton*.

Transition function

- *Deterministic*: For a given current state and an input symbol, if an automaton can only jump to one and only one state then it is a *deterministic automaton*.
- *Nondeterministic*: An automaton that, after reading an input symbol, may jump into any of a number of states, as licensed by its transition relation. Notice that the term transition function is replaced by transition relation: The automaton *non-deterministically* decides to jump into one of the allowed choices. Such automaton are called *nondeterministic automaton*.
- *Alternation*: This idea is quite similar to tree automaton, but orthogonal. The automaton may run its *multiple copies* on the *same* next read symbol. Such automata are called *alternating automaton*. Acceptance condition must satisfy all runs of such *copies* to accept the input.

Acceptance condition

- *Acceptance of finite words*: Same as described in the informal definition above.
- *Acceptance of infinite words*: an *omega automaton* cannot have final states, as infinite words never terminate. Rather, acceptance of the word is decided by looking at the infinite sequence of visited states during the run.
- *Probabilistic acceptance*: An automaton need not strictly accept or reject an input. It may accept the input with some probability between zero and one. For example, quantum finite automaton, geometric automaton and *metric automaton* has probabilistic acceptance.

Different combinations of the above variations produce many variety of automaton.

Automata theory

Automata theory is a subject matter which studies properties of various types of automata. For example, following questions are studied about a given type of automata.

- Which class of formal languages is recognizable by some type of automata? (Recognizable languages)
- Is certain automata *closed* under union, intersection, or complementation of formal languages? (Closure properties)
- How much is a type of automata expressive in terms of recognizing class of formal languages? And, their relative expressive power? (Language Hierarchy)

Automata theory also studies if there exist any effective algorithm or not to solve problems similar to following list.

- Does an automaton accept any input word? (emptiness checking)
- Is it possible to transform a given non-deterministic automaton into deterministic automaton without changing the recognizing language? (Determinization)
- For a given formal language, what is the smallest automaton that recognizes it? (Minimization).

Classes of automata

Following is an incomplete list of some types of automata.

Automata	Recognizable language
Deterministic finite automata (DFA)	regular languages
Nondeterministic finite automata (NFA)	regular languages
Nondeterministic finite automata with ϵ -transitions (FND- ϵ or ϵ -NFA)	regular languages
Pushdown automata (PDA)	context-free languages
Linear bounded automata (LBA)	context-sensitive language
Turing machines	recursively enumerable languages
Timed automata	
Deterministic Büchi automata	ω -limit languages
Nondeterministic Büchi automata	ω -regular languages
Nondeterministic/Deterministic Rabin automata	ω -regular languages
Nondeterministic/Deterministic Streett automata	ω -regular languages
Nondeterministic/Deterministic parity automata	ω -regular languages
Nondeterministic/Deterministic Muller automata	ω -regular languages

Discrete, continuous, and hybrid automata

Normally automata theory describes the states of abstract machines but there are analog automata or continuous automata or hybrid discrete-continuous automata, using analog data, continuous time, or both.

Applications

Each model in automata theory play varied roles in several applied areas. Finite automata is used in text processing, compilers, and hardware design. Context-free grammar is used in programming languages and artificial intelligence. Originally, CFG were used in the study of the human languages. Cellular automata is used in the field of biology, the most common example being John Conway's Game of Life. Some other examples which could be explained using automata theory in biology include mollusk and pine cones growth and pigmentation patterns. Going further, a theory suggesting that the whole universe is computed by some sort of a discrete automaton, is being advocated by some scientist. The idea originated in the work of Konrad Zuse, most importantly his 1969 book *Rechnender Raum* and gave rise to Digital physics.

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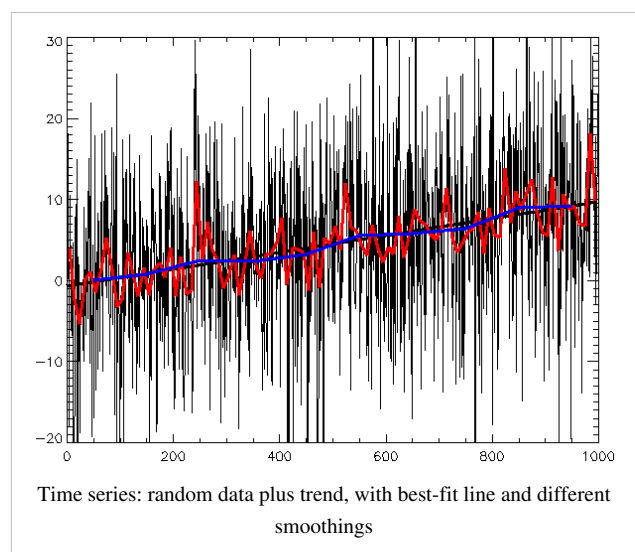
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External links

- Visual Automata Simulator (<http://www.cs.usfca.edu/~jbovet/vas.html>), A tool for simulating, visualizing and transforming finite state automata and Turing Machines, by Jean Bovet
- JFLAP (<http://www.jflap.org>)
- dk.brics.automaton (<http://www.brics.dk/automaton>)
- libfa (<http://www.augeas.net/libfa/index.html>)
- Proyecto SEPa (in Spanish) (<http://www.ucse.edu.ar/fma/sepa/>)
- Exorciser (in German) (<http://www.swisseduc.ch/informatik/exorciser/index.html>)
- Automata Made it on Java, with the Sources so you can see (<http://torturo.com/programas-hechos-en-java/>)

Time series analysis

In statistics, signal processing, econometrics and mathematical finance, a **time series** is a sequence of data points, measured typically at successive times spaced at uniform time intervals. Examples of time series are the daily closing value of the Dow Jones index or the annual flow volume of the Nile River at Aswan. **Time series analysis** comprises methods for analyzing time series data in order to extract meaningful statistics and other characteristics of the data. **Time series forecasting** is the use of a model to forecast future events based on known past events to predict data points before they are measured. An example of time series forecasting in econometrics is predicting the opening price of a stock based on its past performance. Time series are very frequently plotted via line charts.



Time series data have a natural temporal ordering. This makes time series analysis distinct from other common data analysis problems, in which there is no natural ordering of the observations (e.g. explaining people's wages by reference to their education level, where the individuals' data could be entered in any order). Time series analysis is also distinct from spatial data analysis where the observations typically relate to geographical locations (e.g. accounting for house prices by the location as well as the intrinsic characteristics of the houses). A time series model will generally reflect the fact that observations close together in time will be more closely related than observations further apart. In addition, time series models will often make use of the natural one-way ordering of time so that values for a given period will be expressed as deriving in some way from past values, rather than from future values (see time reversibility.)

Methods for time series analysis may be divided into two classes: frequency-domain methods and time-domain methods. The former include auto-correlation, cross-correlation analysis, spectral analysis and recently wavelet analysis; auto-correlation and cross-correlation analysis can also be completed in the time domain.

Analysis

There are several types of data analysis available for time series which are appropriate for different purposes.

General exploration

- Graphical examination of data series
- Autocorrelation analysis to examine serial dependence
- Spectral analysis to examine cyclic behaviour which need not be related to seasonality. For example, sun spot activity varies over 11 year cycles.^{[1] [2]} Other common examples include celestial phenomena, weather patterns, neural activity, commodity prices, and economic activity.

Description

- Separation into components representing trend, seasonality, slow and fast variation, cyclical irregular: see decomposition of time series
- Simple properties of marginal distributions

Prediction and forecasting

- Fully formed statistical models for stochastic simulation purposes, so as to generate alternative versions of the time series, representing what might happen over non-specific time-periods in the future
- Simple or fully formed statistical models to describe the likely outcome of the time series in the immediate future, given knowledge of the most recent outcomes (forecasting).

Models

Models for time series data can have many forms and represent different stochastic processes. When modeling variations in the level of a process, three broad classes of practical importance are the *autoregressive* (AR) models, the *integrated* (I) models, and the *moving average* (MA) models. These three classes depend linearly^[3] on previous data points. Combinations of these ideas produce autoregressive moving average (ARMA) and autoregressive integrated moving average (ARIMA) models. The autoregressive fractionally integrated moving average (ARFIMA) model generalizes the former three. Extensions of these classes to deal with vector-valued data are available under the heading of multivariate time-series models and sometimes the preceding acronyms are extended by including an initial "V" for "vector". An additional set of extensions of these models is available for use where the observed time-series is driven by some "forcing" time-series (which may not have a causal effect on the observed series): the distinction from the multivariate case is that the forcing series may be deterministic or under the experimenter's control. For these models, the acronyms are extended with a final "X" for "exogenous".

Non-linear dependence of the level of a series on previous data points is of interest, partly because of the possibility of producing a chaotic time series. However, more importantly, empirical investigations can indicate the advantage of using predictions derived from non-linear models, over those from linear models.

Among other types of non-linear time series models, there are models to represent the changes of variance along time (heteroskedasticity). These models are called autoregressive conditional heteroskedasticity (ARCH) and the collection comprises a wide variety of representation (GARCH, TARARCH, EGARCH, FIGARCH, CGARCH, etc). Here changes in variability are related to, or predicted by, recent past values of the observed series. This is in contrast to other possible representations of locally varying variability, where the variability might be modelled as being driven by a separate time-varying process, as in a doubly stochastic model.

In recent work on model-free analyses, wavelet transform based methods (for example locally stationary wavelets and wavelet decomposed neural networks) have gained favor. Multiscale (often referred to as multiresolution) techniques decompose a given time series, attempting to illustrate time dependence at multiple scales.

Notation

A number of different notations are in use for time-series analysis. A common notation specifying a time series X that is indexed by the natural numbers is written

$$X = \{X_1, X_2, \dots\}.$$

Another common notation is

$$Y = \{Y_t : t \in T\},$$

where T is the index set.

Conditions

There are two sets of conditions under which much of the theory is built:

- Stationary process
- Ergodicity

However, ideas of stationarity must be expanded to consider two important ideas: strict stationarity and second-order stationarity. Both models and applications can be developed under each of these conditions, although the models in the latter case might be considered as only partly specified.

In addition, time-series analysis can be applied where the series are seasonally stationary or non-stationary. Situations where the amplitudes of frequency components change with time can be dealt with in time-frequency analysis which makes use of a time–frequency representation of a time-series or signal.^[4]

Models

The general representation of an autoregressive model, well-known as $AR(p)$, is

$$Y_t = \alpha_0 + \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + \dots + \alpha_p Y_{t-p} + \varepsilon_t$$

where the term ε_t is the source of randomness and is called white noise. It is assumed to have the following characteristics:

1. $E[\varepsilon_t] = 0$
2. $E[\varepsilon_t^2] = \sigma^2$
3. $E[\varepsilon_t \varepsilon_s] = 0 \quad \forall t \neq s$

With these assumptions, the process is specified up to second-order moments and, subject to conditions on the coefficients, may be second-order stationary.

If the noise also has a normal distribution, it is called normal white noise (denoted here by Normal-WN):

$$\{\varepsilon_t\}_{(t \in T)} : \text{Normal-WN}.$$

In this case the AR process may be strictly stationary, again subject to conditions on the coefficients.

Related tools

Tools for investigating time-series data include:

- Consideration of the autocorrelation function and the spectral density function (also cross-correlation functions and cross-spectral density functions)
- Performing a Fourier transform to investigate the series in the frequency domain
- Use of a filter to remove unwanted noise
- Principal components analysis (or empirical orthogonal function analysis)
- Singular spectrum analysis
- Artificial neural networks
- Time-frequency analysis techniques:
 - Fast Fourier Transform
 - Continuous wavelet transform
 - Short-time Fourier transform
 - Chirplet transform
 - Fractional Fourier transform
- Chaotic analysis
 - Correlation dimension
 - Recurrence plots
 - Recurrence quantification analysis
 - Lyapunov exponents

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External links

- A First Course on Time Series Analysis (<http://statistik.mathematik.uni-wuerzburg.de/timeseries/>) - an open source book on time series analysis with SAS
- Introduction to Time series Analysis (Engineering Statistics Handbook) (<http://www.itl.nist.gov/div898/handbook/pmc/section4/pmc4.htm>) - A practical guide to Time series analysis
- List of Free Software for Time Series Analysis (<http://ces.stat.ucla.edu/software/time-series-analysis>)
- Online Tutorial 'Recurrence Plot' (Flash animation); lots of examples (<http://www.as-internetdienst.de/r67tze4/einbettung.html>)

Lag operator

"Backshift" redirects here. For the linguistic sense see Sequence of tenses.

In time series analysis, the **lag operator** or **backshift operator** operates on an element of a time series to produce the previous element. For example, given some time series

$$X = \{X_1, X_2, \dots\}$$

then

$$LX_t = X_{t-1} \text{ for all } t > 1$$

where L is the lag operator. Sometimes the symbol B for backshift is used instead. Note that the lag operator can be raised to arbitrary integer powers so that

$$L^{-1}X_t = X_{t+1}$$

and

$$L^k X_t = X_{t-k}.$$

Lag polynomials

Also polynomials of the lag operator can be used, and this is a common notation for ARMA models. For example,

$$\varepsilon_t = X_t - \sum_{i=1}^p \varphi_i X_{t-i} = \left(1 - \sum_{i=1}^p \varphi_i L^i\right) X_t$$

specifies an AR(p) model.

A polynomial of lag operators is called a **lag polynomial** so that, for example, the ARMA model can be concisely specified as

$$\varphi X_t = \theta \varepsilon_t$$

where φ and θ respectively represent the lag polynomials,

$$\varphi = 1 - \sum_{i=1}^p \varphi_i L^i$$

and

$$\theta = 1 + \sum_{i=1}^q \theta_i L^i.$$

An **annihilator operator**, denoted $[]_+$, removes the entries of the polynomial with negative power (future values).

Difference operator

In time series analysis, the first difference operator Δ is a special case of lag polynomial.

$$\begin{aligned}\Delta X_t &= X_t - X_{t-1} \\ \Delta X_t &= (1 - L)X_t\end{aligned}$$

Similarly, the second difference operator

$$\begin{aligned}\Delta(\Delta X_t) &= \Delta X_t - \Delta X_{t-1} \\ \Delta^2 X_t &= (1 - L)\Delta X_t \\ \Delta^2 X_t &= (1 - L)(1 - L)X_t \\ \Delta^2 X_t &= (1 - L)^2 X_t\end{aligned}$$

The above approach generalises to the i 'th difference operator $\Delta^i X_t = (1 - L)^i X_t$

Conditional expectation

It is common in stochastic processes to care about the expected value of a variable given a previous information set. Let Ω_t be all information that is common knowledge at time t (this is often subscripted below the expectation operator), then the expected value of X that is some j time-steps in the future can be written equivalently as:

$$E[X_{t+j}|\Omega_t] = E_t[X_{t+j}].$$

With these time-dependent conditional expectations, there is the need to distinguish between the Backshift operator (B) that only adjusts the date of the forecasted variable and the Lag operator (L) that adjusts equally the date of the forecasted variable and the information set:

$$\begin{aligned}L^n E_t[X_{t+j}] &= E_{t-n}[X_{t+j-n}], \\ B^n E_t[X_{t+j}] &= E_t[X_{t+j-n}].\end{aligned}$$

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Shift operator

In mathematics, and in particular functional analysis, the **shift operators** are examples of linear operators, important for their simplicity and natural occurrence. They are used in diverse areas, such as Hardy spaces, the theory of abelian varieties, and the theory of symbolic dynamics, for which the baker's map is an explicit representation. (There is another usage of *shift operator* as a translation operator: see for example Sheffer sequence.) In time series analysis, this operator is called the **lag operator**.

A typical **one-sided shift** operator takes an infinite sequence of numbers

$$(a_1, a_2, \dots)$$

to

$$(0, a_1, a_2, \dots).$$

This operation respects typical convergence conditions, such as absolute convergence of the corresponding infinite series; it therefore gives rise to continuous operators on the standard sequence spaces used in functional analysis, usually with norm 1.

Another way to look at it would be in terms of polynomials: the sequences that eventually end in a string

$$(\dots, 0, 0, 0, \dots)$$

or, in other words, having only a finite number of non-zero entries, are in a 1-1 correspondence with polynomials in an indeterminate T having a_i as coefficient of T^i . The advantage of this representation is then that the *shift operator* becomes multiplication by T : this reveals quickly several aspects of its structure. Spaces of polynomials carry numerous topological structures; shift operators can be constructed by extension on corresponding complete spaces.

The **bilateral shift** operators are the related operators in which the sequences are bi-infinite (functions on the integers, rather than just the natural numbers). One can say that the analogue in this case of the polynomial representation is that by Laurent polynomials. The theory of analytic functions is related to that of polynomials, by allowing infinite power series; on the other hand meromorphic functions have Laurent series that terminate in the direction of negative exponents. In the same way, the one-sided and bilateral shifts have rather different properties. This connection with function theory is made more precise in the context of Hardy spaces.

Action on Hilbert spaces

The unilateral and bilateral shifts have a natural action on Hilbert spaces, giving bounded operators S and U on the ℓ^p sequence spaces $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$ respectively. The unilateral shift S is a proper isometry with range equal to all vectors which vanish in the first coordinate. The bilateral shift U , on the other hand, is a unitary operator. The operator S is a compression of U , in the sense that

$$Ux' = Sx \text{ for each } x \in \ell^2(\mathbb{N}),$$

where x' is the vector in $\ell^2(\mathbb{Z})$ with $x'_i = x_i$ for $i \geq 0$ and $x'_i = 0$ for $i < 0$. This observation is at the heart of the construction of many unitary dilations of isometries.

The spectrum of S is the unit disk while the spectrum of U is the unit circle in the complex plane.

The Wold decomposition says that every isometry on a Hilbert space is of the form

$$S^\alpha \oplus U$$

where S^α is S to the power of some cardinal number α and U is a unitary operator. In turn, the C*-algebra generated by an arbitrary proper isometry is isomorphic to the C*-algebra generated by S .

The shift S is one example of a Fredholm operator; it has Fredholm index -1 .

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Shift space

In symbolic dynamics and related branches of mathematics, a **shift space** or **subshift** is a set of infinite words representing the evolution of a discrete system. In fact, shift spaces and *symbolic dynamical systems* are often considered synonyms.

Notation

Let A be a finite set of states. An *infinite* (respectively *bi-infinite*) *word* over A is a sequence $\mathbf{x} = (x_n)_{n \in M}$, where $M = \mathbb{N}$ (resp. $M = \mathbb{Z}$) and x_n is in A for any integer n . The shift operator σ acts on an infinite or bi-infinite word by shifting all symbols to the left, i.e.,

$$(\sigma(\mathbf{x}))(n) = x_{n+1} \text{ for all } n.$$

In the following we choose $M = \mathbb{N}$ and thus speak of infinite words, but all definitions are naturally generalizable to the bi-infinite case.

Definition

A set of infinite words over A is a *shift space* if it is closed with respect to the natural product topology of $A^{\mathbb{N}}$ and invariant under the shift operator. Thus a set $S \subseteq A^{\mathbb{N}}$ is a subshift if and only if

1. for any (pointwise) convergent sequence $(\mathbf{x}_k)_{k \geq 0}$ of elements of S , the limit $\lim_{k \rightarrow \infty} \mathbf{x}_k$ also belongs to S ; and
2. $\sigma(S) = S$.

A shift space S is sometimes denoted as (S, σ) to emphasize the role of the shift operator.

Some authors^[1] use the term *subshift* for a set of infinite words that is just invariant under the shift, and reserve the term *shift space* for those that are also closed.

Characterization and sofic subshifts

A subset S of $A^{\mathbb{N}}$ is a shift space if and only if there exists a set X of finite words such that S coincides with the set of all infinite words over A having no factor in X .

When X is a regular language, the corresponding subshift is called **sofic**. In particular, if X is finite then S is called a subshift of finite type.

Examples

The first trivial example of shift space (of finite type) is the *full shift* $A^{\mathbb{N}}$.

Let $A = \{a, b\}$. The set of all infinite words over A containing at most one b is a sofic subshift, not of finite type.

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Markov partition

A **Markov partition** is a tool used in dynamical systems theory, allowing the methods of symbolic dynamics to be applied to the study of hyperbolic systems. By using a Markov partition, the system can be made to resemble a discrete-time Markov process, with the long-term dynamical characteristics of the system represented as a Markov shift. The appellation 'Markov' is appropriate because the resulting dynamics of the system obeys the Markov property. The Markov partition thus allows standard techniques from symbolic dynamics to be applied, including the computation of expectation values, correlations, topological entropy, topological zeta functions, Fredholm determinants and the like.

Motivation

Let (M, φ) be a discrete dynamical system. A basic method of studying its dynamics is to find a **symbolic representation**: a faithful encoding of the points of M by sequences of symbols such that the map φ becomes the shift map.

Suppose that M has been divided into a number of pieces E_1, E_2, \dots, E_r , which are thought to be as small and localized, with virtually no overlaps. The behavior of a point x under the iterates of φ can be tracked by recording, for each n , the part E_i which contains $\varphi^n(x)$. This results in an infinite sequence on the alphabet $\{1, 2, \dots, r\}$ which encodes the point. In general, this encoding may be imprecise (the same sequence may represent many different points) and the set of sequences which arise in this way may be difficult to describe. Under certain conditions, which are made explicit in the rigorous definition of a Markov partition, the assignment of the sequence to a point of M becomes an almost one-to-one map whose image is a symbolic dynamical system of a special kind called a shift of finite type. In this case, the symbolic representation is a powerful tool for investigating the properties of the dynamical system (M, φ) .

Formal definition

A Markov partition^[1] is a finite cover of the invariant set of the manifold by a set of curvilinear rectangles $\{E_1, E_2, \dots, E_r\}$ such that

- For any pair of points $x, y \in E_i$, that $W_s(x) \cap W_u(y) \in E_i$
- $\text{Int}E_i \cap \text{Int}E_j = \emptyset$ for $i \neq j$
- If $x \in \text{Int}E_i$ and $\phi(x) \in \text{Int}E_j$, then

$$\phi[W_u(x) \cap E_i] \supset W_u(\phi x) \cap E_j$$

$$\phi[W_s(x) \cap E_i] \subset W_s(\phi x) \cap E_j$$

Here, $W_u(x)$ and $W_s(x)$ are the unstable and stable manifolds of x , respectively, and $\text{Int}E_i$ simply denotes the interior of E_i .

These last two conditions can be understood as a statement of the Markov property for the symbolic dynamics; that is, the movement of a trajectory from one open cover to the next is determined only by the most recent cover, and not the past history of the system. It is this property of the covering that merits the 'Markov' appellation. The resulting dynamics is that of a Markov shift; that this is indeed the case is due to theorems by Yakov Sinai (1968) and Rufus Bowen (1975), thus putting symbolic dynamics on a firm footing.

Examples

Markov partitions have been constructed in several situations.

- Anosov diffeomorphisms of the torus.
- Dynamical billiards, in which case the covering is countable.

Markov partitions make homoclinic and heteroclinic orbits particularly easy to describe.

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Sharkovskii's theorem

In mathematics, **Sharkovskii's theorem**, named after Oleksandr Mikolaiovich Sharkovsky, is a result about discrete dynamical systems. One of the implications of the theorem is that if a continuous discrete dynamical system on the real line has a periodic point of period 3, then it must have periodic points of every other period.

The theorem

Suppose

$$f: \mathbf{R} \rightarrow \mathbf{R}$$

is a continuous function. We say that the number x is a *periodic point of period m* if $f^m(x) = x$ (where f^m denotes the composition of m copies of f) and having *least period m* if furthermore $f^k(x) \neq x$ for all $0 < k < m$. We are interested in the possible periods of periodic points of f . Consider the following ordering of the positive integers:

$$\begin{array}{ccccccccccc}
 3 & 5 & 7 & 9 & 11 & \dots & (2n+1) \cdot 2^0 & \dots \\
 3 \cdot 2 & 5 \cdot 2 & 7 \cdot 2 & 9 \cdot 2 & 11 \cdot 2 & \dots & (2n+1) \cdot 2^1 & \dots \\
 3 \cdot 2^2 & 5 \cdot 2^2 & 7 \cdot 2^2 & 9 \cdot 2^2 & 11 \cdot 2^2 & \dots & (2n+1) \cdot 2^2 & \dots \\
 3 \cdot 2^3 & 5 \cdot 2^3 & 7 \cdot 2^3 & 9 \cdot 2^3 & 11 \cdot 2^3 & \dots & (2n+1) \cdot 2^3 & \dots \\
 & \vdots & & & & & & \\
 \dots & 2^n & \dots & 2^5 & 2^4 & 2^3 & 2 & 1
 \end{array}$$

We start, that is, with the odd numbers in increasing order, then 2 times the odds, 4 times the odds, 8 times the odds, etc., and at the end we put the powers of two in decreasing order. Every positive integer appears exactly once somewhere on this list. Sharkovskii's theorem states that if f has a periodic point of least period m and m precedes n in the above ordering, then f has also a periodic point of least period n .

As a consequence, we see that if f has only finitely many periodic points, then they must all have periods which are powers of two. Furthermore, if there is a periodic point of period three, then there are periodic points of all other periods.

Sharkovskii's theorem does not state that there are *stable* cycles of those periods, just that there are cycles of those periods. For systems such as the logistic map, the bifurcation diagram shows a range of parameter values for which apparently the only cycle has period 3. In fact, there must be cycles of all periods there, but they are not stable and therefore not visible on the computer generated picture.

Interestingly, the above "Sharkovskii ordering" of the positive integers also occurs in a slightly different context in connection with the logistic map: the *stable* cycles appear in this order in the bifurcation diagram, starting with 1 and ending with 3, as the parameter is increased. (Here we ignore a stable cycle if a stable cycle of the same order has occurred earlier.)

The assumption of continuity is important, as the discontinuous function $f: x \mapsto (1-x)^{-1}$, for which every value has period 3, would otherwise be a counterexample.

Generalizations

Sharkovskii's theorem does not immediately apply to dynamical systems on other topological spaces. It is easy to find a circle map with periodic points of period 3 only: take a rotation by 120 degrees, for example. But some generalizations are possible, typically involving the mapping class group of the space minus a periodic orbit.

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External link

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Ergodic system

Ergodic theory is a branch of mathematics that studies dynamical systems with an invariant measure and related problems. Its initial development was motivated by problems of statistical physics.

A central concern of ergodic theory is the behavior of a dynamical system when it is allowed to run for a long time. The first result in this direction is the Poincaré recurrence theorem, which claims that almost all points in any subset of the phase space eventually revisit the set. More precise information is provided by various **ergodic theorems** which assert that, under certain conditions, the time average of a function along the trajectories exists almost everywhere and is related to the space average. Two of the most important examples are ergodic theorems of Birkhoff and von Neumann. For the special class of **ergodic systems**, the time average is the same for almost all initial points: statistically speaking, the system that evolves for a long time "forgets" its initial state. Stronger properties, such as mixing and equidistribution, have also been extensively studied.

The problem of metric classification of systems is another important part of the abstract ergodic theory. An outstanding role in ergodic theory and its applications to stochastic processes is played by the various notions of entropy for dynamical systems.

The concepts of ergodicity and the ergodic hypothesis are central to applications of ergodic theory. The underlying idea is that for certain systems the time average of their properties is equal to the average over the entire space. Applications of ergodic theory to other parts of mathematics usually involve establishing ergodicity properties for systems of special kind. In geometry, methods of ergodic theory have been used to study the geodesic flow on Riemannian manifolds, starting with the results of Eberhard Hopf for Riemann surfaces of negative curvature. Markov chains form a common context for applications in probability theory. Ergodic theory has fruitful connections with harmonic analysis, Lie theory (representation theory, lattices in algebraic groups), and number theory (the theory of diophantine approximations, L-functions).

Ergodic transformations

Ergodic theory is often concerned with **ergodic transformations**.

Let $T: X \rightarrow X$ be a measure-preserving transformation on a measure space (X, Σ, μ) , with $\mu(X) = 1$. A measure-preserving transformation T as above is **ergodic** if for every

$$E \in \Sigma \text{ with } T^{-1}(E) = E \text{ either } \mu(E) = 0 \text{ or } \mu(E) = 1.$$

Examples

- An irrational rotation of the circle \mathbf{R}/\mathbf{Z} , $T: x \rightarrow x+\theta$, where θ is irrational, is ergodic. This transformation has even stronger properties of unique ergodicity, minimality, and equidistribution. By contrast, if $\theta = p/q$ is rational (in lowest terms) then T is periodic, with period q , and thus cannot be ergodic: for any interval I of length a , $0 < a < 1/q$, its orbit under T is a T -invariant mod 0 set that is a union of q intervals of length a , hence it has measure qa strictly between 0 and 1.
- Let G be a compact abelian group, μ the normalized Haar measure, and T a group automorphism of G . Let G^* be the Pontryagin dual group, consisting of the continuous characters of G , and T^* be the corresponding adjoint automorphism of G^* . The automorphism T is ergodic if and only if the equality $(T^*)^n(\chi) = \chi$ is possible only when $n = 0$ or χ is the trivial character of G . In particular, if G is the n -dimensional torus and the automorphism T is represented by an integral matrix A then T is ergodic if and only if no eigenvalue of A is a root of unity.
- A Bernoulli shift is ergodic. More generally, ergodicity of the shift transformation associated with a sequence of i.i.d. random variables and some more general stationary processes follows from Kolmogorov's zero-one law.
- Ergodicity of a continuous dynamical system means that its trajectories "spread around" the phase space. A system with a compact phase space which has a non-constant first integral cannot be ergodic. This applies, in particular, to Hamiltonian systems with a first integral I functionally independent from the Hamilton function H and a compact level set $X = \{(p, q): H(p, q) = E\}$ of constant energy. Liouville's theorem implies the existence of a finite invariant measure on X , but the dynamics of the system is constrained to the level sets of I on X , hence the system possesses invariant sets of positive but less than full measure. A property of continuous dynamical systems that is the opposite of ergodicity is complete integrability.

Ergodic theorems

Let $T: X \rightarrow X$ be a measure-preserving transformation on a measure space (X, Σ, μ) . One may then consider the "time average" of a μ -integrable function f , i.e. $f \in L^1(\mu)$. The "time average" is defined as the average (if it exists) over iterations of T starting from some initial point x .

$$\hat{f}(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x).$$

If $\mu(X)$ is finite and nonzero, we can consider the "space average" or "phase average" of f , defined as

$$\bar{f} = \frac{1}{\mu(X)} \int f d\mu. \quad (\text{For a probability space, } \mu(X) = 1.)$$

In general the time average and space average may be different. But if the transformation is ergodic, and the measure is invariant, then the time average is equal to the space average almost everywhere. This is the celebrated ergodic theorem, in an abstract form due to George David Birkhoff. (Actually, Birkhoff's paper considers not the abstract general case but only the case of dynamical systems arising from differential equations on a smooth manifold.) The equidistribution theorem is a special case of the ergodic theorem, dealing specifically with the distribution of probabilities on the unit interval.

More precisely, the **pointwise** or **strong ergodic theorem** states that the limit in the definition of the time average of f exists for almost every x and that the (almost everywhere defined) limit function \hat{f} is integrable:

$$\hat{f} \in L^1(\mu).$$

Furthermore, \hat{f} is T -invariant, that is to say

$$\hat{f} \circ T = \hat{f}$$

holds almost everywhere, and if $\mu(X)$ is finite, then the normalization is the same:

$$\int \hat{f} d\mu = \int f d\mu.$$

In particular, if T is ergodic, then \hat{f} must be a constant (almost everywhere), and so one has that

$$\bar{f} = \hat{f}$$

almost everywhere. Joining the first to the last claim and assuming that $\mu(X)$ is finite and nonzero, one has that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \frac{1}{\mu(X)} \int f d\mu$$

for almost all x , i.e., for all x except for a set of measure zero.

For an ergodic transformation, the time average equals the space average almost surely.

As an example, assume that the measure space (X, Σ, μ) models the particles of a gas as above, and let $f(x)$ denotes the velocity of the particle at position x . Then the pointwise ergodic theorems says that the average velocity of all particles at some given time is equal to the average velocity of one particle over time.

Probabilistic formulation: Birkhoff–Khinchin theorem

Birkhoff–Khinchin theorem. Let f be measurable, $E(|f|) < +\infty$, and T be a measure-preserving map. Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = E(f|\mathcal{C}) \text{ a.s.,}$$

where $E(f|\mathcal{C})$ is the conditional expectation given the σ -algebra \mathcal{C} of invariant sets of T .

Corollary (**Pointwise ergodic theorem**) In particular, if T is also ergodic, then \mathcal{C} is the trivial σ -algebra, and thus

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = E(f) \text{ a.s.}$$

Mean ergodic theorem

Another form of the ergodic theorem, **von Neumann's mean ergodic theorem**, holds in Hilbert spaces.^[1]

Let U be a unitary operator on a Hilbert space H ; more generally, an isometric linear operator (that is, a not necessarily surjective linear operator satisfying $\|Ux\| = \|x\|$ for all $x \in H$, or equivalently, satisfying $U^*U = I$, but not necessarily $UU^* = I$). Let P be the orthogonal projection onto $\{\psi \in H | U\psi = \psi\} = \text{Ker}(I - U)$.

Then, for any $x \in H$, we have:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n x = Px,$$

where the limit is with respect to the norm on H . In other words, the sequence of averages

$$\frac{1}{N} \sum_{n=0}^{N-1} U^n$$

converges to P in the strong operator topology.

This theorem specializes to the case in which the Hilbert space H consists of L^2 functions on a measure space and U is an operator of the form

$$Uf(x) = f(Tx)$$

where T is a measure-preserving endomorphism of X , thought of in applications as representing a time-step of a discrete dynamical system.^[2] The ergodic theorem then asserts that the average behavior of a function f over sufficiently large time-scales is approximated by the orthogonal component of f which is time-invariant.

In another form of the mean ergodic theorem, let U_t be a strongly continuous one-parameter group of unitary operators on H . Then the operator

$$\frac{1}{T} \int_0^T U_t dt$$

converges in the strong operator topology as $T \rightarrow \infty$. In fact, this result also extends to the case of strongly continuous one-parameter semigroup of contractive operators on a reflexive space.

Remark: Some intuition for the mean ergodic theorem can be developed by considering the case where complex numbers of unit length are regarded as unitary transformations on the complex plane (by left multiplication). If we pick a single complex number of unit length (which we think of as U), it is intuitive that its powers will fill up the circle. Since the circle is symmetric around 0, it makes sense that the averages of the powers of U will converge to 0. Also, 0 is the only fixed point of U , and so the projection onto the space of fixed points must be the zero operator (which agrees with the limit just described).

Convergence of the ergodic means in the L^p norms

Let (X, Σ, μ) be as above a probability space with a measure preserving transformation T , and let $1 \leq p \leq \infty$. The conditional expectation with respect to the sub- σ -algebra Σ_T of the T -invariant sets is a linear projector E_T of norm 1 of the Banach space $L^p(X, \Sigma, \mu)$ onto its closed subspace $L^p(X, \Sigma_T, \mu)$. The latter may also be characterized as the space of all T -invariant L^p -functions on X . The ergodic means, as linear operators on $L^p(X, \Sigma, \mu)$ also have unit operator norm; and, as a simple consequence of the Birkhoff–Khinchin theorem, converge to the projector E_T in the strong operator topology of L^p if $1 \leq p < \infty$, and in the weak operator topology if $p = \infty$. More is true if $1 < p \leq \infty$: then the Wiener–Yoshida–Kakutani ergodic dominated convergence theorem states that the ergodic means of $f \in L^p$ are dominated in L^p ; however, if $f \in L^1$, the ergodic means may fail to be equidominated in L^1 . Finally, if f is assumed to be in the Zygmund class, that is $|f| \log^+ |f|$ is integrable, then the ergodic means are even dominated in L^1 .

Sojourn time

Let (X, Σ, μ) be a measure space such that $\mu(X)$ is finite and nonzero. The time spent in a measurable set A is called the **sojourn time**. An immediate consequence of the ergodic theorem is that, in an ergodic system, the relative measure of A is equal to the mean sojourn time:

$$\frac{\mu(A)}{\mu(X)} = \frac{1}{\mu(X)} \int \chi_A d\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_A(T^k x)$$

for all x except for a set of measure zero, where χ_A is the indicator function of A .

Let the **occurrence times** of a measurable set A be defined as the set k_1, k_2, k_3, \dots , of times k such that $T^k(x)$ is in A , sorted in increasing order. The differences between consecutive occurrence times $R_i = k_i - k_{i-1}$ are called the **recurrence times** of A . Another consequence of the ergodic theorem is that the average recurrence time of A is inversely proportional to the measure of A , assuming that the initial point x is in A , so that $k_0 = 0$.

$$\frac{R_1 + \dots + R_n}{n} \rightarrow \frac{\mu(X)}{\mu(A)} \quad (\text{almost surely})$$

(See almost surely.) That is, the smaller A is, the longer it takes to return to it.

Ergodic flows on manifolds

The ergodicity of the geodesic flow on compact Riemann surfaces of variable negative curvature and on compact manifolds of constant negative curvature of any dimension was proved by Eberhard Hopf in 1939, although special cases had been studied earlier: see for example, Hadamard's billiards (1898) and Artin billiard (1924). The relation between geodesic flows on Riemann surfaces and one-parameter subgroups on $SL(2, \mathbf{R})$ was described in 1952 by S. V. Fomin and I. M. Gelfand. The article on Anosov flows provides an example of ergodic flows on $SL(2, \mathbf{R})$ and on Riemann surfaces of negative curvature. Much of the development described there generalizes to hyperbolic manifolds, since they can be viewed as quotients of the hyperbolic space by the action of a lattice in the semisimple Lie group $SO(n, 1)$. Ergodicity of the geodesic flow on Riemannian symmetric spaces was demonstrated by F. I. Mautner in 1957. In 1967 D. V. Anosov and Ya. G. Sinai proved ergodicity of the geodesic flow on compact manifolds of variable negative sectional curvature. A simple criterion for the ergodicity of a homogeneous flow on a homogeneous space of a semisimple Lie group was given by Calvin C. Moore in 1966. Many of the theorems and results from this area of study are typical of rigidity theory.

In the 1930s G. A. Hedlund proved that the horocycle flow on a compact hyperbolic surface is minimal and ergodic. Unique ergodicity of the flow was established by Hillel Furstenberg in 1972. Ratner's theorems provide a major generalization of ergodicity for unipotent flows on the homogeneous spaces of the form $\Gamma \backslash G$, where G is a Lie group and Γ is a lattice in G .

In the last 20 years, there have been many works trying to find a measure-classification theorem similar to Ratner's theorems but for diagonalizable actions, motivated by conjectures of Furstenberg and Margulis. An important partial result (solving those conjectures with an extra assumption of positive entropy) was proved by Elon Lindenstrauss, and he was awarded the Fields medal in 2010 for this result.

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External links

- Ergodic Theory (29 October 2007) (<http://www.cscs.umich.edu/~crshalizi/notebooks/ergodic-theory.html>)
Notes by Cosma Rohilla Shalizi

Ergodic theory

Ergodic theory is a branch of mathematics that studies dynamical systems with an invariant measure and related problems. Its initial development was motivated by problems of statistical physics.

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Birkhoff–Khinchin theorem. Let f be measurable, $E(|f|) < +\infty$, and T be a measure-preserving map. Then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = E(f|\mathcal{C}) \text{ a.s.},$$

where $E(f|\mathcal{C})$ is the conditional expectation given the σ -algebra \mathcal{C} of invariant sets of T .

Corollary (**Pointwise ergodic theorem**) In particular, if T is also ergodic, then \mathcal{C} is the trivial σ -algebra, and thus

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = E(f) \text{ a.s.}$$

Mean ergodic theorem

Another form of the ergodic theorem, **von Neumann's mean ergodic theorem**, holds in Hilbert spaces.^[1]

Let U be a unitary operator on a Hilbert space H ; more generally, an isometric linear operator (that is, a not necessarily surjective linear operator satisfying $\|Ux\| = \|x\|$ for all $x \in H$, or equivalently, satisfying $U^*U = I$, but not necessarily $UU^* = I$). Let P be the orthogonal projection onto $\{\psi \in H | U\psi = \psi\} = \text{Ker}(I - U)$.

Then, for any $x \in H$, we have:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} U^n x = Px,$$

where the limit is with respect to the norm on H . In other words, the sequence of averages

$$\frac{1}{N} \sum_{n=0}^{N-1} U^n$$

converges to P in the strong operator topology.

This theorem specializes to the case in which the Hilbert space H consists of L^2 functions on a measure space and U is an operator of the form

$$Uf(x) = f(Tx)$$

where T is a measure-preserving endomorphism of X , thought of in applications as representing a time-step of a discrete dynamical system.^[2] The ergodic theorem then asserts that the average behavior of a function f over sufficiently large time-scales is approximated by the orthogonal component of f which is time-invariant.

In another form of the mean ergodic theorem, let U_t be a strongly continuous one-parameter group of unitary operators on H . Then the operator

$$\frac{1}{T} \int_0^T U_t dt$$

converges in the strong operator topology as $T \rightarrow \infty$. In fact, this result also extends to the case of strongly continuous one-parameter semigroup of contractive operators on a reflexive space.

Remark: Some intuition for the mean ergodic theorem can be developed by considering the case where complex numbers of unit length are regarded as unitary transformations on the complex plane (by left multiplication). If we pick a single complex number of unit length (which we think of as U), it is intuitive that its powers will fill up the circle. Since the circle is symmetric around 0, it makes sense that the averages of the powers of U will converge to 0. Also, 0 is the only fixed point of U , and so the projection onto the space of fixed points must be the zero operator (which agrees with the limit just described).

Convergence of the ergodic means in the L^p norms

Let (X, Σ, μ) be as above a probability space with a measure preserving transformation T , and let $1 \leq p \leq \infty$. The conditional expectation with respect to the sub- σ -algebra Σ_T of the T -invariant sets is a linear projector E_T of norm 1 of the Banach space $L^p(X, \Sigma, \mu)$ onto its closed subspace $L^p(X, \Sigma_T, \mu)$. The latter may also be characterized as the space of all T -invariant L^p -functions on X . The ergodic means, as linear operators on $L^p(X, \Sigma, \mu)$ also have unit operator norm; and, as a simple consequence of the Birkhoff–Khinchin theorem, converge to the projector E_T in the strong operator topology of L^p if $1 \leq p < \infty$, and in the weak operator topology if $p = \infty$. More is true if $1 < p \leq \infty$: then the Wiener–Yoshida–Kakutani ergodic dominated convergence theorem states that the ergodic means of $f \in L^p$ are dominated in L^p ; however, if $f \in L^1$, the ergodic means may fail to be equidominated in L^1 . Finally, if f is assumed to be in the Zygmund class, that is $|f| \log^+ |f|$ is integrable, then the ergodic means are even dominated in L^1 .

Sojourn time

Let (X, Σ, μ) be a measure space such that $\mu(X)$ is finite and nonzero. The time spent in a measurable set A is called the **sojourn time**. An immediate consequence of the ergodic theorem is that, in an ergodic system, the relative measure of A is equal to the mean sojourn time:

$$\frac{\mu(A)}{\mu(X)} = \frac{1}{\mu(X)} \int \chi_A d\mu = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \chi_A(T^k x)$$

for all x except for a set of measure zero, where χ_A is the indicator function of A .

Let the **occurrence times** of a measurable set A be defined as the set k_1, k_2, k_3, \dots , of times k such that $T^k(x)$ is in A , sorted in increasing order. The differences between consecutive occurrence times $R_i = k_i - k_{i-1}$ are called the **recurrence times** of A . Another consequence of the ergodic theorem is that the average recurrence time of A is inversely proportional to the measure of A , assuming that the initial point x is in A , so that $k_0 = 0$.

$$\frac{R_1 + \dots + R_n}{n} \rightarrow \frac{\mu(X)}{\mu(A)} \quad (\text{almost surely})$$

(See almost surely.) That is, the smaller A is, the longer it takes to return to it.

Ergodic flows on manifolds

The ergodicity of the geodesic flow on compact Riemann surfaces of variable negative curvature and on compact manifolds of constant negative curvature of any dimension was proved by Eberhard Hopf in 1939, although special cases had been studied earlier: see for example, Hadamard's billiards (1898) and Artin billiard (1924). The relation between geodesic flows on Riemann surfaces and one-parameter subgroups on $SL(2, \mathbf{R})$ was described in 1952 by S. V. Fomin and I. M. Gelfand. The article on Anosov flows provides an example of ergodic flows on $SL(2, \mathbf{R})$ and on Riemann surfaces of negative curvature. Much of the development described there generalizes to hyperbolic manifolds, since they can be viewed as quotients of the hyperbolic space by the action of a lattice in the semisimple Lie group $SO(n, 1)$. Ergodicity of the geodesic flow on Riemannian symmetric spaces was demonstrated by F. I. Mautner in 1957. In 1967 D. V. Anosov and Ya. G. Sinai proved ergodicity of the geodesic flow on compact manifolds of variable negative sectional curvature. A simple criterion for the ergodicity of a homogeneous flow on a homogeneous space of a semisimple Lie group was given by Calvin C. Moore in 1966. Many of the theorems and results from this area of study are typical of rigidity theory.

In the 1930s G. A. Hedlund proved that the horocycle flow on a compact hyperbolic surface is minimal and ergodic. Unique ergodicity of the flow was established by Hillel Furstenberg in 1972. Ratner's theorems provide a major generalization of ergodicity for unipotent flows on the homogeneous spaces of the form $\Gamma \backslash G$, where G is a Lie group and Γ is a lattice in G .

In the last 20 years, there have been many works trying to find a measure-classification theorem similar to Ratner's theorems but for diagonalizable actions, motivated by conjectures of Furstenberg and Margulis. An important partial result (solving those conjectures with an extra assumption of positive entropy) was proved by Elon Lindenstrauss, and he was awarded the Fields medal in 2010 for this result.

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External links

- Ergodic Theory (29 October 2007) (<http://www.cscs.umich.edu/~crshalizi/notebooks/ergodic-theory.html>)
Notes by Cosma Rohilla Shalizi

Measure-preserving dynamical system

In mathematics, a **measure-preserving dynamical system** is an object of study in the abstract formulation of dynamical systems, and ergodic theory in particular.

Definition

A measure-preserving dynamical system is defined as a probability space and a measure-preserving transformation on it. In more detail, it is a system

$$(X, \mathcal{B}, \mu, T)$$

with the following structure:

- X is a set,
- \mathcal{B} is a σ -algebra over X ,
- $\mu : \mathcal{B} \rightarrow [0, 1]$ is a probability measure, so that $\mu(X) = 1$, and
- $T : X \rightarrow X$ is a measurable transformation which preserves the measure μ , i. e. each $A \in \mathcal{B}$ satisfies

$$\mu(T^{-1}A) = \mu(A).$$

This definition can be generalized to the case in which T is not a single transformation that is iterated to give the dynamics of the system, but instead is a monoid (or even a group) of transformations $T_s : X \rightarrow X$ parametrized by $s \in \mathbb{Z}$ (or \mathbb{R} , or $\mathbb{N} \cup \{0\}$, or $[0, +\infty)$), where each transformation T_s satisfies the same requirements as

T above. In particular, the transformations obey the rules

- $T_0 = \text{id}_X : X \rightarrow X$, the identity function on X ;
- $T_s \circ T_t = T_{t+s}$, whenever all the terms are well-defined;
- $T_s^{-1} = T_{-s}$, whenever all the terms are well-defined.

The earlier, simpler case fits into this framework by defining $T_s := T^s$ for $s \in \mathbb{N}$.

The existence of invariant measures for certain maps and Markov processes is established by the Krylov–Bogolyubov theorem.

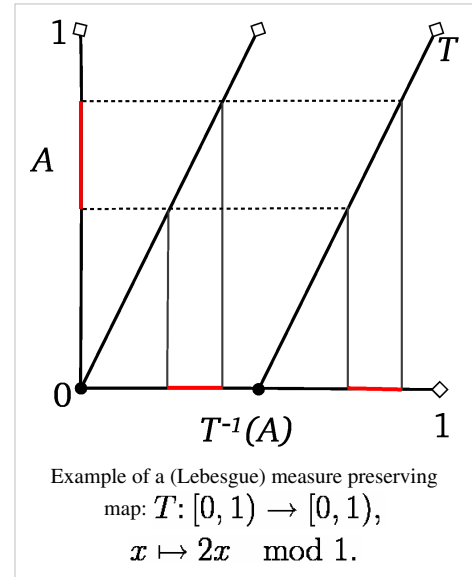
Examples

Examples include:

- μ could be the normalized angle measure $d\theta/2\pi$ on the unit circle, and T a rotation. See equidistribution theorem;
- the Bernoulli scheme;
- the interval exchange transformation;
- with the definition of an appropriate measure, a subshift of finite type;
- the base flow of a random dynamical system.

Homomorphisms

The concept of a homomorphism and an isomorphism may be defined.



Consider two dynamical systems (X, \mathcal{A}, μ, T) and (Y, \mathcal{B}, ν, S) . Then a mapping

$$\phi: X \rightarrow Y$$

is a **homomorphism of dynamical systems** if it satisfies the following three properties:

1. The map ϕ is measurable,
2. For each $B \in \mathcal{B}$, one has $\mu(\phi^{-1}B) = \nu(B)$,
3. For μ -almost all $x \in X$, one has $\phi(Tx) = S(\phi x)$.

The system (Y, \mathcal{B}, ν, S) is then called a **factor** of (X, \mathcal{A}, μ, T) .

The map ϕ is an **isomorphism of dynamical systems** if, in addition, there exists another mapping

$$\psi: Y \rightarrow X$$

that is also a homomorphism, which satisfies

1. For μ -almost all $x \in X$, one has $x = \psi(\phi x)$
2. For ν -almost all $y \in Y$, one has $y = \phi(\psi y)$.

Generic points

A point $x \in X$ is called a **generic point** if the orbit of the point is distributed uniformly according to the measure.

Symbolic names and generators

Consider a dynamical system (X, \mathcal{B}, T, μ) , and let $\mathcal{Q} = \{Q_1, \dots, Q_k\}$ be a partition of X into k measurable pair-wise disjoint pieces. Given a point $x \in X$, clearly x belongs to only one of the Q_i . Similarly, the iterated point $T^n x$ can belong to only one of the parts as well. The **symbolic name** of x , with regards to the partition \mathcal{Q} , is the sequence of integers $\{a_n\}$ such that

$$T^n x \in Q_{a_n}.$$

The set of symbolic names with respect to a partition is called the symbolic dynamics of the dynamical system. A partition \mathcal{Q} is called a **generator** or **generating partition** if μ -almost every point x has a unique symbolic name.

Operations on partitions

Given a partition $Q = \{Q_1, \dots, Q_k\}$ and a dynamical system (X, \mathcal{B}, T, μ) , we define T -pullback of Q as

$$T^{-1}Q = \{T^{-1}Q_1, \dots, T^{-1}Q_k\}.$$

Further, given two partitions $Q = \{Q_1, \dots, Q_k\}$ and $R = \{R_1, \dots, R_m\}$, we define their *refinement* $Q \vee R$ as

$$Q \vee R = \{Q_i \cap R_j \mid i = 1, \dots, k, j = 1, \dots, m, \mu(Q_i \cap R_j) > 0\}.$$

With these two constructs we may define *refinement of an iterated pullback*

$$\begin{aligned} \bigvee_{n=0}^N T^{-n}Q &= \{Q_{i_0} \cap T^{-1}Q_{i_1} \cap \dots \cap T^{-N}Q_{i_N} \\ &\quad \mid i_\ell = 1, \dots, k, \ell = 0, \dots, N, \\ &\quad \mu(Q_{i_0} \cap T^{-1}Q_{i_1} \cap \dots \cap T^{-N}Q_{i_N}) > 0\} \end{aligned}$$

which plays crucial role in the construction of the measure-theoretic entropy of a dynamical system.

Measure-theoretic entropy

The entropy of a partition Q is defined as

$$H(Q) = - \sum_{m=1}^k \mu(Q_m) \log \mu(Q_m).$$

The measure-theoretic entropy of a dynamical system (X, \mathcal{B}, T, μ) with respect to a partition $Q = \{Q_1, \dots, Q_k\}$ is then defined as

$$h_\mu(T, Q) = \lim_{N \rightarrow \infty} \frac{1}{N} H \left(\bigvee_{n=0}^N T^{-n}Q \right).$$

Finally, the **Kolmogorov–Sinai** or **measure-theoretic entropy** of a dynamical system (X, \mathcal{B}, T, μ) is defined as

$$h_\mu(T) = \sup_Q h_\mu(T, Q).$$

where the supremum is taken over all finite measurable partitions. A theorem of Yakov G. Sinai in 1959 shows that the supremum is actually obtained on partitions that are generators. Thus, for example, the entropy of the Bernoulli process is $\log 2$, since every real number has a unique binary expansion. That is, one may partition the unit interval into the intervals $[0, 1/2)$ and $[1/2, 1]$. Every real number x is either less than $1/2$ or not; and likewise so is the fractional part of $2^n x$.

If the space X is compact and endowed with a topology, or is a metric space, then the topological entropy may also be defined.

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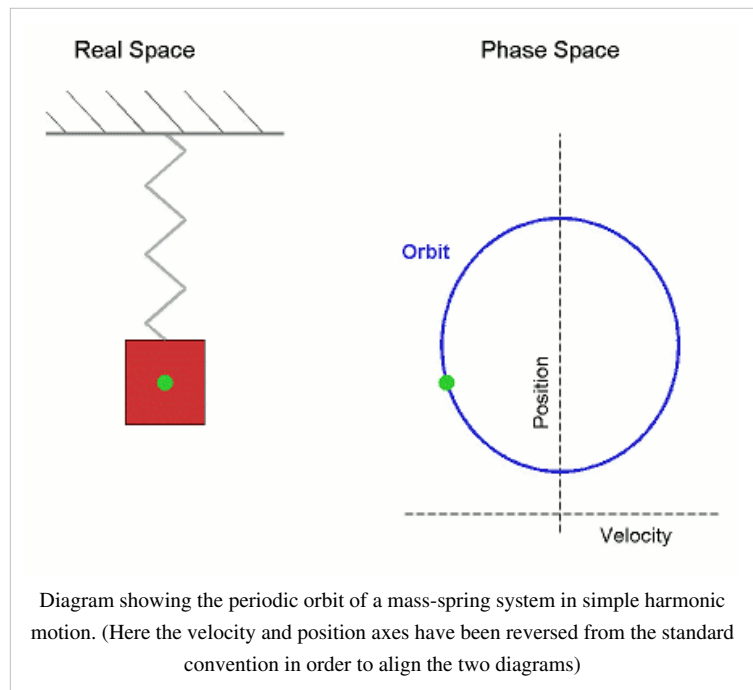
Periodic orbit

In mathematics, in the study of dynamical systems, an **orbit** is a collection of points related by the evolution function of the dynamical system. The orbit is a subset of the phase space and the set of all orbits is a partition of the phase space, that is different orbits do not intersect in the phase space. Understanding the properties of orbits by using topological method is one of the objectives of the modern theory of dynamical systems.

For discrete-time dynamical systems the orbits are sequences, for real dynamical systems the orbits are curves and for holomorphic dynamical systems the orbits are Riemann surfaces.

Definition

Given a dynamical system (T, M, Φ) with T a group, M a set and Φ the evolution function



$$\Phi : U \rightarrow M \text{ where } U \subset T \times M$$

we define

$$I(x) := \{t \in T : (t, x) \in U\},$$

then the set

$$\gamma_x := \{\Phi(t, x) : t \in I(x)\}$$

is called **orbit** through x . An orbit which consists of a single point is called **constant orbit**. A non-constant orbit is called **closed** or **periodic** if there exists a t in T so that

$$\Phi(t, x) = x$$

for every point x on the orbit.

Real dynamical system

Given a real dynamical system (R, M, Φ) , $I(x)$ is an open interval in the real numbers, that is $I(x) = (t_x^-, t_x^+)$. For any x in M

$$\gamma_x^+ := \{\Phi(t, x) : t \in (0, t_x^+)\}$$

is called **positive semi-orbit** through x and

$$\gamma_x^- := \{\Phi(t, x) : t \in (t_x^-, 0)\}$$

is called **negative semi-orbit** through x .

Discrete time dynamical system

For discrete time dynamical system :

forward orbit of x is a set :

$$\gamma_x^+ \stackrel{\text{def}}{=} \{\Phi(t, x) : t \geq 0\}$$

backward orbit of x is a set :

$$\gamma_x^- \stackrel{\text{def}}{=} \{\Phi(-t, x) : t \geq 0\}$$

and **orbit** of x is a set :

$$\gamma_x \stackrel{\text{def}}{=} \gamma_x^- \cup \gamma_x^+$$

where :

- Φ is an evolution function $\Phi : X \rightarrow X$ which is here an iterated function,
- set X is **dynamical space**,
- t is number of iteration, which is natural number and $t \in T$
- x is initial state of system and $x \in X$

Usually different notation is used :

- $\Phi(t, x)$ is noted as $\Phi^t(x)$
- $x_t = \Phi^t(x)$ with x_0 is a x from above notation.

Notes

It is often the case that the evolution function can be understood to compose the elements of a group, in which case the group-theoretic orbits of the group action are the same thing as the dynamical orbits.

Examples

- The orbit of an equilibrium point is a constant orbit

Stability of orbits

A basic classification of orbits is

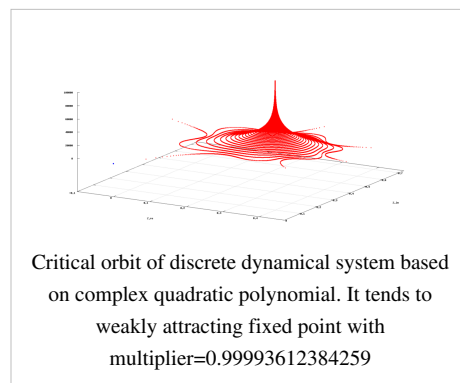
- constant orbits or fixed points
- periodic orbits
- non-constant and non-periodic orbits

An orbit can fail to be closed in two ways. It could be an **asymptotically periodic** orbit if it converges to a periodic orbit. Such orbits are not closed because they never truly repeat, but they become arbitrarily close to a repeating orbit. An orbit can also be chaotic. These orbits come arbitrarily close to the initial point, but fail to ever converge to a periodic orbit. They exhibit sensitive dependence on initial conditions, meaning that small differences in the initial value will cause large differences in future points of the orbit.

There are other properties of orbits that allow for different classifications. An orbit can be hyperbolic if nearby points approach or diverge from the orbit exponentially fast.

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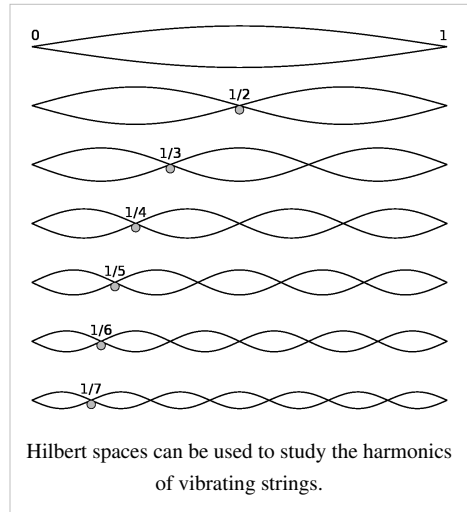


Hilbert space

The mathematical concept of a **Hilbert space**, named after David Hilbert, generalizes the notion of Euclidean space. It extends the methods of vector algebra and calculus from the two-dimensional Euclidean plane and three-dimensional space to spaces with any finite or infinite number of dimensions. A Hilbert space is an abstract vector space possessing the structure of an inner product that allows length and angle to be measured. Furthermore, Hilbert spaces are required to be complete, a property that stipulates the existence of enough limits in the space to allow the techniques of calculus to be used.

Hilbert spaces arise naturally and frequently in mathematics, physics, and engineering, typically as infinite-dimensional function spaces. The earliest Hilbert spaces were studied from this point of view in the first decade of the 20th century by David Hilbert, Erhard Schmidt, and Frigyes Riesz. They are indispensable tools in the theories of partial differential equations, quantum mechanics, Fourier analysis (which includes applications to signal processing and heat transfer) and ergodic theory which forms the mathematical underpinning of the study of thermodynamics. John von Neumann coined the term "Hilbert space" for the abstract concept underlying many of these diverse applications. The success of Hilbert space methods ushered in a very fruitful era for functional analysis. Apart from the classical Euclidean spaces, examples of Hilbert spaces include spaces of square-integrable functions, spaces of sequences, Sobolev spaces consisting of generalized functions, and Hardy spaces of holomorphic functions.

Geometric intuition plays an important role in many aspects of Hilbert space theory. Exact analogs of the Pythagorean theorem and parallelogram law hold in a Hilbert space. At a deeper level, perpendicular projection onto a subspace (the analog of "dropping the altitude" of a triangle) plays a significant role in optimization problems and other aspects of the theory. An element of a Hilbert space can be uniquely specified by its coordinates with respect to a set of coordinate axes (an orthonormal basis), in analogy with Cartesian coordinates in the plane. When that set of axes is countably infinite, this means that the Hilbert space can also usefully be thought of in terms of infinite sequences that are square-summable. Linear operators on a Hilbert space are likewise fairly concrete objects: in good cases, they are simply transformations that stretch the space by different factors in mutually perpendicular directions in a sense that is made precise by the study of their spectrum.



Definition and illustration

Motivating example: Euclidean space

One of the most familiar examples of a Hilbert space is the Euclidean space consisting of three-dimensional vectors, denoted by \mathbf{R}^3 , and equipped with the dot product. The dot product takes two vectors \mathbf{x} and \mathbf{y} , and produces a real number $\mathbf{x} \cdot \mathbf{y}$. If \mathbf{x} and \mathbf{y} are represented in Cartesian coordinates, then the dot product is defined by

$$(x_1, x_2, x_3) \cdot (y_1, y_2, y_3) = x_1 y_1 + x_2 y_2 + x_3 y_3.$$

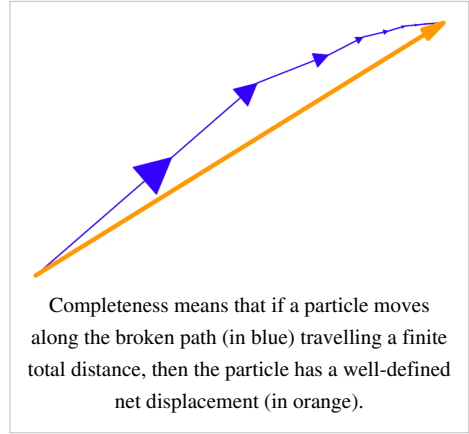
The dot product satisfies the properties:

1. It is symmetric in \mathbf{x} and \mathbf{y} : $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$.
2. It is linear in its first argument: $(a\mathbf{x}_1 + b\mathbf{x}_2) \cdot \mathbf{y} = a\mathbf{x}_1 \cdot \mathbf{y} + b\mathbf{x}_2 \cdot \mathbf{y}$ for any scalars a, b , and vectors $\mathbf{x}_1, \mathbf{x}_2$, and \mathbf{y} .
3. It is positive definite: for all vectors \mathbf{x} , $\mathbf{x} \cdot \mathbf{x} \geq 0$ with equality if and only if $\mathbf{x} = 0$.

An operation on pairs of vectors that, like the dot product, satisfies these three properties is known as a (real) inner product. A vector space equipped with such an inner product is known as a (real) inner product space. Every finite-dimensional inner product space is also a Hilbert space. The basic feature of the dot product that connects it with Euclidean geometry is that it is related to both the length (or norm) of a vector, denoted $\|\mathbf{x}\|$, and to the angle θ between two vectors \mathbf{x} and \mathbf{y} by means of the formula

$$\mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| \cos \theta.$$

Multivariable calculus in Euclidean space relies on the ability to compute limits, and to have useful criteria for concluding that limits exist. A mathematical series



$$\sum_{n=0}^{\infty} \mathbf{x}_n$$

consisting of vectors in \mathbf{R}^3 is absolutely convergent provided that the sum of the lengths converges as an ordinary series of real numbers:^[1]

$$\sum_{k=0}^{\infty} \|\mathbf{x}_k\| < \infty.$$

Just as with a series of scalars, a series of vectors that converges absolutely also converges to some limit vector \mathbf{L} in the Euclidean space, in the sense that

$$\left\| \mathbf{L} - \sum_{k=0}^N \mathbf{x}_k \right\| \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

This property expresses the *completeness* of Euclidean space: that a series which converges absolutely also converges in the ordinary sense.

Definition

A **Hilbert space** H is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.^[2] To say that H is a complex inner product space means that H is a complex vector space on which there is an inner product $\langle x, y \rangle$ associating a complex number to each pair of elements x, y of H that satisfies the following properties:

- $\langle y, x \rangle$ is the complex conjugate of $\langle x, y \rangle$:

$$\langle y, x \rangle = \overline{\langle x, y \rangle}.$$

- $\langle x, y \rangle$ is linear in its first argument.^[3] For all complex numbers a and b ,

$$\langle ax_1 + bx_2, y \rangle = a\langle x_1, y \rangle + b\langle x_2, y \rangle.$$

- The inner product $\langle \bullet, \bullet \rangle$ is positive definite:

$$\langle x, x \rangle \geq 0$$

where the case of equality holds precisely when $x = 0$.

It follows from properties 1 and 2 that a complex inner product is antilinear in its second argument, meaning that

$$\langle x, ay_1 + by_2 \rangle = \bar{a}\langle x, y_1 \rangle + \bar{b}\langle x, y_2 \rangle.$$

A real inner product space is defined in the same way, except that H is a real vector space and the inner product takes real values. Such an inner product will be bilinear: that is, linear in each argument.

The norm defined by the inner product $\langle \bullet, \bullet \rangle$ is the real-valued function

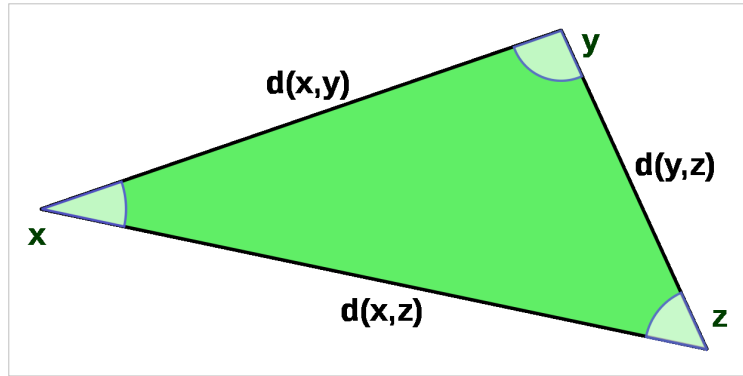
$$\|x\| = \sqrt{\langle x, x \rangle},$$

and the distance between two points x, y in H is defined in terms of the norm by

$$d(x, y) = \|x - y\| = \sqrt{\langle x - y, x - y \rangle}.$$

That this function is a distance function means (1) that it is symmetric in x and y , (2) that the distance between x and itself is zero, and otherwise the distance between x and y must be positive, and (3) that the triangle inequality holds, meaning that the length of one leg of a triangle xyz cannot exceed the sum of the lengths of the other two legs:

$$d(x, z) \leq d(x, y) + d(y, z).$$



This last property is ultimately a consequence of the more fundamental Cauchy–Schwarz inequality, which asserts

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

with equality if and only if x and y are linearly dependent.

Relative to a distance function defined in this way, any inner product space is a metric space, and sometimes is known as a **pre-Hilbert space**.^[4] A pre-Hilbert space is a Hilbert space if in addition it is complete. Completeness is expressed using a form of the Cauchy criterion for sequences in H : a pre-Hilbert space H is complete if every Cauchy sequence converges with respect to this norm to an element in the space. Completeness can be characterized by the following equivalent condition: if a series of vectors $\sum_{k=0}^{\infty} u_k$ converges absolutely in the sense that

$$\sum_{k=0}^{\infty} \|u_k\| < \infty,$$

then the series converges in H , in the sense that the partial sums converge to an element of H .

As a complete normed space, Hilbert spaces are by definition also Banach spaces. As such they are topological vector spaces, in which topological notions like the openness and closedness of subsets are well-defined. Of special importance is the notion of a closed linear subspace of a Hilbert space which, with the inner product induced by restriction, is also complete (being a closed set in a complete metric space) and therefore a Hilbert space in its own right.

Second example: sequence spaces

The sequence space ℓ^2 consists of all infinite sequences $\mathbf{z} = (z_1, z_2, \dots)$ of complex numbers such that the series

$$\sum_{n=1}^{\infty} |z_n|^2$$

converges. The inner product on ℓ^2 is defined by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \sum_{n=1}^{\infty} z_n \overline{w_n},$$

with the latter series converging as a consequence of the Cauchy–Schwarz inequality.

Completeness of the space holds provided that whenever a series of elements from ℓ^2 converges absolutely (in norm), then it converges to an element of ℓ^2 . The proof is basic in mathematical analysis, and permits mathematical series of elements of the space to be manipulated with the same ease as series of complex numbers (or vectors in a finite-dimensional Euclidean space).^[5]

History

Prior to the development of Hilbert spaces, other generalizations of Euclidean spaces were known to mathematicians and physicists. In particular, the idea of an abstract linear space had gained some traction towards the end of the 19th century:^[6] this is a space whose elements can be added together and multiplied by scalars (such as real or complex numbers) without necessarily identifying these elements with "geometric" vectors, such as position and momentum vectors in physical systems. Other objects studied by mathematicians at the turn of the 20th century, in particular spaces of sequences (including series) and spaces of functions,^[7] can naturally be thought of as linear spaces. Functions, for instance, can be added together or multiplied by constant scalars, and these operations obey the algebraic laws satisfied by addition and scalar multiplication of spatial vectors.

In the first decade of the 20th century, parallel developments led to the introduction of Hilbert spaces. The first of these was the observation, which arose during David Hilbert and Erhard Schmidt's study of integral equations,^[8] that two square-integrable real-valued functions f and g on an interval $[a, b]$ have an *inner product*

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx$$

which has many of the familiar properties of the Euclidean dot product. In particular, the idea of an orthogonal family of functions has meaning. Schmidt exploited the similarity of this inner product with the usual dot product to prove an analog of the spectral decomposition for an operator of the form

$$f(x) \mapsto \int_a^b K(x, y)f(y) dy$$

where K is a continuous function symmetric in x and y . The resulting eigenfunction expansion expresses the function K as a series of the form

$$K(x, y) = \sum_n \lambda_n \varphi_n(x) \varphi_n(y)$$



David Hilbert

where the functions φ_n are orthogonal in the sense that $\langle \varphi_n, \varphi_m \rangle = 0$ for all $n \neq m$. The individual terms in this series are sometimes referred to as elementary product solutions. However, there are eigenfunction expansions which fail to converge in a suitable sense to a square-integrable function: the missing ingredient, which ensures convergence, is completeness.^[9]

The second development was the Lebesgue integral, an alternative to the Riemann integral introduced by Henri Lebesgue in 1904.^[10] The Lebesgue integral made it possible to integrate a much broader class of functions. In 1907, Frigyes Riesz and Ernst Sigismund Fischer independently proved that the space L^2 of square Lebesgue-integrable functions is a complete metric space.^[11] As a consequence of the interplay between geometry and completeness, the 19th century results of Joseph Fourier, Friedrich Bessel and Marc-Antoine Parseval on trigonometric series easily carried over to these more general spaces, resulting in a geometrical and analytical apparatus now usually known as the Riesz-Fischer theorem.^[12]

Further basic results were proved in the early 20th century. For example, the Riesz representation theorem was independently established by Maurice Fréchet and Frigyes Riesz in 1907.^[13] John von Neumann coined the term *abstract Hilbert space* in his work on unbounded Hermitian operators.^[14] Although other mathematicians such as Hermann Weyl and Norbert Wiener had already studied particular Hilbert spaces in great detail, often from a physically motivated point of view, von Neumann gave the first complete and axiomatic treatment of them.^[15] Von Neumann later used them in his seminal work on the foundations of quantum mechanics,^[16] and in his continued work with Eugene Wigner. The name "Hilbert space" was soon adopted by others, for example by Hermann Weyl in his book on quantum mechanics and the theory of groups.^[17]

The significance of the concept of a Hilbert space was underlined with the realization that it offers one of the best mathematical formulations of quantum mechanics.^[18] In short, the states of a quantum mechanical system are vectors in a certain Hilbert space, the observables are hermitian operators on that space, the symmetries of the system are unitary operators, and measurements are orthogonal projections. The relation between quantum mechanical symmetries and unitary operators provided an impetus for the development of the unitary representation theory of groups, initiated in the 1928 work of Hermann Weyl.^[17] On the other hand, in the early 1930s it became clear that certain properties of classical dynamical systems can be analyzed using Hilbert space techniques in the framework of ergodic theory.^[19]

The algebra of observables in quantum mechanics is naturally an algebra of operators defined on a Hilbert space, according to Werner Heisenberg's matrix mechanics formulation of quantum theory. Von Neumann began investigating operator algebras in the 1930s, as rings of operators on a Hilbert space. The kind of algebras studied by von Neumann and his contemporaries are now known as von Neumann algebras. In the 1940s, Israel Gelfand, Mark Naimark and Irving Segal gave a definition of a kind of operator algebras called C^* -algebras that on the one hand made no reference to an underlying Hilbert space, and on the other extrapolated many of the useful features of the operator algebras that had previously been studied. The spectral theorem for self-adjoint operators in particular that underlies much of the existing Hilbert space theory was generalized to C^* -algebras. These techniques are now basic in abstract harmonic analysis and representation theory.

Examples

Lebesgue spaces

Lebesgue spaces are function spaces associated to measure spaces (X, M, μ) , where X is a set, M is a σ -algebra of subsets of X , and μ is a countably additive measure on M . Let $L^2(X, \mu)$ be the space of those complex-valued measurable functions on X for which the Lebesgue integral of the square of the absolute value of the function is finite, i.e., for a function f in $L^2(X, \mu)$,

$$\int_X |f|^2 d\mu < \infty,$$

and where functions are identified if and only if they differ only on a set of measure zero.

The inner product of functions f and g in $L^2(X, \mu)$ is then defined as

$$\langle f, g \rangle = \int_X f(t) \overline{g(t)} d\mu(t).$$

For f and g in L^2 , this integral exists because of the Cauchy–Schwarz inequality, and defines an inner product on the space. Equipped with this inner product, L^2 is in fact complete.^[20] The Lebesgue integral is essential to ensure completeness: on domains of real numbers, for instance, not enough functions are Riemann integrable.^[21]

The Lebesgue spaces appear in many natural settings. The spaces $L^2(\mathbf{R})$ and $L^2([0,1])$ of square-integrable functions with respect to the Lebesgue measure on the real line and unit interval, respectively, are natural domains on which to define the Fourier transform and Fourier series. In other situations, the measure may be something other than the ordinary Lebesgue measure on the real line. For instance, if w is any positive measurable function, the space of all measurable functions f on the interval $[0,1]$ satisfying

$$\int_0^1 |f(t)|^2 w(t) dt < \infty$$

is called the weighted L^2 space $L^2_{w, \mu}([0,1])$, and w is called the weight function. The inner product is defined by

$$\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} w(t) dt.$$

The weighted space $L^2_{w, \mu}([0,1])$ is identical with the Hilbert space $L^2([0,1], \mu)$ where the measure μ of a Lebesgue-measurable set A is defined by

$$\mu(A) = \int_A w(t) dt.$$

Weighted L^2 spaces like this are frequently used to study orthogonal polynomials, because different families of orthogonal polynomials are orthogonal with respect to different weighting functions.

Sobolev spaces

Sobolev spaces, denoted by H^s or $W^{s,2}$, are Hilbert spaces. These are a special kind of function space in which differentiation may be performed, but which (unlike other Banach spaces such as the Hölder spaces) support the structure of an inner product. Because differentiation is permitted, Sobolev spaces are a convenient setting for the theory of partial differential equations.^[22] They also form the basis of the theory of direct methods in the calculus of variations.^[23]

For s a non-negative integer and $\Omega \subset \mathbf{R}^n$, the Sobolev space $H^s(\Omega)$ contains L^2 functions whose weak derivatives of order up to s are also L^2 . The inner product in $H^s(\Omega)$ is

$$\langle f, g \rangle = \int_{\Omega} f(x) \overline{g(x)} dx + \int_{\Omega} Df \cdot D\overline{g}(x) dx + \cdots + \int_{\Omega} D^s f(x) \cdot D^s \overline{g}(x) dx$$

where the dot indicates the dot product in the Euclidean space of partial derivatives of each order. Sobolev spaces can also be defined when s is not an integer.

Sobolev spaces are also studied from the point of view of spectral theory, relying more specifically on the Hilbert space structure. If Ω is a suitable domain, then one can define the Sobolev space $H^s(\Omega)$ as the space of Bessel potentials;^[24] roughly,

$$H^s(\Omega) = \{(1 - \Delta)^{-s/2} f \mid f \in L^2(\Omega)\}.$$

Here Δ is the Laplacian and $(1 - \Delta)^{-s/2}$ is understood in terms of the spectral mapping theorem. Apart from providing a workable definition of Sobolev spaces for non-integer s , this definition also has particularly desirable properties under the Fourier transform that make it ideal for the study of pseudodifferential operators. Using these methods on a compact Riemannian manifold, one can obtain for instance the Hodge decomposition which is the basis of Hodge theory.^[25]

Spaces of holomorphic functions

Hardy spaces

The Hardy spaces are function spaces, arising in complex analysis and harmonic analysis, whose elements are certain holomorphic functions in a complex domain.^[26] Let U denote the unit disc in the complex plane. Then the Hardy space $H^2(U)$ is defined to be the space of holomorphic functions f on U such that the means

$$M_r(f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta$$

remain bounded for $r < 1$. The norm on this Hardy space is defined by

$$\|f\|_2 = \lim_{r \rightarrow 1} \sqrt{M_r(f)}.$$

Hardy spaces in the disc are related to Fourier series. A function f is in $H^2(U)$ if and only if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

Thus $H^2(U)$ consists of those functions which are L^2 on the circle, and whose negative frequency Fourier coefficients vanish.

Bergman spaces

The Bergman spaces are another family of Hilbert spaces of holomorphic functions.^[27] Let D be a bounded open set in the complex plane (or a higher dimensional complex space) and let $L^{2,h}(D)$ be the space of holomorphic functions f in D that are also in $L^2(D)$ in the sense that

$$\|f\|^2 = \int_D |f(z)|^2 d\mu(z) < \infty,$$

where the integral is taken with respect to the Lebesgue measure in D . Clearly $L^{2,h}(D)$ is a subspace of $L^2(D)$; in fact, it is a closed subspace, and so a Hilbert space in its own right. This is a consequence of the estimate, valid on compact subsets K of D , that

$$\sup_{z \in K} |f(z)| \leq C_K \|f\|_2,$$

which in turn follows from Cauchy's integral formula. Thus convergence of a sequence of holomorphic functions in $L^2(D)$ implies also compact convergence, and so the limit function is also holomorphic. Another consequence of this inequality is that the linear functional that evaluates a function f at a point of D is actually continuous on $L^{2,h}(D)$. The Riesz representation theorem implies that the evaluation functional can be represented as an element of $L^{2,h}(D)$. Thus, for every $z \in D$, there is a function $\eta_z \in L^{2,h}(D)$ such that

$$f(z) = \int_D f(\zeta) \overline{\eta_z(\zeta)} d\mu(\zeta)$$

for all $f \in L^{2,h}(D)$. The integrand

$$K(\zeta, z) = \overline{\eta_z(\zeta)}$$

is known as the Bergman kernel of D . This integral kernel satisfies a reproducing property

$$f(z) = \int_D f(\zeta) K(\zeta, z) d\mu(\zeta).$$

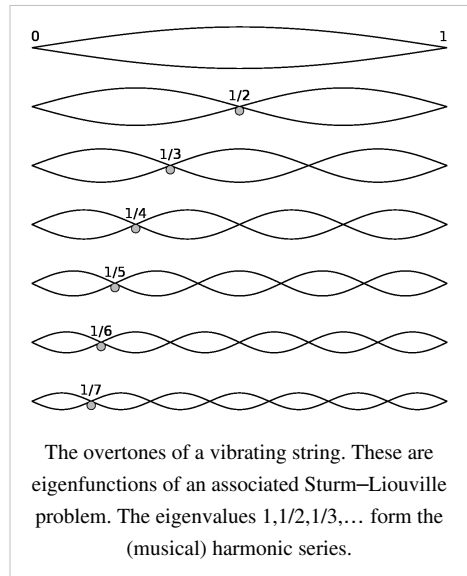
A Bergman space is an example of a reproducing kernel Hilbert space, which is a Hilbert space of functions along with a kernel $K(\zeta, z)$ that verifies a reproducing property analogous to this one. The Hardy space $H^2(D)$ also admits a reproducing kernel, known as the Szegő kernel.^[28] Reproducing kernels are common in other areas of mathematics as well. For instance, in harmonic analysis the Poisson kernel is a reproducing kernel for the Hilbert space of square-integrable harmonic functions in the unit ball. That the latter is a Hilbert space at all is a consequence of the mean value theorem for harmonic functions.

Applications

Many of the applications of Hilbert spaces exploit the fact that Hilbert spaces support generalizations of simple geometric concepts like projection and change of basis from their usual finite dimensional setting. In particular, the spectral theory of continuous self-adjoint linear operators on a Hilbert space generalizes the usual spectral decomposition of a matrix, and this often plays a major role in applications of the theory to other areas of mathematics and physics.

Sturm–Liouville theory

In the theory of ordinary differential equations, spectral methods on a suitable Hilbert space are used to study the behavior of eigenvalues and eigenfunctions of differential equations. For example, the Sturm–Liouville problem arises in the study of the harmonics of waves in a violin string or a drum, and is a central problem in ordinary differential equations.^[29] The problem is a differential equation of the form



$$-\frac{d}{dx} \left[p(x) \frac{dy}{dx} \right] + q(x)y = \lambda w(x)y$$

for an unknown function y on an interval $[a, b]$, satisfying general homogeneous Robin boundary conditions

$$\begin{cases} \alpha y(a) + \alpha' y'(a) = 0 \\ \beta y(b) + \beta' y'(b) = 0. \end{cases}$$

The functions p , q , and w are given in advance, and the problem is to find the function y and constants λ for which the equation has a solution. The problem only has solutions for certain values of λ , called eigenvalues of the system,

and this is a consequence of the spectral theorem for compact operators applied to the integral operator defined by the Green's function for the system. Furthermore, another consequence of this general result is that the eigenvalues λ of the system can be arranged in an increasing sequence tending to infinity.^[30]

Partial differential equations

Hilbert spaces form a basic tool in the study of partial differential equations.^[22] For many classes of partial differential equations, such as linear elliptic equations, it is possible to consider a generalized solution (known as a weak solution) by enlarging the class of functions. Many weak formulations involve the class of Sobolev functions, which is a Hilbert space. A suitable weak formulation reduces to a geometrical problem the analytic problem of finding a solution or, often what is more important, showing that a solution exists and is unique for given boundary data. For linear elliptic equations, one geometrical result that ensures unique solvability for a large class of problems is the Lax–Milgram theorem. This strategy forms the rudiment of the Galerkin method (a finite element method) for numerical solution of partial differential equations.^[31]

A typical example is the Poisson equation $-\Delta u = g$ with Dirichlet boundary conditions in a bounded domain Ω in \mathbf{R}^2 . The weak formulation consists of finding a function u such that, for all continuously differentiable functions v in Ω vanishing on the boundary:

$$\int_{\Omega} \nabla u \cdot \nabla v = \int_{\Omega} g v.$$

This can be recast in terms of the Hilbert space $H_0^1(\Omega)$ consisting of functions u such that u , along with its weak partial derivatives, are square integrable on Ω , and which vanish on the boundary. The question then reduces to finding u in this space such that for all v in this space

$$a(u, v) = b(v)$$

where a is a continuous bilinear form, and b is a continuous linear functional, given respectively by

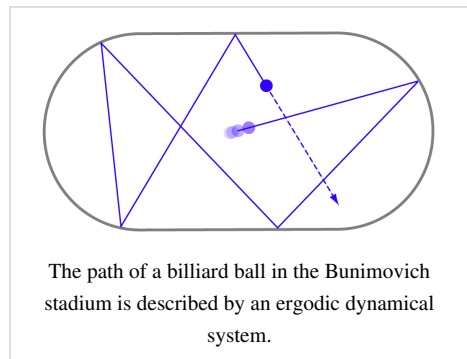
$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v, \quad b(v) = \int_{\Omega} g v.$$

Since the Poisson equation is elliptic, it follows from Poincaré's inequality that the bilinear form a is coercive. The Lax–Milgram theorem then ensures the existence and uniqueness of solutions of this equation.

Hilbert spaces allow for many elliptic partial differential equations to be formulated in a similar way, and the Lax–Milgram theorem is then a basic tool in their analysis. With suitable modifications, similar techniques can be applied to parabolic partial differential equations and certain hyperbolic partial differential equations.

Ergodic theory

The field of ergodic theory is the study of the long-term behavior of chaotic dynamical systems. The prototypical case of a field to which ergodic theory is applicable is that of thermodynamics in which, although the microscopic state of a system is extremely complicated—it is impossible to understand the ensemble of individual collisions between particles of matter—the average behavior over sufficiently long time intervals is tractable. The laws of thermodynamics are assertions about such average behavior. In particular, one formulation of the zeroth law of thermodynamics asserts that over sufficiently long timescales, the only functionally independent measurement that one can make of a thermodynamic system in equilibrium is its total energy, in the form of temperature.



An ergodic dynamical system is one for which, apart from the energy—measured by the Hamiltonian—there are no other functionally independent conserved quantities on the phase space. More explicitly, suppose that the energy E is fixed, and let Ω_E be the subset of the phase space consisting of all states of energy E (an energy surface), and let T_t denote the evolution operator on the phase space. The dynamical system is ergodic if there are no continuous non-constant functions on Ω_E such that

$$f(T_t w) = f(w)$$

for all w on Ω_E and all time t . Liouville's theorem implies that there exists a measure μ on the energy surface that is invariant under the time translation. As a result, time translation is a unitary transformation of the Hilbert space $L^2(\Omega_E, \mu)$ consisting of square-integrable functions on the energy surface Ω_E with respect to the inner product

$$\langle f, g \rangle_{L^2(\Omega_E, \mu)} = \int_E f \bar{g} d\mu.$$

The von Neumann mean ergodic theorem^[19] states the following:

- If U_t is a (strongly continuous) one-parameter semigroup of unitary operators on a Hilbert space H , and P is the orthogonal projection onto the space of common fixed points of U_t , $\{x \in H \mid U_t x = x \text{ for all } t > 0\}$, then

$$Px = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T U_t x dt.$$

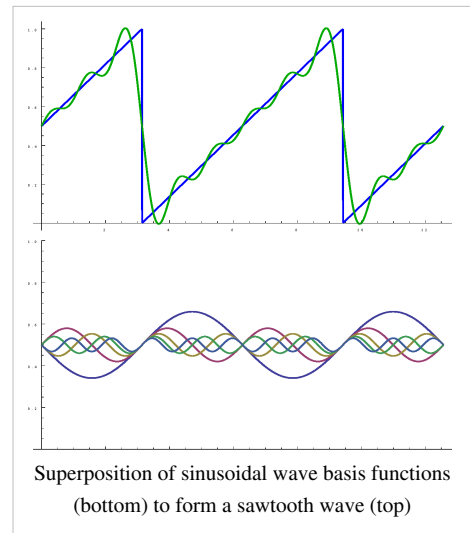
For an ergodic system, the fixed set of the time evolution consists only of the constant functions, so the ergodic theorem implies the following:^[32] for any function $f \in L^2(\Omega_E, \mu)$,

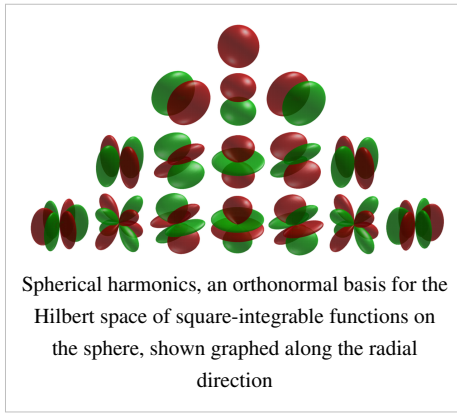
$$L^2\text{-}\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(T_t w) dt = \int_{\Omega_E} f(y) d\mu(y).$$

That is, the long time average of an observable f is equal to its expectation value over an energy surface.

Fourier analysis

One of the basic goals of Fourier analysis is to decompose a function into a (possibly infinite) linear combination of given basis functions: the associated Fourier series. The classical Fourier series associated to a function f defined on the interval $[0,1]$ is a series of the form





$$\sum_{n=-\infty}^{\infty} a_n e^{2\pi i n \theta}$$

where

$$a_n = \int_0^1 f(\theta) e^{-2\pi i n \theta} d\theta.$$

The example of adding up the first few terms in a Fourier series for a sawtooth function is shown in the figure. The basis functions are sine waves with wavelengths λ/n (n =integer) shorter than the wavelength λ of the sawtooth itself (except for $n=1$, the *fundamental* wave). All basis functions have nodes at the nodes of the sawtooth, but all but the fundamental have additional nodes. The oscillation of the summed terms about the sawtooth is called the Gibbs phenomenon.

A significant problem in classical Fourier series asks in what sense the Fourier series converges, if at all, to the function f . Hilbert space methods provide one possible answer to this question.^[33] The functions $e_n(\theta) = e^{2\pi i n \theta}$ form an orthogonal basis of the Hilbert space $L^2([0,1])$. Consequently, any square-integrable function can be expressed as a series

$$f(\theta) = \sum_n a_n e_n(\theta), \quad a_n = \langle f, e_n \rangle$$

and, moreover, this series converges in the Hilbert space sense (that is, in the L^2 mean).

The problem can also be studied from the abstract point of view: every Hilbert space has an orthonormal basis, and every element of the Hilbert space can be written in a unique way as a sum of multiples of these basis elements. The coefficients appearing on these basis elements are sometimes known abstractly as the Fourier coefficients of the element of the space.^[34] The abstraction is especially useful when it is more natural to use different basis functions for a space such as $L^2([0,1])$. In many circumstances, it is desirable not to decompose a function into trigonometric functions, but rather into orthogonal polynomials or wavelets for instance,^[35] and in higher dimensions into spherical harmonics.^[36]

For instance, if e_n are any orthonormal basis functions of $L^2[0,1]$, then a given function in $L^2[0,1]$ can be approximated as a finite linear combination^[37]

$$f(x) \approx f_n(x) = a_1 e_1(x) + a_2 e_2(x) + \cdots + a_n e_n(x)$$

The coefficients $\{a_j\}$ are selected to make the magnitude of the difference $\|f - f_n\|^2$ as small as possible. Geometrically, the best approximation is the orthogonal projection of f onto the subspace consisting of all linear combinations of the $\{e_j\}$, and can be calculated by^[38]

$$a_j = \int_0^1 \overline{e_j(x)} f(x) dx.$$

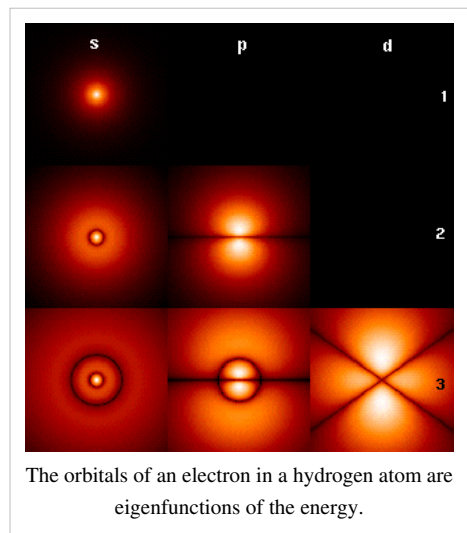
That this formula minimizes the difference $\|f - f_n\|^2$ is a consequence of Bessel's inequality and Parseval's formula.

In various applications to physical problems, a function can be decomposed into physically meaningful eigenfunctions of a differential operator (typically the Laplace operator): this forms the foundation for the spectral study of functions, in reference to the spectrum of the differential operator.^[39] A concrete physical application involves the problem of hearing the shape of a drum: given the fundamental modes of vibration that a drumhead is capable of producing, can one infer the shape of the drum itself?^[40] The mathematical formulation of this question involves the Dirichlet eigenvalues of the Laplace equation in the plane, that represent the fundamental modes of vibration in direct analogy with the integers that represent the fundamental modes of vibration of the violin string.

Spectral theory also underlies certain aspects of the Fourier transform of a function. Whereas Fourier analysis decomposes a function defined on a compact set into the discrete spectrum of the Laplacian (which corresponds to the vibrations of a violin string or drum), the Fourier transform of a function is the decomposition of a function defined on all of Euclidean space into its components in the continuous spectrum of the Laplacian. The Fourier transformation is also geometrical, in a sense made precise by the Plancherel theorem, that asserts that it is an isometry of one Hilbert space (the "time domain") with another (the "frequency domain"). This isometry property of the Fourier transformation is a recurring theme in abstract harmonic analysis, as evidenced for instance by the Plancherel theorem for spherical functions occurring in noncommutative harmonic analysis.

Quantum mechanics

In the mathematically rigorous formulation of quantum mechanics, developed by Paul Dirac^[41] and John von Neumann^[42], the possible states (more precisely, the pure states) of a quantum mechanical system are represented by unit vectors (called *state vectors*) residing in a complex separable Hilbert space, known as the state space, well defined up to a complex number of norm 1 (the phase factor). In other words, the possible states are points in the projectivization of a Hilbert space, usually called the complex projective space. The exact nature of this Hilbert space is dependent on the system; for example, the position and momentum states for a single non-relativistic spin zero particle is the space of all square-integrable functions, while the states for the spin of a single proton are unit elements of the two-dimensional complex Hilbert space of spinors. Each observable is represented by a self-adjoint linear operator acting on the state space. Each eigenstate of an observable corresponds to an eigenvector of the operator, and the associated eigenvalue corresponds to the value of the observable in that eigenstate.



The orbitals of an electron in a hydrogen atom are eigenfunctions of the energy.

The time evolution of a quantum state is described by the Schrödinger equation, in which the Hamiltonian, the operator corresponding to the total energy of the system, generates time evolution.

The inner product between two state vectors is a complex number known as a probability amplitude. During an ideal measurement of a quantum mechanical system, the probability that a system collapses from a given initial state to a particular eigenstate is given by the square of the absolute value of the probability amplitudes between the initial and final states. The possible results of a measurement are the eigenvalues of the operator—which explains the choice of self-adjoint operators, for all the eigenvalues must be real. The probability distribution of an observable in a given state can be found by computing the spectral decomposition of the corresponding operator.

For a general system, states are typically not pure, but instead are represented as statistical mixtures of pure states, or mixed states, given by density matrices: self-adjoint operators of trace one on a Hilbert space. Moreover, for general quantum mechanical systems, the effects of a single measurement can influence other parts of a system in a manner that is described instead by a positive operator valued measure. Thus the structure both of the states and observables

in the general theory is considerably more complicated than the idealization for pure states.

Heisenberg's uncertainty principle is represented by the statement that the operators corresponding to certain observables do not commute, and gives a specific form that the commutator must have.

Properties

Pythagorean identity

Two vectors u and v in a Hilbert space H are orthogonal when $\langle u, v \rangle = 0$. The notation for this is $u \perp v$. More generally, when S is a subset in H , the notation $u \perp S$ means that u is orthogonal to every element from S .

When u and v are orthogonal, one has

$$\|u + v\|^2 = \langle u + v, u + v \rangle = \langle u, u \rangle + 2\operatorname{Re}\langle u, v \rangle + \langle v, v \rangle = \|u\|^2 + \|v\|^2.$$

By induction on n , this is extended to any family u_1, \dots, u_n of n orthogonal vectors,

$$\|u_1 + \dots + u_n\|^2 = \|u_1\|^2 + \dots + \|u_n\|^2.$$

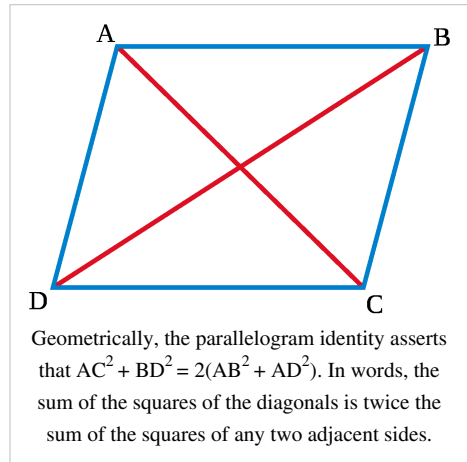
Whereas the Pythagorean identity as stated is valid in any inner product space, completeness is required for the extension of the Pythagorean identity to series. A series $\sum u_k$ of *orthogonal* vectors converges in H if and only if the series of squares of norms converges, and

$$\left\| \sum_{k=0}^{\infty} u_k \right\|^2 = \sum_{k=0}^{\infty} \|u_k\|^2.$$

Furthermore, the sum of a series of orthogonal vectors is independent of the order in which it is taken.

Parallelogram identity and polarization

By definition, every Hilbert space is also a Banach space. Furthermore, in every Hilbert space the following parallelogram identity holds:



$$\|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

Conversely, every Banach space in which the parallelogram identity holds is a Hilbert space, and the inner product is uniquely determined by the norm by the polarization identity.^[43] For real Hilbert spaces, the polarization identity is

$$\langle u, v \rangle = \frac{1}{4} (\|u + v\|^2 - \|u - v\|^2).$$

For complex Hilbert spaces, it is

$$\langle u, v \rangle = \frac{1}{4} (\|u + v\|^2 - \|u - v\|^2 + i\|u + iv\|^2 - i\|u - iv\|^2).$$

The parallelogram law implies that any Hilbert space is a uniformly convex Banach space.^[44]

Best approximation

If C is a non-empty closed convex subset of a Hilbert space H and x a point in H , there exists a unique point $y \in C$ which minimizes the distance between x and points in C ,^[45]

$$y \in C, \quad \|x - y\| = \text{dist}(x, C) = \min\{\|x - z\| : z \in C\}.$$

This is equivalent to saying that there is a point with minimal norm in the translated convex set $D = C - x$. The proof consists in showing that every minimizing sequence $(d_n) \subset D$ is Cauchy (using the parallelogram identity) hence converges (using completeness) to a point in D that has minimal norm. More generally, this holds in any uniformly convex Banach space.^[46]

When this result is applied to a closed subspace F of H , it can be shown that the point $y \in F$ closest to x is characterized by^[47]

$$y \in F, \quad x - y \perp F.$$

This point y is the *orthogonal projection* of x onto F , and the mapping $P_F : x \rightarrow y$ is linear (see Orthogonal complements and projections). This result is especially significant in applied mathematics, especially numerical analysis, where it forms the basis of least squares methods.

In particular, when F is not equal to H , one can find a non-zero vector v orthogonal to F (select x not in F and $v = x - y$). A very useful criterion is obtained by applying this observation to the closed subspace F generated by a subset S of H .

A subset S of H spans a dense vector subspace if (and only if) the vector 0 is the sole vector $v \in H$ orthogonal to S .

Duality

The dual space H^* is the space of all continuous linear functions from the space H into the base field. It carries a natural norm, defined by

$$\|\varphi\| = \sup_{\|x\|=1, x \in H} |\varphi(x)|.$$

This norm satisfies the parallelogram law, and so the dual space is also an inner product space. The dual space is also complete, and so it is a Hilbert space in its own right.

The Riesz representation theorem affords a convenient description of the dual. To every element u of H , there is a unique element φ_u of H^* , defined by

$$\varphi_u(x) = \langle x, u \rangle.$$

The mapping $u \mapsto \varphi_u$ is an antilinear mapping from H to H^* . The Riesz representation theorem states that this mapping is an antilinear isomorphism.^[48] Thus to every element φ of the dual H^* there exists one and only one u_φ in H such that

$$\langle x, u_\varphi \rangle = \varphi(x)$$

for all $x \in H$. The inner product on the dual space H^* satisfies

$$\langle \varphi, \psi \rangle = \langle u_\psi, u_\varphi \rangle.$$

The reversal of order on the right-hand side restores linearity in φ from the antilinearity of u_φ . In the real case, the antilinear isomorphism from H to its dual is actually an isomorphism, and so real Hilbert spaces are naturally isomorphic to their own duals.

The representing vector u_φ is obtained in the following way. When $\varphi \neq 0$, the kernel $F = \ker \varphi$ is a closed vector subspace of H , not equal to H , hence there exists a non-zero vector v orthogonal to F . The vector u is a suitable scalar multiple λv of v . The requirement that $\varphi(v) = \langle v, u \rangle$ yields

$$u = \langle v, v \rangle^{-1} \overline{\varphi(v)} v.$$

This correspondence $\varphi \leftrightarrow u$ is exploited by the bra-ket notation popular in physics. It is common in physics to assume that the inner product, denoted by $\langle x|y\rangle$, is linear on the right,

$$\langle x|y\rangle = \langle y, x\rangle.$$

The result $\langle x|y\rangle$ can be seen as the action of the linear functional $\langle x|$ (the *bra*) on the vector $|y\rangle$ (the *ket*).

The Riesz representation theorem relies fundamentally not just on the presence of an inner product, but also on the completeness of the space. In fact, the theorem implies that the topological dual of any inner product space can be identified with its completion. An immediate consequence of the Riesz representation theorem is also that a Hilbert space H is reflexive, meaning that the natural map from H into its double dual space is an isomorphism.

Weakly convergent sequences

In a Hilbert space H , a sequence $\{x_n\}$ is weakly convergent to a vector $x \in H$ when

$$\lim_n \langle x_n, v \rangle = \langle x, v \rangle$$

for every $v \in H$.

For example, any orthonormal sequence $\{f_n\}$ converges weakly to 0, as a consequence of Bessel's inequality. Every weakly convergent sequence $\{x_n\}$ is bounded, by the uniform boundedness principle.

Conversely, every bounded sequence in a Hilbert space admits weakly convergent subsequences (Alaoglu's theorem).^[49] This fact may be used to prove minimization results for continuous convex functionals, in the same way that the Bolzano-Weierstrass theorem is used for continuous functions on \mathbf{R}^d . Among several variants, one simple statement is as follows:^[50]

If $f: H \rightarrow \mathbf{R}$ is a convex continuous function such that $f(x)$ tends to $+\infty$ when $\|x\|$ tends to ∞ , then f admits a minimum at some point $x_0 \in H$.

This fact (and its various generalizations) are fundamental for direct methods in the calculus of variations. Minimization results for convex functionals are also a direct consequence of the slightly more abstract fact that closed bounded convex subsets in a Hilbert space H are weakly compact, since H is reflexive. The existence of weakly convergent subsequences is a special case of the Eberlein-Šmulian theorem.

Banach space properties

Any general property of Banach spaces continues to hold for Hilbert spaces. The open mapping theorem states that a continuous surjective linear transformation from one Banach space to another is an open mapping meaning that it sends open sets to open sets. A corollary is the bounded inverse theorem, that a continuous and bijective linear function from one Banach space to another is an isomorphism (that is, a continuous linear map whose inverse is also continuous). This theorem is considerably simpler to prove in the case of Hilbert spaces than in general Banach spaces.^[51] The open mapping theorem is equivalent to the closed graph theorem, which asserts that a function from one Banach space to another is continuous if and only if its graph is a closed set.^[52] In the case of Hilbert spaces, this is basic in the study of unbounded operators (see closed operator).

The (geometrical) Hahn-Banach theorem asserts that a closed convex set can be separated from any point outside it by means of a hyperplane of the Hilbert space. This is an immediate consequence of the best approximation property: if y is the element of a closed convex set F closest to x , then the separating hyperplane is the plane perpendicular to the segment xy passing through its midpoint.^[53]

Operators on Hilbert spaces

Bounded operators

The continuous linear operators $A : H_1 \rightarrow H_2$ from a Hilbert space H_1 to a second Hilbert space H_2 are *bounded* in the sense that they map bounded sets to bounded sets. Conversely, if an operator is bounded, then it is continuous. The space of such bounded linear operators has a norm, the operator norm given by

$$\|A\| = \sup \{ \|Ax\| : \|x\| \leq 1 \}.$$

The sum and the composite of two bounded linear operators is again bounded and linear. For y in H_2 , the map that sends $x \in H_1$ to $\langle Ax, y \rangle$ is linear and continuous, and according to the Riesz representation theorem can therefore be represented in the form

$$\langle x, A^*y \rangle = \langle Ax, y \rangle$$

for some vector A^*y in H_1 . This defines another bounded linear operator $A^* : H_2 \rightarrow H_1$, the adjoint of A . One can see that $A^{**} = A$.

The set $B(H)$ of all bounded linear operators on H , together with the addition and composition operations, the norm and the adjoint operation, is a C^* -algebra, which is a type of operator algebra.

An element A of $B(H)$ is called *self-adjoint* or *Hermitian* if $A^* = A$. If A is Hermitian and $\langle Ax, x \rangle \geq 0$ for every x , then A is called *non-negative*, written $A \geq 0$; if equality holds only when $x = 0$, then A is called *positive*. The set of self adjoint operators admits a partial order, in which $A \geq B$ if $A - B \geq 0$. If A has the form B^*B for some B , then A is non-negative; if B is invertible, then A is positive. A converse is also true in the sense that, for a non-negative operator A , there exists a unique non-negative square root B such that

$$A = B^2 = B^*B.$$

In a sense made precise by the spectral theorem, self-adjoint operators can usefully be thought of as operators that are "real". An element A of $B(H)$ is called *normal* if $A^*A = AA^*$. Normal operators decompose into the sum of a self-adjoint operators and an imaginary multiple of a self adjoint operator

$$A = \frac{A + A^*}{2} + i \frac{(A - A^*)}{2i}$$

that commute with each other. Normal operators can also usefully be thought of in terms of their real and imaginary parts.

An element U of $B(H)$ is called *unitary* if U is invertible and its inverse is given by U^* . This can also be expressed by requiring that U be onto and $\langle Ux, Uy \rangle = \langle x, y \rangle$ for all x and y in H . The unitary operators form a group under composition, which is the isometry group of H .

An element of $B(H)$ is *compact* if it sends bounded sets to relatively compact sets. Equivalently, a bounded operator T is compact if, for any bounded sequence $\{x_k\}$, the sequence $\{Tx_k\}$ has a convergent subsequence. Many integral operators are compact, and in fact define a special class of operators known as Hilbert–Schmidt operators that are especially important in the study of integral equations. Fredholm operators are those which differ from a compact operator by a multiple of the identity, and are equivalently characterized as operators with a finite dimensional kernel and cokernel. The index of a Fredholm operator T is defined by

$$\text{index } T = \dim \ker T - \dim \text{coker } T.$$

The index is homotopy invariant, and plays a deep role in differential geometry via the Atiyah–Singer index theorem.

Unbounded operators

Unbounded operators are also tractable in Hilbert spaces, and have important applications to quantum mechanics.^[54] An unbounded operator T on a Hilbert space H is defined to be a linear operator whose domain $D(T)$ is a linear subspace of H . Often the domain $D(T)$ is a dense subspace of H , in which case T is known as a densely defined operator.

The adjoint of a densely defined unbounded operator is defined in essentially the same manner as for bounded operators. Self-adjoint unbounded operators play the role of the *observables* in the mathematical formulation of quantum mechanics. Examples of self-adjoint unbounded operators on the Hilbert space $L^2(\mathbf{R})$ are:^[55]

- A suitable extension of the differential operator

$$(Af)(x) = i \frac{d}{dx} f(x),$$

where i is the imaginary unit and f is a differentiable function of compact support.

- The multiplication-by- x operator:

$$(Bf)(x) = xf(x).$$

These correspond to the momentum and position observables, respectively. Note that neither A nor B is defined on all of H , since in the case of A the derivative need not exist, and in the case of B the product function need not be square integrable. In both cases, the set of possible arguments form dense subspaces of $L^2(\mathbf{R})$.

Constructions

Direct sums

Two Hilbert spaces H_1 and H_2 can be combined into another Hilbert space, called the (orthogonal) direct sum,^[56] and denoted

$$H_1 \oplus H_2,$$

consisting of the set of all ordered pairs (x_1, x_2) where $x_i \in H_i$, $i = 1, 2$, and inner product defined by

$$\langle (x_1, x_2), (y_1, y_2) \rangle_{H_1 \oplus H_2} = \langle x_1, y_1 \rangle_{H_1} + \langle x_2, y_2 \rangle_{H_2}.$$

More generally, if H_i is a family of Hilbert spaces indexed by $i \in I$, then the direct sum of the H_i , denoted

$$\bigoplus_{i \in I} H_i$$

consists of the set of all indexed families

$$x = (x_i \in H_i | i \in I) \in \prod_{i \in I} H_i$$

in the Cartesian product of the H_i such that

$$\sum_{i \in I} \|x_i\|^2 < \infty.$$

The inner product is defined by

$$\langle x, y \rangle = \sum_{i \in I} \langle x_i, y_i \rangle_{H_i}.$$

Each of the H_i is included as a closed subspace in the direct sum of all of the H_i . Moreover, the H_i are pairwise orthogonal. Conversely, if there is a system of closed subspaces V_i , $i \in I$, in a Hilbert space H which are pairwise orthogonal and whose union is dense in H , then H is canonically isomorphic to the direct sum of V_i . In this case, H is called the internal direct sum of the V_i . A direct sum (internal or external) is also equipped with a family of orthogonal projections E_i onto the i th direct summand H_i . These projections are bounded, self-adjoint, idempotent operators which satisfy the orthogonality condition

$$E_i E_j = 0, \quad i \neq j.$$

The spectral theorem for compact self-adjoint operators on a Hilbert space H states that H splits into an orthogonal direct sum of the eigenspaces of an operator, and also gives an explicit decomposition of the operator as a sum of projections onto the eigenspaces. The direct sum of Hilbert spaces also appears in quantum mechanics as the Fock space of a system containing a variable number of particles, where each Hilbert space in the direct sum corresponds to an additional degree of freedom for the quantum mechanical system. In representation theory, the Peter-Weyl theorem guarantees that any unitary representation of a compact group on a Hilbert space splits as the direct sum of finite-dimensional representations.

Tensor products

If H_1 and H_2 , then one defines an inner product on the (ordinary) tensor product as follows. On simple tensors, let

$$\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle = \langle x_1, y_1 \rangle \langle x_2, y_2 \rangle.$$

This formula then extends by sesquilinearity to an inner product on $H_1 \otimes H_2$. The Hilbertian tensor product of H_1 and H_2 , sometimes denoted by $H_1 \widehat{\otimes} H_2$, is the Hilbert space obtained by completing $H_1 \otimes H_2$ for the metric associated to this inner product.^[57]

An example is provided by the Hilbert space $L^2([0, 1])$. The Hilbertian tensor product of two copies of $L^2([0, 1])$ is isometrically and linearly isomorphic to the space $L^2([0, 1]^2)$ of square-integrable functions on the square $[0, 1]^2$. This isomorphism sends a simple tensor $f_1 \otimes f_2$ to the function

$$(s, t) \mapsto f_1(s) f_2(t)$$

on the square.

This example is typical in the following sense.^[58] Associated to every simple tensor product $x_1 \otimes x_2$ is the rank one operator

$$x^* \in H_1^* \rightarrow x^*(x_1) x_2$$

from the (continuous) dual H_1^* to H_2 . This mapping defined on simple tensors extends to a linear identification between $H_1 \otimes H_2$ and the space of finite rank operators from H_1^* to H_2 . This extends to a linear isometry of the Hilbertian tensor product $H_1 \widehat{\otimes} H_2$ with the Hilbert space $HS(H_1^*, H_2)$ of Hilbert-Schmidt operators from H_1^* to H_2 .

Orthonormal bases

The notion of an orthonormal basis from linear algebra generalizes over to the case of Hilbert spaces.^[59] In a Hilbert space H , an orthonormal basis is a family $\{e_k\}_{k \in B}$ of elements of H satisfying the conditions:

1. *Orthogonality*: Every two different elements of B are orthogonal: $\langle e_k, e_j \rangle = 0$ for all k, j in B with $k \neq j$.
2. *Normalization*: Every element of the family has norm 1: $\|e_k\| = 1$ for all k in B .
3. *Completeness*: The linear span of the family $e_k, k \in B$, is dense in H .

A system of vectors satisfying the first two conditions is called an orthonormal system or an orthonormal set (or an orthonormal sequence if B is countable). Such a system is always linearly independent. Completeness of an orthonormal system of vectors of a Hilbert space can be equivalently restated as:

$$\text{if } \langle v, e_k \rangle = 0 \text{ for all } k \in B \text{ and some } v \in H \text{ then } v = 0.$$

This is related to the fact that the only vector orthogonal to a dense linear subspace is the zero vector, for if S is any orthonormal set and v is orthogonal to S , then v is orthogonal to the closure of the linear span of S , which is the whole space.

Examples of orthonormal bases include:

- the set $\{(1,0,0), (0,1,0), (0,0,1)\}$ forms an orthonormal basis of \mathbf{R}^3 with the dot product;

- the sequence $\{f_n : n \in \mathbf{Z}\}$ with $f_n(x) = \exp(2\pi i n x)$ forms an orthonormal basis of the complex space $L^2([0,1])$;

In the infinite-dimensional case, an orthonormal basis will not be a basis in the sense of linear algebra; to distinguish the two, the latter basis is also called a Hamel basis. That the span of the basis vectors is dense implies that every vector in the space can be written as the sum of an infinite series, and the orthogonality implies that this decomposition is unique.

Sequence spaces

The space ℓ^2 of square-summable sequences of complex numbers is the set of infinite sequences

$$(c_1, c_2, c_3, \dots)$$

of complex numbers such that

$$|c_1|^2 + |c_2|^2 + |c_3|^2 + \dots < \infty.$$

This space has an orthonormal basis:

$$e_1 = (1, 0, 0, \dots)$$

$$e_2 = (0, 1, 0, \dots)$$

\vdots

More generally, if B is any set, then one can form a Hilbert space of sequences with index set B , defined by

$$\ell^2(B) = \left\{ x : B \rightarrow \mathbb{C} \mid \sum_{b \in B} |x(b)|^2 < \infty \right\}.$$

The summation over B is here defined by

$$\sum_{b \in B} |x(b)|^2 = \sup \sum_{n=1}^N |x(b_n)|^2$$

the supremum being taken over all finite subsets of B . It follows that, in order for this sum to be finite, every element of $\ell^2(B)$ has only countably many nonzero terms. This space becomes a Hilbert space with the inner product

$$\langle x, y \rangle = \sum_{b \in B} x(b) \overline{y(b)}$$

for all x and y in $\ell^2(B)$. Here the sum also has only countably many nonzero terms, and is unconditionally convergent by the Cauchy–Schwarz inequality.

An orthonormal basis of $\ell^2(B)$ is indexed by the set B , given by

$$e_b(b') = \begin{cases} 1 & \text{if } b = b' \\ 0 & \text{otherwise.} \end{cases}$$

Bessel's inequality and Parseval's formula

Let f_1, \dots, f_n be a finite orthonormal system in H . For an arbitrary vector x in H , let

$$y = \sum_{j=1}^n \langle x, f_j \rangle f_j.$$

Then $\langle x, f_k \rangle = \langle y, f_k \rangle$ for every $k = 1, \dots, n$. It follows that $x - y$ is orthogonal to each f_k , hence $x - y$ is orthogonal to y . Using the Pythagorean identity twice, it follows that

$$\|x\|^2 = \|x - y\|^2 + \|y\|^2 \geq \|y\|^2 = \sum_{j=1}^n |\langle x, f_j \rangle|^2.$$

Let $\{f_i\}$, $i \in I$, be an arbitrary orthonormal system in H . Applying the preceding inequality to every finite subset J of I gives the *Bessel inequality*^[60]

$$\sum_{i \in I} |\langle x, f_i \rangle|^2 \leq \|x\|^2, \quad x \in H$$

(according to the definition of the sum of an arbitrary family of non-negative real numbers).

Geometrically, Bessel's inequality implies that the orthogonal projection of x onto the linear subspace spanned by the f_i has norm that does not exceed that of x . In two dimensions, this is the assertion that the length of the leg of a right triangle may not exceed the length of the hypotenuse.

Bessel's inequality is a stepping stone to the more powerful Parseval identity which governs the case when Bessel's inequality is actually an equality. If $\{e_k\}_{k \in B}$ is an orthonormal basis of H , then every element x of H may be written as

$$x = \sum_{k \in B} \langle x, e_k \rangle e_k.$$

Even if B is uncountable, Bessel's inequality guarantees that the expression is well-defined and consists only of countably many nonzero terms. This sum is called the *Fourier expansion* of x , and the individual coefficients $\langle x, e_k \rangle$ are the *Fourier coefficients* of x . Parseval's formula is then

$$\|x\|^2 = \sum_{k \in B} |\langle x, e_k \rangle|^2.$$

Conversely, if $\{e_k\}$ is an orthonormal set such that Parseval's identity holds for every x , then $\{e_k\}$ is an orthonormal basis.

Hilbert dimension

As a consequence of Zorn's lemma, every Hilbert space admits an orthonormal basis; furthermore, any two orthonormal bases of the same space have the same cardinality, called the Hilbert dimension of the space.^[61] For instance, since $\ell^2(B)$ has an orthonormal basis indexed by B , its Hilbert dimension is the cardinality of B (which may be a finite integer, or a countable or uncountable cardinal number).

As a consequence of Parseval's identity, if $\{e_k\}_{k \in B}$ is an orthonormal basis of H , then the map $\Phi : H \rightarrow \ell^2(B)$ defined by $\Phi(x) = (\langle x, e_k \rangle)_{k \in B}$ is an isometric isomorphism of Hilbert spaces: it is a bijective linear mapping such that

$$\langle \Phi(x), \Phi(y) \rangle_{\ell^2(B)} = \langle x, y \rangle_H$$

for all x and y in H . The cardinal number of B is the Hilbert dimension of H . Thus every Hilbert space is isometrically isomorphic to a sequence space $\ell^2(B)$ for some set B .

Separable spaces

A Hilbert space is separable if and only if it admits a countable orthonormal basis. All infinite-dimensional separable Hilbert spaces are therefore isometrically isomorphic to ℓ^2 .

In the past, Hilbert spaces were often required to be separable as part of the definition.^[62] Most spaces used in physics are separable, and since these are all isomorphic to each other, one often refers to any infinite-dimensional separable Hilbert space as "the Hilbert space" or just "Hilbert space".^[63] Even in quantum field theory, most of the Hilbert spaces are in fact separable, as stipulated by the Wightman axioms. However, it is sometimes argued that non-separable Hilbert spaces are also important in quantum field theory, roughly because the systems in the theory possess an infinite number of degrees of freedom and any infinite Hilbert tensor product (of spaces of dimension greater than one) is non-separable.^[64] For instance, a bosonic field can be naturally thought of as an element of a tensor product whose factors represent harmonic oscillators at each point of space. From this perspective, the natural state space of a boson might seem to be a non-separable space.^[64] However, it is only a small separable subspace of the full tensor product that can contain physically meaningful fields (on which the observables can be defined). Another non-separable Hilbert space models the state of an infinite collection of particles in an unbounded region of space. An orthonormal basis of the space is indexed by the density of the particles, a continuous parameter, and since

the set of possible densities is uncountable, the basis is not countable.^[64]

Orthogonal complements and projections

If S is a subset of a Hilbert space H , the set of vectors orthogonal to S is defined by

$$S^\perp = \{x \in H : \langle x, s \rangle = 0 \ \forall s \in S\}.$$

S^\perp is a closed subspace of H and so forms itself a Hilbert space. If V is a closed subspace of H , then V^\perp is called the *orthogonal complement* of V . In fact, every x in H can then be written uniquely as $x = v + w$, with v in V and w in V^\perp . Therefore, H is the internal Hilbert direct sum of V and V^\perp .

The linear operator $P_V : H \rightarrow H$ which maps x to v is called the *orthogonal projection* onto V . There is a natural one-to-one correspondence between the set of all closed subspaces of H and the set of all bounded self-adjoint operators P such that $P^2 = P$. Specifically,

Theorem. The orthogonal projection P_V is a self-adjoint linear operator on H of norm ≤ 1 with the property $P_V^2 = P_V$. Moreover, any self-adjoint linear operator E such that $E^2 = E$ is of the form P_V , where V is the range of E . For every x in H , $P_V(x)$ is the unique element v of V which minimizes the distance $\|x - v\|$.

This provides the geometrical interpretation of $P_V(x)$: it is the best approximation to x by elements of V .^[65]

An operator P such that $P = P^2 = P^*$ is called an orthogonal projection. The orthogonal projection P_V onto a closed subspace V of H is the adjoint of the inclusion mapping

$$i_V : V \rightarrow H,$$

meaning that

$$\langle i_V x, y \rangle = \langle x, P_V y \rangle$$

for all $x \in H$ and $y \in V$. Projections P_U and P_V are called mutually orthogonal if $P_U P_V = 0$. This is equivalent to U and V being orthogonal as subspaces of H . As a result, the sum of the two projections P_U and P_V is only a projection if U and V are orthogonal to each other, and in that case $P_U + P_V = P_{U+V}$. The composite $P_U P_V$ is generally not a projection; in fact, the composite is a projection if and only if the two projections commute, and in that case $P_U P_V = P_{U \cap V}$.

The operator norm of a projection P onto a non-zero closed subspace is equal to one:

$$\|P\| = \sup_{x \in H, x \neq 0} \frac{\|Px\|}{\|x\|} = 1.$$

Every closed subspace V of a Hilbert space is therefore the image of an operator P of norm one such that $P^2 = P$. In fact this property characterizes Hilbert spaces.^[66]

- A Banach space of dimension higher than 2 is (isometrically) a Hilbert space if and only if, to every closed subspace V , there is an operator P_V of norm one whose image is V such that $P_V^2 = P_V$.

While this result characterizes the metric structure of a Hilbert space, the structure of a Hilbert space as a topological vector space can itself be characterized in terms of the presence of complementary subspaces.^[67]

- A Banach space X is topologically and linearly isomorphic to a Hilbert space if and only if, to every closed subspace V , there is a closed subspace W such that X is equal to the internal direct sum $V \oplus W$.

The orthogonal complement satisfies some more elementary results. It is a monotone function in the sense that if $U \subset V$, then $V^\perp \subseteq U^\perp$ with equality holding if and only if V is contained in the closure of U . This result is a special case of the Hahn–Banach theorem. The closure of a subspace can be completely characterized in terms of the orthogonal complement: If V is a subspace of H , then the closure of V is equal to $V^{\perp\perp}$. The orthogonal complement is thus a Galois connection on the partial order of subspaces of a Hilbert space. In general, the orthogonal complement of a sum of subspaces is the intersection of the orthogonal complements.^[68] $(\sum_i V_i)^\perp = \bigcap_i V_i^\perp$. If the V_i are in addition closed, then $\overline{\sum_i V_i} = (\bigcap_i V_i)^\perp$.

Spectral theory

There is a well-developed spectral theory for self-adjoint operators in a Hilbert space, that is roughly analogous to the study of symmetric matrices over the reals or self-adjoint matrices over the complex numbers.^[69] In the same sense, one can obtain a "diagonalization" of a self-adjoint operator as a suitable sum (actually an integral) of orthogonal projection operators.

The spectrum of an operator T , denoted $\sigma(T)$ is the set of complex numbers λ such that $T - \lambda$ lacks a continuous inverse. If T is bounded, then the spectrum is always a compact set in the complex plane, and lies inside the disc $|z| \leq \|T\|$. If T is self-adjoint, then the spectrum is real. In fact, it is contained in the interval $[m, M]$ where

$$m = \inf_{\|x\|=1} \langle Tx, x \rangle, \quad M = \sup_{\|x\|=1} \langle Tx, x \rangle.$$

Moreover, m and M are both actually contained within the spectrum.

The eigenspaces of an operator T are given by

$$H_\lambda = \ker(T - \lambda).$$

Unlike with finite matrices, not every element of the spectrum of T must be an eigenvalue: the linear operator $T - \lambda$ may only lack an inverse because it is not surjective. Elements of the spectrum of an operator in the general sense are known as *spectral values*. Since spectral values need not be eigenvalues, the spectral decomposition is often more subtle than in finite dimensions.

However, the spectral theorem of a self-adjoint operator T takes a particularly simple form if, in addition, T is assumed to be a compact operator. The spectral theorem for compact self-adjoint operators states:^[70]

- A compact self-adjoint operator T has only countably (or finitely) many spectral values. The spectrum of T has no limit point in the complex plane except possibly zero. The eigenspaces of T decompose H into an orthogonal direct sum:

$$H = \bigoplus_{\lambda \in \sigma(T)} H_\lambda.$$

Moreover, if E_λ denotes the orthogonal projection onto the eigenspace H_λ , then

$$T = \sum_{\lambda \in \sigma(T)} \lambda E_\lambda$$

where the sum converges with respect to the norm on $B(H)$.

This theorem plays a fundamental role in the theory of integral equations, as many integral operators are compact, in particular those that arise from Hilbert-Schmidt operators.

The general spectral theorem for self-adjoint operators involves a kind of operator-valued Riemann–Stieltjes integral, rather than an infinite summation.^[71] The *spectral family* associated to T associates to each real number λ an operator E_λ , which is the projection onto the nullspace of the operator $(T - \lambda)^+$, where the positive part of a self-adjoint operator is defined by

$$A^+ = \frac{1}{2} (\sqrt{A^2} + A).$$

The operators E_λ are monotone increasing relative to the partial order defined on self-adjoint operators; the eigenvalues correspond precisely to the jump discontinuities. One has the spectral theorem, which asserts

$$T = \int_{\mathbb{R}} \lambda dE_\lambda.$$

The integral is understood as a Riemann–Stieltjes integral, convergent with respect to the norm on $B(H)$. In particular, one has the ordinary scalar-valued integral representation

$$\langle Tx, y \rangle = \int_{\mathbb{R}} \lambda d\langle E_\lambda x, y \rangle.$$

A somewhat similar spectral decomposition holds for normal operators, although because the spectrum may now contain non-real complex numbers, the operator-valued Stieltjes measure dE_λ must instead be replaced by a resolution of the identity.

A major application of spectral methods is the spectral mapping theorem, which allows one to apply to a self-adjoint operator T any continuous complex function f defined on the spectrum of T by forming the integral

$$f(T) = \int_{\sigma(T)} f(\lambda) dE_\lambda.$$

The resulting continuous functional calculus has applications in particular to pseudodifferential operators.^[72]

The spectral theory of *unbounded* self-adjoint operators is only marginally more difficult than for bounded operators. The spectrum of an unbounded operator is defined in precisely the same way as for bounded operators: λ is a spectral value if the resolvent operator

$$R_\lambda = (T - \lambda)^{-1}$$

fails to be a well-defined continuous operator. The self-adjointness of T still guarantees that the spectrum is real. Thus the essential idea of working with unbounded operators is to look instead at the resolvent R_λ where λ is non-real. This is a *bounded* normal operator, which admits a spectral representation that can then be transferred to a spectral representation of T itself. A similar strategy is used, for instance, to study the spectrum of the Laplace operator: rather than address the operator directly, one instead looks at an associated resolvent such as a Riesz potential or Bessel potential.

A precise version of the spectral theorem which holds in this case is:^[73]

Given a densely defined self-adjoint operator T on a Hilbert space H , there corresponds a unique resolution of the identity E on the Borel sets of \mathbf{R} , such that

$$\langle Tx, y \rangle = \int_{\mathbf{R}} \lambda dE_{x,y}(\lambda)$$

for all $x \in D(T)$ and $y \in H$. The spectral measure E is concentrated on the spectrum of T .

There is also a version of the spectral theorem that applies to unbounded normal operators.

Notes

- [1] Marsden 1974, §2.8
- [2] The mathematical material in this section can be found in any good textbook on functional analysis, such as Dieudonné (1960), Hewitt & Stromberg (1965), Reed & Simon (1980) or Rudin (1980).
- [3] In some conventions, inner products are linear in their second arguments instead.
- [4] Dieudonné 1960, §6.2
- [5] Dieudonné 1960
- [6] Largely from the work of Hermann Grassmann, at the urging of August Ferdinand Möbius (Boyer & Merzbach 1991, pp. 584–586). The first modern axiomatic account of abstract vector spaces ultimately appeared in Giuseppe Peano's 1888 account (Grattan-Guinness 2000, §5.2.2; O'Connor & Robertson 1996).
- [7] A detailed account of the history of Hilbert spaces can be found in Bourbaki 1987.
- [8] Schmidt 1908
- [9] Titchmarsh 1946, §IX.1
- [10] Lebesgue 1904. Further details on the history of integration theory can be found in Bourbaki (1987) and Saks (2005).
- [11] Bourbaki 1987.
- [12] Dunford & Schwartz 1958, §IV.16
- [13] In Dunford & Schwartz (1958, §IV.16), the result that every linear functional on $L^2[0,1]$ is represented by integration is jointly attributed to Fréchet (1907) and Riesz (1907). The general result, that the dual of a Hilbert space is identified with the Hilbert space itself, can be found in Riesz (1934).
- [14] von Neumann 1929.
- [15] Kline 1972, p. 1092
- [16] Hilbert, Nordheim & von Neumann 1927.
- [17] Weyl 1931.

- [18] Prugovečki 1981, pp. 1–10.
- [19] von Neumann 1932
- [20] Halmos 1957, Section 42.
- [21] Hewitt & Stromberg 1965.
- [22] Bers, John & Schechter 1981.
- [23] Giusti 2003.
- [24] Stein 1970
- [25] Details can be found in Warner (1983).
- [26] A general reference on Hardy spaces is the book Duren (1970).
- [27] Krantz 2002, §1.4
- [28] Krantz 2002, §1.5
- [29] Young 1988, Chapter 9.
- [30] The eigenvalues of the Fredholm kernel are $1/\lambda$, which tend to zero.
- [31] More detail on finite element methods from this point of view can be found in Brenner & Scott (2005).
- [32] Reed & Simon 1980
- [33] A treatment of Fourier series from this point of view is available, for instance, in Rudin (1987) or Folland (2009).
- [34] Halmos 1957, §5
- [35] Bachman, Narici & Beckenstein 2000
- [36] Stein & Weiss 1971, §IV.2.
- [37] Lancos 1988, pp. 212–213
- [38] Lanczos 1988, Equation 4-3.10
- [39] The classic reference for spectral methods is Courant & Hilbert 1953. A more up-to-date account is Reed & Simon 1975.
- [40] Kac 1966
- [41] Dirac 1930
- [42] von Neumann 1955
- [43] Young 1988, p. 23.
- [44] Clarkson 1936.
- [45] Rudin 1987, Theorem 4.10
- [46] Dunford & Schwartz 1958, II.4.29
- [47] Rudin 1987, Theorem 4.11
- [48] Weidmann 1980, Theorem 4.8
- [49] Weidmann 1980, §4.5
- [50] Buttazzo, Giaquinta & Hildebrandt 1998, Theorem 5.17
- [51] Halmos 1982, Problem 52, 58
- [52] Rudin 1973
- [53] Trèves 1967, Chapter 18
- [54] See Prugovečki (1981), Reed & Simon (1980, Chapter VIII) and Folland (1989).
- [55] Prugovečki 1981, III, §1.4
- [56] Dunford & Schwartz 1958, IV.4.17-18
- [57] Weidmann 1980, §3.4
- [58] Kadison & Ringrose 1983, Theorem 2.6.4
- [59] Dunford & Schwartz 1958, §IV.4.
- [60] For the case of finite index sets, see, for instance, Halmos 1957, §5. For infinite index sets, see Weidmann 1980, Theorem 3.6.
- [61] Levitan 2001. Many authors, such as Dunford & Schwartz (1958, §IV.4), refer to this just as the dimension. Unless the Hilbert space is finite dimensional, this is not the same thing as its dimension as a linear space (the cardinality of a Hamel basis).
- [62] Prugovečki 1981, I, §4.2
- [63] von Neumann (1955) defines a Hilbert space via a countable Hilbert basis, which amounts to an isometric isomorphism with ℓ^2 . The convention still persists in most rigorous treatments of quantum mechanics; see for instance Sobrino 1996, Appendix B.
- [64] Streater & Wightman 1964, pp. 86–87
- [65] Young 1988, Theorem 15.3
- [66] Kakutani 1939
- [67] Lindenstrauss & Tzafriri 1971
- [68] Halmos 1957, §12
- [69] A general account of spectral theory in Hilbert spaces can be found in Riesz & Sz Nagy (1990). A more sophisticated account in the language of C^* -algebras is in Rudin (1973) or Kadison & Ringrose (1997)
- [70] See, for instance, Riesz & Sz Nagy (1990, Chapter VI) or Weidmann 1980, Chapter 7. This result was already known to Schmidt (1907) in the case of operators arising from integral kernels.
- [71] Riesz & Sz Nagy 1990, §§107–108
- [72] Shubin 1987

[73] Rudin 1973, Theorem 13.30.

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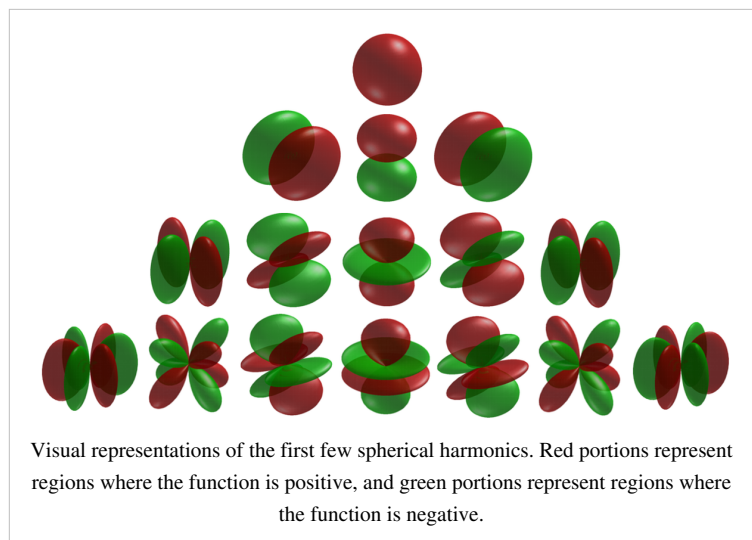
External links

- Hilbert Space at Mathworld (<http://mathworld.wolfram.com/HilbertSpace.html>)
- 245B, notes 5: Hilbert spaces (<http://terrytao.wordpress.com/2009/01/17/254a-notes-5-hilbert-spaces/>) by Terence Tao

Spherical harmonics

In mathematics, **spherical harmonics** are the angular portion of a set of solutions to Laplace's equation. Represented in a system of spherical coordinates, Laplace's spherical harmonics Y_ℓ^m are a specific set of spherical harmonics that forms an orthogonal system, first introduced by Pierre Simon de Laplace.^[1] Spherical harmonics are important in many theoretical and practical applications, particularly in the computation of atomic orbital electron configurations, representation of gravitational fields, geoids, and the magnetic fields of planetary bodies and

stars, and characterization of the cosmic microwave background radiation. In 3D computer graphics, spherical harmonics play a special role in a wide variety of topics including indirect lighting (ambient occlusion, global illumination, precomputed radiance transfer, etc.) and recognition of 3D shapes.



History

Spherical harmonics were first investigated in connection with the Newtonian potential of Newton's law of universal gravitation in three dimensions. In 1782, Pierre-Simon de Laplace had, in his *Mécanique Céleste*, determined that the gravitational potential at a point \mathbf{x} associated to a set of point masses m_i located at points \mathbf{x}_i was given by

$$V(\mathbf{x}) = \sum_i \frac{m_i}{|\mathbf{x}_i - \mathbf{x}|}.$$

Each term in the above summation is an individual Newtonian potential for a point mass. Just prior to that time, Adrien-Marie Legendre had investigated the expansion of the Newtonian potential in powers of $r = |\mathbf{x}|$ and $r_1 = |\mathbf{x}_1|$. He discovered that if $r \leq r_1$ then

$$\frac{1}{|\mathbf{x}_1 - \mathbf{x}|} = P_0(\cos \gamma) \frac{1}{r_1} + P_1(\cos \gamma) \frac{r}{r_1^2} + P_2(\cos \gamma) \frac{r^2}{r_1^3} + \dots$$

where γ is the angle between the vectors \mathbf{x} and \mathbf{x}_1 . The functions P_i are the Legendre polynomials, and they are a special case of spherical harmonics. Subsequently, in his 1782 memoir, Laplace investigated these coefficients using spherical coordinates to represent the angle γ between \mathbf{x}_1 and \mathbf{x} . (See Applications of Legendre polynomials in physics for a more detailed analysis.)

In 1867, William Thomson (Lord Kelvin) and Peter Guthrie Tait introduced the solid spherical harmonics in their *Treatise on Natural Philosophy*, and also first introduced the name of "spherical harmonics" for these functions. The solid harmonics were homogeneous solutions of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

By examining Laplace's equation in spherical coordinates, Thomson and Tait recovered Laplace's spherical harmonics. The term "Laplace's coefficients" was employed by William Whewell to describe the particular system of solutions introduced along these lines, whereas others reserved this designation for the zonal spherical harmonics that had properly been introduced by Laplace and Legendre.

The 19th century development of Fourier series made possible the solution of a wide variety of physical problems in rectangular domains, such as the solution of the heat equation and wave equation. This could be achieved by expansion of functions in series of trigonometric functions. Whereas the trigonometric functions in a Fourier series represent the fundamental modes of vibration in a string, the spherical harmonics represent the fundamental modes of vibration of a sphere in much the same way. Many aspects of the theory of Fourier series could be generalized by taking expansions in spherical harmonics rather than trigonometric functions. This was a boon for problems possessing spherical symmetry, such as those of celestial mechanics originally studied by Laplace and Legendre.

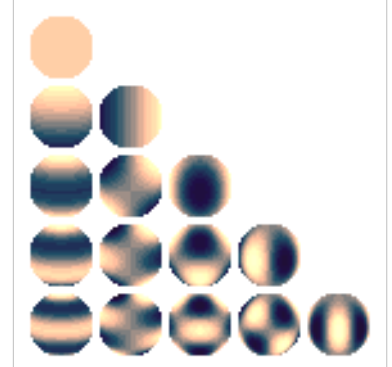
The prevalence of spherical harmonics already in physics set the stage for their later importance in the 20th century birth of quantum mechanics. The spherical harmonics are eigenfunctions of the square of the orbital angular momentum operator

$$-i\hbar \mathbf{r} \times \nabla,$$

and therefore they represent the different quantized configurations of atomic orbitals.

Laplace's spherical harmonics

Laplace's equation imposes that the divergence of the gradient of a scalar field f is zero. In spherical coordinates this is:^[2]



Real (Laplace) spherical harmonics Y_ℓ^m for $\ell = 0$ to 4 (top to bottom) and $m = 0$ to 4 (left to right). The negative order harmonics Y_ℓ^{-m} are rotated about the z axis by $90^\circ/m$ with respect to the positive order ones.

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2} = 0.$$

Consider the problem of finding solutions of the form $f(r, \theta, \varphi) = R(r)Y(\theta, \varphi)$. By separation of variables, two differential equations result by imposing Laplace's equation:

$$\frac{1}{R} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = \lambda, \quad \frac{1}{Y} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial Y}{\partial \theta} \right) + \frac{1}{Y} \frac{1}{\sin^2 \theta} \frac{\partial^2 Y}{\partial \varphi^2} = -\lambda.$$

The second equation can be simplified under the assumption that Y has the form $Y(\theta, \varphi) = \Theta(\theta)\Phi(\varphi)$. Applying separation of variables again to the second equation gives way to the pair of differential equations

$$\frac{1}{\Phi(\varphi)} \frac{d^2 \Phi(\varphi)}{d\varphi^2} = -m^2$$

$$\lambda \sin^2(\theta) + \frac{\sin(\theta)}{\Theta(\theta)} \frac{d}{d\theta} \left[\sin(\theta) \frac{d\Theta}{d\theta} \right] = m^2$$

for some number m . A priori, m is a complex constant, but because Φ must be a periodic function whose period evenly divides 2π , m is necessarily an integer and Φ is a linear combination of the complex exponentials $e^{\pm im\varphi}$. The solution function $Y(\theta, \varphi)$ is regular at the poles of the sphere, where $\theta=0, \pi$. Imposing this regularity in the solution Θ of the second equation at the boundary points of the domain is a Sturm–Liouville problem that forces the parameter λ to be of the form $\lambda = \ell(\ell+1)$ for some non-negative integer with $\ell \geq |m|$; this is also explained below in terms of the orbital angular momentum. Furthermore, a change of variables $t = \cos \theta$ transforms this equation into the Legendre equation, whose solution is a multiple of the associated Legendre polynomial $P_\ell^m(\cos \theta)$. Finally, the equation for R has solutions of the form $R(r) = Ar^\ell + Br^{-\ell-1}$; requiring the solution to be regular throughout \mathbf{R}^3 forces $B = 0$.^[3]

Here the solution was assumed to have the special form $Y(\theta, \varphi) = \Theta(\theta)\Phi(\varphi)$. For a given value of ℓ , there are $2\ell+1$ independent solutions of this form, one for each integer m with $-\ell \leq m \leq \ell$. These angular solutions are a product of trigonometric functions, here represented as a complex exponential, and associated Legendre polynomials:

$$Y_\ell^m(\theta, \varphi) = N e^{im\varphi} P_\ell^m(\cos \theta),$$

which fulfill

$$r^2 \nabla^2 Y_\ell^m(\theta, \varphi) = -\ell(\ell+1) Y_\ell^m(\theta, \varphi).$$

Here Y_ℓ^m is called a spherical harmonic function of degree ℓ and order m , P_ℓ^m is an associated Legendre polynomial, N is a normalization constant, and θ and φ represent colatitude and longitude, respectively. In particular, the colatitude θ , or polar angle, ranges from 0 at the North Pole to π at the South Pole, assuming the value of $\pi/2$ at the Equator, and the longitude φ , or azimuth, may assume all values with $0 \leq \varphi < 2\pi$. For a fixed integer ℓ , every solution $Y(\theta, \varphi)$ of the eigenvalue problem

$$r^2 \nabla^2 Y = -\ell(\ell + 1)Y$$

is a linear combination of Y_ℓ^m . In fact, for any such solution, $r^\ell Y(\theta, \varphi)$ is the expression in spherical coordinates of a homogeneous polynomial that is harmonic (see below), and so counting dimensions shows that there are $2\ell + 1$ linearly independent such polynomials.

The general solution to Laplace's equation in a ball centered at the origin is a linear combination of the spherical harmonic functions multiplied by the appropriate scale factor r^ℓ ,

$$f(r, \theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_\ell^m r^\ell Y_\ell^m(\theta, \varphi),$$

where the f_ℓ^m are constants and the factors $r^\ell Y_\ell^m$ are known as solid harmonics. Such an expansion is valid in the ball

$$r < R = 1 / \limsup_{\ell \rightarrow \infty} |f_\ell^m|^{1/\ell}.$$

Orbital angular momentum

In quantum mechanics, Laplace's spherical harmonics are understood in terms of the orbital angular momentum^[4]

$$\mathbf{L} = -i\hbar \mathbf{x} \times \nabla = L_x \mathbf{i} + L_y \mathbf{j} + L_z \mathbf{k}.$$

The \hbar is conventional in quantum mechanics; it is convenient to work in units in which $\hbar = 1$. The spherical harmonics are eigenfunctions of the square of the orbital angular momentum

$$\begin{aligned} \mathbf{L}^2 &= -r^2 \nabla^2 + \left(r \frac{\partial}{\partial r} + 1 \right) r \frac{\partial}{\partial r} \\ &= -\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \sin \theta \frac{\partial}{\partial \theta} - \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}. \end{aligned}$$

Laplace's spherical harmonics are the joint eigenfunctions of the square of the orbital angular momentum and the generator of rotations about the azimuthal axis:

$$\begin{aligned} L_z &= -i \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) \\ &= -i \frac{\partial}{\partial \varphi}. \end{aligned}$$

These operators commute, and are densely defined self-adjoint operators on the Hilbert space of functions f square-integrable with respect to the normal distribution on \mathbf{R}^3 :

$$\frac{1}{(2\pi)^{3/2}} \int_{\mathbf{R}^3} |f(x)|^2 e^{-|x|^2/2} dx < \infty.$$

Furthermore, \mathbf{L}^2 is a positive operator.

If Y is a joint eigenfunction of \mathbf{L}^2 and L_z , then by definition

$$\mathbf{L}^2 Y = \lambda Y$$

$$L_z Y = m Y$$

for some real numbers m and λ . Here m must in fact be an integer, for Y must be periodic in the coordinate φ with period a number that evenly divides 2π . Furthermore, since

$$\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2$$

and each of L_x, L_y, L_z are self-adjoint, it follows that $\lambda \geq m^2$.

Denote this joint eigenspace by $E_{\lambda, m}$, and define the raising and lowering operators by

$$L_+ = L_x + iL_y$$

$$L_- = L_x - iL_y$$

Then L_+ and L_- commute with \mathbf{L}^2 , and the Lie algebra generated by L_+, L_-, L_z is the special linear Lie algebra, with commutation relations

$$[L_z, L_+] = L_+, \quad [L_z, L_-] = -L_-, \quad [L_+, L_-] = 2L_z.$$

Thus $L_+ : E_{\lambda, m} \rightarrow E_{\lambda, m+1}$ (it is a "raising operator") and $L_- : E_{\lambda, m} \rightarrow E_{\lambda, m-1}$ (it is a "lowering operator"). In particular, $L_+^{k''} : E_{\lambda, m} \rightarrow E_{\lambda, m+k}$ must be zero for k sufficiently large, because the inequality $\lambda \geq m^2$ must hold in each of the nontrivial joint eigenspaces. Let $Y \in E_{\lambda, m}$ be a nonzero joint eigenfunction, and let k be the least integer such that

$$L_+^k Y = 0.$$

Then, since

$$L_- L_+ = \mathbf{L}^2 - L_z^2 - L_z$$

it follows that

$$0 = L_- L_+^k Y = (\lambda - (m+k)^2 - (m+k))Y.$$

Thus $\lambda = \ell(\ell+1)$ for the positive integer $\ell = m+k$.

Conventions

Orthogonality and normalization

Several different normalizations are in common use for the Laplace spherical harmonic functions. Throughout the section, we use the standard convention that (see associated Legendre polynomials)

$$P_\ell^{-m} = (-1)^m \frac{(\ell-m)!}{(\ell+m)!} P_\ell^m$$

which is the natural normalization given by Rodrigues' formula.

In physics and seismology, the Laplace spherical harmonics are generally defined as

$$Y_\ell^m(\theta, \varphi) = \sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\varphi}$$

which are orthonormal

$$\int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} Y_\ell^m Y_{\ell'}^{m'*} d\Omega = \delta_{\ell\ell'} \delta_{mm'},$$

where $\delta_{aa} = 1$, $\delta_{ab} = 0$ if $a \neq b$, (see Kronecker delta) and $d\Omega = \sin\theta d\varphi d\theta$. This normalization is used in quantum mechanics because it ensures that probability is normalized, i.e. $\int |Y_\ell^m|^2 d\Omega = 1$. The disciplines of geodesy and spectral analysis use

$$Y_\ell^m(\theta, \varphi) = \sqrt{(2\ell+1) \frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos \theta) e^{im\varphi}$$

which possess unit power

$$\frac{1}{4\pi} \int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} Y_\ell^m Y_{\ell'}^{m'*} d\Omega = \delta_{\ell\ell'} \delta_{mm'}.$$

The magnetics community, in contrast, uses Schmidt semi-normalized harmonics

$$Y_\ell^m(\theta, \varphi) = \sqrt{\frac{(\ell - m)!}{(\ell + m)!}} P_\ell^m(\cos \theta) e^{im\varphi}$$

which have the normalization

$$\int_{\theta=0}^{\pi} \int_{\varphi=0}^{2\pi} Y_\ell^m Y_{\ell'}^{m'*} d\Omega = \frac{4\pi}{(2\ell + 1)} \delta_{\ell\ell'} \delta_{mm'}.$$

In quantum mechanics this normalization is often used as well, and is named Racah's normalization after Giulio Racah.

It can be shown that all of the above normalized spherical harmonic functions satisfy

$$Y_\ell^{m*}(\theta, \varphi) = (-1)^m Y_\ell^{-m}(\theta, \varphi),$$

where the superscript * denotes complex conjugation. Alternatively, this equation follows from the relation of the spherical harmonic functions with the Wigner D-matrix.

Condon–Shortley phase

One source of confusion with the definition of the spherical harmonic functions concerns a phase factor of $(-1)^m$, commonly referred to as the Condon–Shortley phase in the quantum mechanical literature. In the quantum mechanics community, it is common practice to either include this phase factor in the definition of the associated Legendre polynomials, or to append it to the definition of the spherical harmonic functions. There is no requirement to use the Condon–Shortley phase in the definition of the spherical harmonic functions, but including it can simplify some quantum mechanical operations, especially the application of raising and lowering operators. The geodesy and magnetics communities never include the Condon–Shortley phase factor in their definitions of the spherical harmonic functions.

Real form

A real basis of spherical harmonics can be defined in terms of their complex analogues by setting

$$Y_{\ell m} = \begin{cases} \frac{1}{\sqrt{2}} (Y_\ell^m + (-1)^m Y_\ell^{-m}) = \sqrt{2} N_{(\ell, m)} P_\ell^m(\cos \theta) \cos m\varphi & \text{if } m > 0 \\ Y_\ell^0 & \text{if } m = 0 \\ \frac{1}{i\sqrt{2}} (Y_\ell^{-m} - (-1)^m Y_\ell^m) = \sqrt{2} N_{(\ell, m)} P_\ell^{-m}(\cos \theta) \sin m\varphi & \text{if } m < 0. \end{cases}$$

where $N_{(\ell, m)}$ denotes the normalization constant as a function of ℓ and m . The real form requires only associated Legendre polynomials $P_\ell^{|m|}$ of non-negative $|m|$. The harmonics with $m > 0$ are said to be of cosine type, and those with $m < 0$ of sine type. These real spherical harmonics are sometimes known as *tesseral spherical harmonics*.^[5] These functions have the same normalization properties as the complex ones above. See here for a list of real spherical harmonics up to and including $\ell = 5$. Note, however, that the listed functions differ by the phase $(-1)^m$ from the phase given in this article.

Spherical harmonics expansion

The Laplace spherical harmonics form a complete set of orthonormal functions and thus form an orthonormal basis of the Hilbert space of square-integrable functions. On the unit sphere, any square-integrable function can thus be expanded as a linear combination of these:

$$f(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell}^m Y_{\ell}^m(\theta, \varphi).$$

This expansion holds in the sense of mean-square convergence — convergence in L^2 of the sphere — which is to say that

$$\lim_{N \rightarrow \infty} \int_0^{2\pi} \int_0^{\pi} \left| f(\theta, \varphi) - \sum_{\ell=0}^N \sum_{m=-\ell}^{\ell} f_{\ell}^m Y_{\ell}^m(\theta, \varphi) \right|^2 \sin \theta d\theta d\varphi = 0.$$

The expansion coefficients are the analogs of Fourier coefficients, and can be obtained by multiplying the above equation by the complex conjugate of a spherical harmonic, integrating over the solid angle Ω , and utilizing the above orthogonality relationships. This is justified rigorously by basic Hilbert space theory. For the case of orthonormalized harmonics, this gives:

$$f_{\ell}^m = \int_{\Omega} f(\theta, \varphi) Y_{\ell}^{m*}(\theta, \varphi) d\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin \theta f(\theta, \varphi) Y_{\ell}^{m*}(\theta, \varphi).$$

If the coefficients decay in ℓ sufficiently rapidly — for instance, exponentially — then the series also converges uniformly to f .

A real square-integrable function f can be expanded in terms of the real harmonics $Y_{\ell m}$ above as a sum

$$f(\theta, \varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} Y_{\ell m}(\theta, \varphi).$$

Convergence of the series holds again in the same sense.

Spectrum analysis

Power spectrum in signal processing

The total power of a function f is defined in the signal processing literature as the integral of the function squared, divided by the area of its domain. Using the orthonormality properties of the real unit-power spherical harmonic functions, it is straightforward to verify that the total power of a function defined on the unit sphere is related to its spectral coefficients by a generalization of Parseval's theorem:

$$\frac{1}{4\pi} \int_{\Omega} f(\Omega)^2 d\Omega = \sum_{\ell=0}^{\infty} S_{ff}(\ell),$$

where

$$S_{ff}(\ell) = \sum_{m=-\ell}^{\ell} f_{\ell m}^2$$

is defined as the angular power spectrum. In a similar manner, one can define the cross-power of two functions as

$$\frac{1}{4\pi} \int_{\Omega} f(\Omega) g(\Omega) d\Omega = \sum_{\ell=0}^{\infty} S_{fg}(\ell),$$

where

$$S_{fg}(\ell) = \sum_{m=-\ell}^{\ell} f_{\ell m} g_{\ell m}$$

is defined as the cross-power spectrum. If the functions f and g have a zero mean (i.e., the spectral coefficients f_{00} and g_{00} are zero), then $S_{ff}(\ell)$ and $S_{fg}(\ell)$ represent the contributions to the function's variance and covariance for degree ℓ , respectively. It is common that the (cross-)power spectrum is well approximated by a power law of the form

$$S_{ff}(\ell) = C \ell^\beta.$$

When $\beta = 0$, the spectrum is "white" as each degree possesses equal power. When $\beta < 0$, the spectrum is termed "red" as there is more power at the low degrees with long wavelengths than higher degrees. Finally, when $\beta > 0$, the spectrum is termed "blue". The condition on the order of growth of $S_{ff}(\ell)$ is related to the order of differentiability of f in the next section.

Differentiability properties

One can also understand the differentiability properties of the original function f in terms of the asymptotics of $S_{ff}(\ell)$. In particular, if $S_{ff}(\ell)$ decays faster than any rational function of ℓ as $\ell \rightarrow \infty$, then f is infinitely differentiable. If, furthermore, $S_{ff}(\ell)$ decays exponentially, then f is actually real analytic on the sphere.

The general technique is to use the theory of Sobolev spaces. Statements relating the growth of the $S_{ff}(\ell)$ to differentiability are then similar to analogous results on the growth of the coefficients of Fourier series. Specifically, if

$$\sum_{\ell=0}^{\infty} (1 + \ell^2)^s S_{ff}(\ell) < \infty,$$

then f is in the Sobolev space $H^s(S^2)$. In particular, the Sobolev embedding theorem implies that f is infinitely differentiable provided that

$$S_{ff}(\ell) = O(\ell^{-s}) \quad \text{as } \ell \rightarrow \infty$$

for all s .

Algebraic properties

Addition theorem

A mathematical result of considerable interest and use is called the *addition theorem* for spherical harmonics. This is a generalization of the trigonometric identity

$$\cos(\theta' - \theta) = \cos \theta' \cos \theta + \sin \theta' \sin \theta$$

in which the role of the trigonometric functions appearing on the right-hand side is played by the spherical harmonics and that of the left-hand side is played by the Legendre polynomials.

Consider two unit vectors \mathbf{x} and \mathbf{y} , having spherical coordinates (θ, φ) and (θ', φ') , respectively. The addition theorem states

{{{}}}

$$P_\ell(\mathbf{x} \cdot \mathbf{y}) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta', \varphi') Y_{\ell m}(\theta, \varphi). \quad (1)$$

{{{}}}

where P_ℓ is the Legendre polynomial of degree ℓ . This expression is valid for both real and complex harmonics.^[6] The result can be proven analytically, using the properties of the Poisson kernel in the unit ball, or geometrically by applying a rotation to the vector \mathbf{y} so that it points along the z -axis, and then directly calculating the right-hand side.^[7]

In particular, when $\mathbf{x} = \mathbf{y}$, this gives Unsöld's theorem^[8]

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta, \varphi) Y_{\ell m}(\theta, \varphi) = \frac{2\ell + 1}{4\pi}$$

which generalizes the identity $\cos^2\theta + \sin^2\theta = 1$ to two dimensions.

In the expansion (1), the left-hand side $P_\ell(\mathbf{x} \cdot \mathbf{y})$ is a constant multiple of the degree ℓ zonal spherical harmonic. From this perspective, one has the following generalization to higher dimensions. Let Y_j be an arbitrary orthonormal basis of the space \mathbf{H}_ℓ of degree ℓ spherical harmonics on the n -sphere. Then $Z_{\mathbf{x}}^{(\ell)}$, the degree ℓ zonal harmonic corresponding to the unit vector \mathbf{x} , decomposes as^[9]

{{{}}}

$$Z_{\mathbf{x}}^{(\ell)}(\mathbf{y}) = \sum_{j=1}^{\dim(\mathbf{H}_\ell)} \overline{Y_j(\mathbf{x})} Y_j(\mathbf{y}) \quad (2)$$

{{{}}}

Furthermore, the zonal harmonic $Z_{\mathbf{x}}^{(\ell)}(\mathbf{y})$ is given as a constant multiple of the appropriate Gegenbauer polynomial: {{{}}}

$$Z_{\mathbf{x}}^{(\ell)}(\mathbf{y}) = C_\ell^{((n-1)/2)}(\mathbf{x} \cdot \mathbf{y}) \quad (3)$$

{{{}}} Combining (2) and (3) gives (1) in dimension $n = 2$ when \mathbf{x} and \mathbf{y} are represented in spherical coordinates. Finally, evaluating at $\mathbf{x} = \mathbf{y}$ gives the functional identity

$$\frac{\dim \mathbf{H}_\ell}{\omega_{n-1}} = \sum_{j=1}^{\dim(\mathbf{H}_\ell)} |Y_j(\mathbf{x})|^2$$

where ω_{n-1} is the volume of the $(n-1)$ -sphere.

Clebsch-Gordan coefficients

The Clebsch-Gordan coefficients are the coefficients appearing in the expansion of the product of two spherical harmonics in terms of spherical harmonics itself. A variety of techniques are available for doing essentially the same calculation, including the Wigner 3-jm symbol, the Racah coefficients, and the Slater integrals. Abstractly, the Clebsch-Gordan coefficients express the tensor product of two irreducible representations of the rotation group as a sum of irreducible representations: suitably normalized, the coefficients are then the multiplicities.

Parity

The spherical harmonics have well defined parity in the sense that they are either even or odd with respect to reflection about the origin. Reflection about the origin is represented by the operator $P\Psi(\vec{r}) = \Psi(-\vec{r})$. For the spherical angles, $\{\theta, \phi\}$ this corresponds to the replacement $\{\pi - \theta, \pi + \phi\}$. The associated Legendre polynomials gives $(-1)^{\ell-m}$ and from the exponential function we have $(-1)^m$, giving together for the spherical harmonics a parity of $(-1)^\ell$:

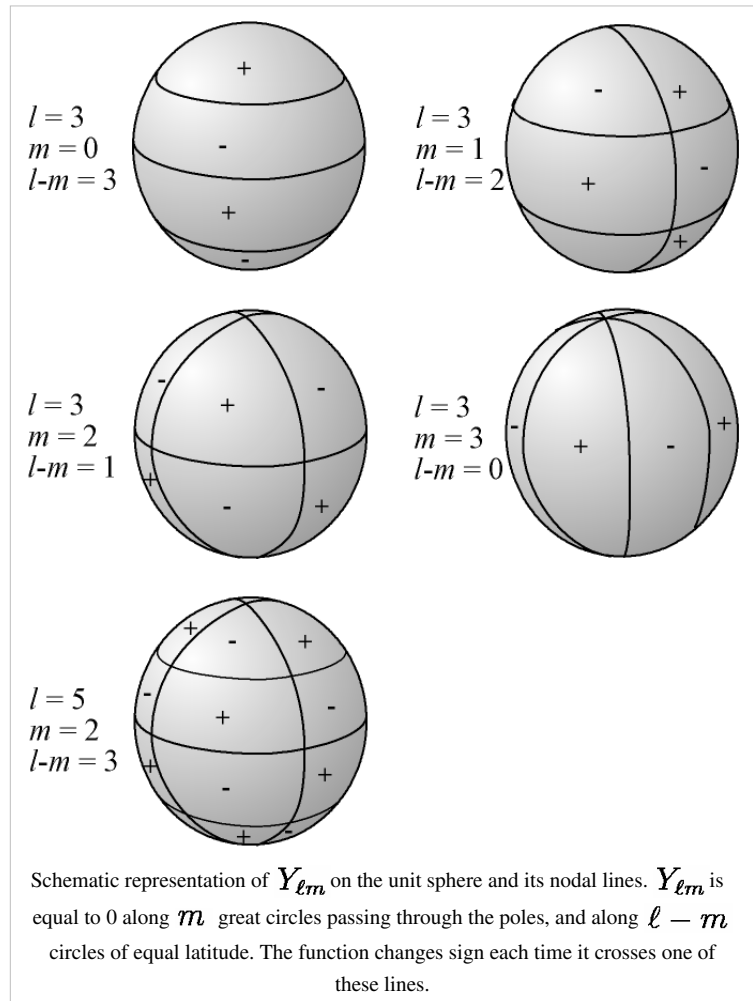
$$Y_\ell^m(\theta, \phi) \rightarrow Y_\ell^m(\pi - \theta, \pi + \phi) = (-1)^\ell Y_\ell^m(\theta, \phi)$$

This remains true for spherical harmonics in higher dimensions: applying a point reflection to a spherical harmonic of degree ℓ changes the sign by a factor of $(-1)^\ell$.

Visualization of the spherical harmonics

The Laplace spherical harmonics Y_ℓ^m can be visualized by considering their "nodal lines", that is, the set of points on the sphere where $Y_\ell^m = 0$. Nodal lines of Y_ℓ^m are composed of circles: some are latitudes and others are longitudes. One can determine the number of nodal lines of each type by counting the number of zeros of Y_ℓ^m in the latitudinal and longitudinal directions independently. For the latitudinal direction, the associated Legendre polynomials possess $\ell - |m|$ zeros, whereas for the longitudinal direction, the trigonometric sin and cos functions possess $2|m|$ zeros.

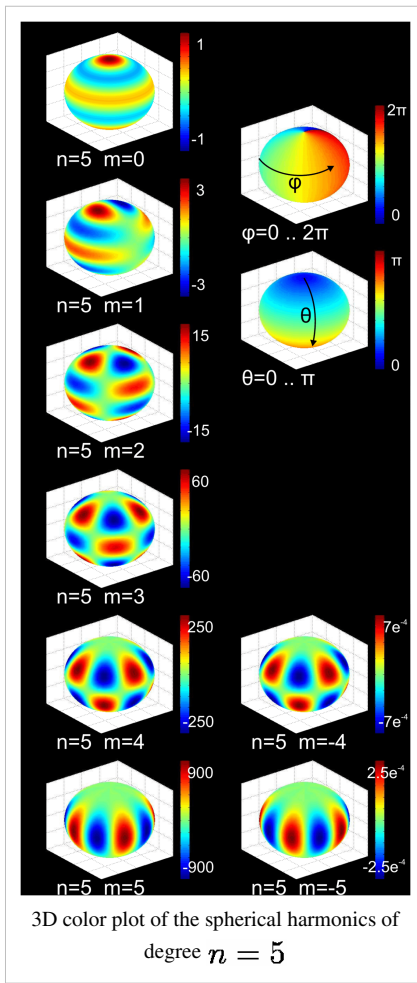
When the spherical harmonic order m is zero (upper-left in the figure), the spherical harmonic functions do not depend upon longitude, and are referred to as **zonal**. Such spherical harmonics are a special case of zonal spherical functions. When $\ell = |m|$ (bottom-right in the figure), there are no zero crossings in latitude, and the functions are referred to as **sectoral**. For the other cases, the functions checker the sphere, and they are referred to as **tesseral**.



More general spherical harmonics of degree ℓ are not necessarily those of the Laplace basis Y_ℓ^m , and their nodal sets can be of a fairly general kind.^[10]

List of spherical harmonics

Analytic expressions for the first few orthonormalized Laplace spherical harmonics that use the Condon-Shortley phase convention:



$$Y_0^0(\theta, \varphi) = \frac{1}{2} \sqrt{\frac{1}{\pi}}$$

$$Y_1^{-1}(\theta, \varphi) = \frac{1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta e^{-i\varphi}$$

$$Y_1^0(\theta, \varphi) = \frac{1}{2} \sqrt{\frac{3}{\pi}} \cos \theta$$

$$Y_1^1(\theta, \varphi) = \frac{-1}{2} \sqrt{\frac{3}{2\pi}} \sin \theta e^{i\varphi}$$

$$Y_2^{-2}(\theta, \varphi) = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{-2i\varphi}$$

$$Y_2^{-1}(\theta, \varphi) = \frac{1}{2} \sqrt{\frac{15}{2\pi}} \sin \theta \cos \theta e^{-i\varphi}$$

$$Y_2^0(\theta, \varphi) = \frac{1}{4} \sqrt{\frac{5}{\pi}} (3 \cos^2 \theta - 1)$$

$$Y_2^1(\theta, \varphi) = \frac{-1}{2} \sqrt{\frac{15}{2\pi}} \sin \theta \cos \theta e^{i\varphi}$$

$$Y_2^2(\theta, \varphi) = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{2i\varphi}$$

Higher dimensions

The classical spherical harmonics are defined as functions on the unit sphere S^2 inside three-dimensional Euclidean space. Spherical harmonics can be generalized to higher dimensional Euclidean space \mathbf{R}^n as follows.^[11] Let \mathbf{P}_ℓ denote the space of homogeneous polynomials of degree ℓ in n variables. That is, a polynomial P is in \mathbf{P}_ℓ provided that

$$P(\lambda \mathbf{x}) = \lambda^\ell P(\mathbf{x}).$$

Let \mathbf{A}_ℓ denote the subspace of \mathbf{P}_ℓ consisting of all harmonic polynomials; these are the solid spherical harmonics. Let \mathbf{H}_ℓ denote the space of functions on the unit sphere

$$S^{n-1} = \{\mathbf{x} \in \mathbf{R}^n \mid |\mathbf{x}| = 1\}$$

obtained by restriction from \mathbf{A}_ℓ .

The following properties hold:

- The spaces \mathbf{H}_ℓ are dense in the set of continuous functions on S^{n-1} with respect to the uniform topology, by the Stone-Weierstrass theorem. As a result, they are also dense in the space $L^2(S^{n-1})$ of square-integrable functions on the sphere.
- For all $f \in \mathbf{H}_\ell$, one has

$$\Delta_{S^{n-1}} f = -\ell(\ell + n - 2)f.$$

where $\Delta_{S^{n-1}}$ is the Laplace-Beltrami operator on S^{n-1} . This operator is the analog of the angular part of the Laplacian in three dimensions; to wit, the Laplacian in n dimensions decomposes as

$$\nabla^2 = r^{1-n} \frac{\partial}{\partial r} r^{n-1} \frac{\partial}{\partial r} + r^{-2} \Delta_{S^{n-1}}.$$

- It follows from the Stokes theorem and the preceding property that the spaces \mathbf{H}_ℓ are orthogonal with respect to the inner product from $L^2(S^{n-1})$. That is to say,

$$\int_{S^{n-1}} f \bar{g} d\Omega = 0$$

for $f \in \mathbf{H}_\ell$ and $g \in \mathbf{H}_k$ for $k \neq \ell$.

- Conversely, the spaces \mathbf{H}_ℓ are precisely the eigenspaces of $\Delta_{S^{n-1}}$. In particular, an application of the spectral theorem to the Riesz potential $\Delta_{S^{n-1}}^{-1}$ gives another proof that the spaces \mathbf{H}_ℓ are pairwise orthogonal and complete in $L^2(S^{n-1})$.
- Every homogeneous polynomial $P \in \mathbf{P}_\ell$ can be uniquely written in the form

$$P(\mathbf{x}) = P_\ell(\mathbf{x}) + |\mathbf{x}|^2 P_{\ell-2} + \cdots + \begin{cases} |\mathbf{x}|^\ell P_0 & \ell \text{ even} \\ |\mathbf{x}|^{\ell-1} P_1(\mathbf{x}) & \ell \text{ odd} \end{cases}$$

where $P_j \in \mathbf{A}_j$. In particular,

$$\dim \mathbf{H}_\ell = \binom{n+\ell-1}{n-1} - \binom{n+\ell-3}{n-1}.$$

An orthogonal basis of spherical harmonics in higher dimensions can be constructed inductively by the method of separation of variables, by solving the Sturm-Liouville problem for the spherical Laplacian

$$\Delta_{S^{n-1}} = \sin^{2-n} \phi \frac{\partial}{\partial \phi} \sin^{n-2} \phi \frac{\partial}{\partial \phi} + \sin^{-2} \phi \Delta_{S^{n-2}}$$

where ϕ is the axial coordinate in a spherical coordinate system on S^{n-1} .

Connection with representation theory

The space \mathbf{H}_ℓ of spherical harmonics of degree ℓ is a representation of the symmetry group of rotations around a point ($\text{SO}(3)$) and its double-cover $\text{SU}(2)$. Indeed, rotations act on the two-dimensional sphere, and thus also on \mathbf{H}_ℓ by function composition

$$\psi \mapsto \psi \circ \rho$$

for ψ a spherical harmonic and ρ a rotation. The representation \mathbf{H}_ℓ is an irreducible representation of $\text{SO}(3)$.

The elements of \mathbf{H}_ℓ arise as the restrictions to the sphere of elements of \mathbf{A}_ℓ : harmonic polynomials homogeneous of degree ℓ on three-dimensional Euclidean space \mathbf{R}^3 . By polarization of $\psi \in \mathbf{A}_\ell$, there are coefficients $\psi_{i_1 \dots i_\ell}$ symmetric on the indices, uniquely determined by the requirement

$$\psi(x_1, \dots, x_n) = \sum_{i_1 \dots i_\ell} \psi_{i_1 \dots i_\ell} x_{i_1} \cdots x_{i_\ell}.$$

The condition that ψ be harmonic is equivalent to the assertion that the tensor $\psi_{i_1 \dots i_\ell}$ must be trace free on every pair of indices. Thus as an irreducible representation of $\text{SO}(3)$, \mathbf{H}_ℓ is isomorphic to the space of traceless symmetric tensors of degree ℓ .

More generally, the analogous statements hold in higher dimensions: the space \mathbf{H}_ℓ of spherical harmonics on the n -sphere is the irreducible representation of $\text{SO}(n+1)$ corresponding to the traceless symmetric ℓ -tensors. However, whereas every irreducible tensor representation of $\text{SO}(2)$ and $\text{SO}(3)$ is of this kind, the special orthogonal groups in higher dimensions have additional irreducible representations that do not arise in this manner.

The special orthogonal groups have additional spin representations that are not tensor representations, and are *typically* not spherical harmonics. An exception are the spin representation of $\text{SO}(3)$: strictly speaking these are representations of the double cover $\text{SU}(2)$ of $\text{SO}(3)$. In turn, $\text{SU}(2)$ is identified with the group of unit quaternions, and so coincides with the 3-sphere. The spaces of spherical harmonics on the 3-sphere are certain spin representations of $\text{SO}(3)$, with respect to the action by quaternionic multiplication.

Generalizations

The angle-preserving symmetries of the two-sphere are described by the group of Möbius transformations $\text{PSL}(2, \mathbf{C})$. With respect to this group, the sphere is equivalent to the usual Riemann sphere. The group $\text{PSL}(2, \mathbf{C})$ is isomorphic to the (proper) Lorentz group, and its action on the two-sphere agrees with the action of the Lorentz group on the celestial sphere in Minkowski space. The analog of the spherical harmonics for the Lorentz group is given by the hypergeometric series; furthermore, the spherical harmonics can be re-expressed in terms of the hypergeometric series, as $\text{SO}(3) = \text{PSU}(2)$ is a subgroup of $\text{PSL}(2, \mathbf{C})$.

More generally, hypergeometric series can be generalized to describe the symmetries of any symmetric space; in particular, hypergeometric series can be developed for any Lie group.^{[12] [13] [14] [15]}

Notes

- [1] A historical account of various approaches to spherical harmonics in three-dimensions can be found in Chapter IV of MacRobert 1967. The term "Laplace spherical harmonics" is in common use; see Courant & Hilbert 1962 and Meijer & Bauer 2004.
- [2] The approach to spherical harmonics taken here is found in (Courant & Hilbert 1966, §V.8, §VII.5).
- [3] Physical applications often take the solution that vanishes at infinity, making $A = 0$. This does not affect the angular portion of the spherical harmonics.
- [4] Edmonds 1957, §2.5
- [5] Watson & Whittaker 1927, p. 392.
- [6] This is valid for any orthonormal basis of spherical harmonics of degree ℓ . For unit power harmonics it is necessary to remove the factor of 4π .
- [7] Watson & Whittaker 1927, p. 395
- [8] Unsöld 1927
- [9] Stein & Weiss 1971, §IV.2

- [10] Eremenko, Jakobson & Nadirashvili 2007
- [11] Solomentsev 2001; Stein & Weiss 1971, §IV.2
- [12] N. Vilenkin, *Special Functions and the Theory of Group Representations*, Am. Math. Soc. Transl., vol. 22, (1968).
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External links

- Interactive calculator of spherical harmonics on Tal Carmon's Research Homepage (<http://wm.eecs.umich.edu:8180/webMathematica/tcarmon/sh2.jsp>)
- Spherical harmonics applied to Acoustic Field analysis on Trinnov Audio's research page (<http://www.trinnov.com/en/about-us/research/overview>)
- Spherical Harmonics (<http://demonstrations.wolfram.com/SphericalHarmonics/>) by Stephen Wolfram and Nodal Domains of Spherical Harmonics (<http://demonstrations.wolfram.com/NodalDomainsOfSphericalHarmonics/>) by Michael Trott, the Wolfram Demonstrations Project
- An accessible introduction to spherical harmonics (by J. B. Calvert) (<http://mysite.du.edu/~jcalvert/math/harmonic/harmonic.htm>)
- Citizendium:Spherical harmonics
- OpenGL Spherical harmonics demo (<http://www.paulsprojects.net/opengl/sh/sh.html>)
- Allen McNamara's spherical harmonics animations (http://mcnamara.asu.edu/sphere_harmonics/index.html)
- Thorsten Becker's spherical harmonics animations (<http://geodynamics.usc.edu/~becker/teaching-sh.html>)

Software

- Spherical harmonics generator in OpenGL (<http://adomas.org/shg/>)
 - SHTOOLS: Fortran 95 software archive (<http://www.ipgp.jussieu.fr/~wieczor/SHTOOLS/SHTOOLS.html>)
 - HEALPIX: Fortran 90 and C++ software archive (<http://healpix.jpl.nasa.gov/>)
 - SpherePack: Fortran 77 software archive (<http://www.cisl.ucar.edu/css/software/spherepack/>)
 - SpharmonicKit: C software archive (<http://www.cs.dartmouth.edu/~geelong/sphere/>)
 - Frederik J Simons: Matlab software archive (<http://geoweb.princeton.edu/people/simons/software.html>)
 - NFFT: C subroutine library (fast spherical Fourier transform for arbitrary nodes) (<http://www-user.tu-chemnitz.de/~potts/nfft/>)
 - Shansyn: spherical harmonics package for GMT/netcdf grd files (<http://www.spice-rtn.org/library/software/shansyn>)
 - SHAPE: Spherical HArmonic Parameterization Explorer (<http://www.embl-heidelberg.de/~khairy/links.html>)
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Quantum computer

A **quantum computer** is a device for computation that makes direct use of quantum mechanical phenomena, such as superposition and entanglement, to perform operations on data. Quantum computers are different from traditional computers based on transistors. The basic principle behind quantum computation is that quantum properties can be used to represent data and perform operations on these data.^[1] A theoretical model is the quantum Turing machine, also known as the universal quantum computer.

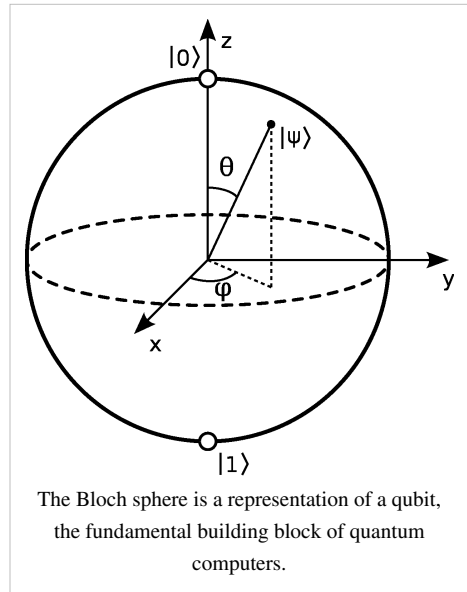
Although quantum computing is still in its infancy, experiments have been carried out in which quantum computational operations were executed on a very small number of qubits (quantum bits). Both practical and theoretical research continues, and many national government and military funding agencies support quantum computing research to develop quantum computers for both civilian and national security purposes, such as cryptanalysis.^[2]

If large-scale quantum computers can be built, they will be able to solve certain problems much faster than any classical computer using the best currently known algorithms (for example integer factorization using Shor's algorithm or the simulation of quantum many-body systems). Furthermore, there exist quantum algorithms, such as Simon's algorithm, which run exponentially faster than any possible probabilistic classical algorithm.^[3] Given enough resources, a classical computer can simulate an arbitrary quantum computer. Hence, ignoring computational and space constraints, a quantum computer is not capable of solving any problem which a classical computer cannot.^[4]

Basis

A classical computer has a memory made up of bits, where each bit represents either a one or a zero. A quantum computer maintains a sequence of qubits. A single qubit can represent a one, a zero, or, crucially, any quantum superposition of these; moreover, a pair of qubits can be in any quantum superposition of 4 states, and three qubits in any superposition of 8. In general a quantum computer with n qubits can be in an arbitrary superposition of up to 2^n different states simultaneously (this compares to a normal computer that can only be in *one* of these 2^n states at any one time). A quantum computer operates by manipulating those qubits with a fixed sequence of quantum logic gates. The sequence of gates to be applied is called a quantum algorithm.

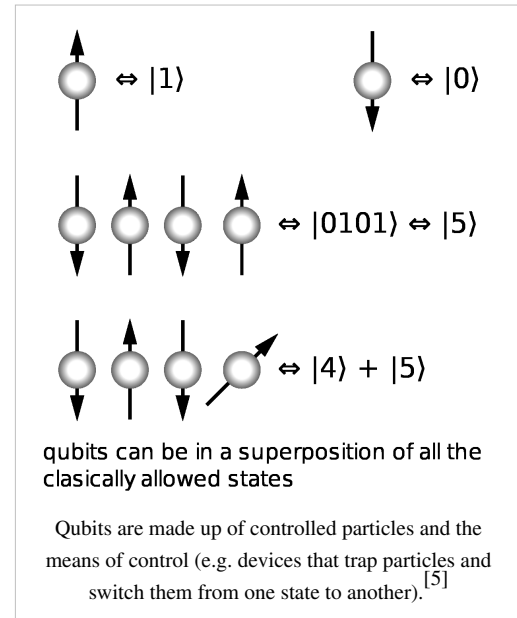
An example of an implementation of qubits for a quantum computer could start with the use of particles with two spin states: "down" and "up" (typically written $|\downarrow\rangle$ and $|\uparrow\rangle$, or $|0\rangle$ and $|1\rangle$). But in fact any system possessing an observable quantity A which is *conserved* under time evolution and such that A has at least two discrete and sufficiently spaced consecutive eigenvalues, is a suitable candidate for implementing a qubit. This is true because any such system can be mapped onto an effective spin-1/2 system.



Bits vs. qubits

A quantum computer with a given number of qubits is exponentially more complex than a classical computer with the same number of bits because describing the state of n qubits requires 2^n complex coefficients. Measuring the qubits would produce a classical state of only n bits, but such an action would also destroy the quantum state. We can think of the system as being exactly one of the n -bit strings—we just do not know which one. For example, a 300-qubit quantum computer has a state described by 2^{300} (approximately 10^{90}) complex numbers, more than the number of atoms in the observable universe.

For example: Consider first a classical computer that operates on a three-bit register. The state of the computer at any time is a probability distribution over the $2^3 = 8$ different three-bit strings 000, 001, 010, 011, 100, 101, 110, 111. If it is a deterministic computer, then it is in exactly one of these states with probability 1. However, if it is a probabilistic computer, then there is a possibility of it being in any *one* of a number of different states. We can describe this probabilistic state by eight nonnegative numbers a, b, c, d, e, f, g, h (where a = probability computer is in state 000, b = probability computer is in state 001, etc.). There is a restriction that these probabilities sum to 1.



The state of a three-qubit quantum computer is similarly described by an eight-dimensional vector (a, b, c, d, e, f, g, h) , called a ket. However, instead of adding to one, the sum of the *squares* of the coefficient magnitudes, $|a|^2 + |b|^2 + \dots + |h|^2$, must equal one. Moreover, the coefficients are complex numbers. Since states are represented by complex wavefunctions, two states being added together will undergo interference, which is a key difference between quantum computing and probabilistic classical computing.^[6]

If you measure the three qubits, you will observe a three-bit string. The probability of measuring a given string is the squared magnitude of that string's coefficient (i.e., the probability of measuring 000 = $|a|^2$, the probability of measuring 001 = $|b|^2$, etc.). Thus, measuring a quantum state described by complex coefficients (a, b, \dots, h) gives the classical probability distribution $(|a|^2, |b|^2, \dots, |h|^2)$ and we say that the quantum state "collapses" to a classical state as a result of making the measurement.

Note that an eight-dimensional vector can be specified in many different ways depending on what basis is chosen for the space. The basis of bit strings (e.g., 000, 001, ..., 111) is known as the computational basis. Other possible bases are unit-length, orthogonal vectors and the eigenvectors of the Pauli-x operator. Ket notation is often used to make the choice of basis explicit. For example, the state (a, b, c, d, e, f, g, h) in the computational basis can be written as:

$$a |000\rangle + b |001\rangle + c |010\rangle + d |011\rangle + e |100\rangle + f |101\rangle + g |110\rangle + h |111\rangle$$

$$\text{where, e.g., } |010\rangle = (0, 0, 1, 0, 0, 0, 0, 0)$$

The computational basis for a single qubit (two dimensions) is $|0\rangle = (1, 0)$ and $|1\rangle = (0, 1)$.

Using the eigenvectors of the Pauli-x operator, a single qubit is $|+\rangle = \frac{1}{\sqrt{2}}(1, 1)$ and $|-\rangle = \frac{1}{\sqrt{2}}(1, -1)$.

Operation

While a classical three-bit state and a quantum three-qubit state are both eight-dimensional vectors, they are manipulated quite differently for classical or quantum computation. For computing in either case, the system must be initialized, for example into the all-zeros string, $|000\rangle$, corresponding to the vector $(1,0,0,0,0,0,0,0)$. In classical randomized computation, the system evolves according to the application of stochastic matrices, which preserve that the probabilities add up to one (i.e., preserve the L1 norm). In quantum computation, on the other hand, allowed operations are unitary matrices, which are effectively rotations (they preserve that the sum of the squares add up to one, the Euclidean or L2 norm). (Exactly what unitaries can be applied depend on the physics of the quantum device.) Consequently, since rotations can be undone by rotating backward, quantum computations are reversible. (Technically, quantum operations can be probabilistic combinations of unitaries, so quantum computation really does generalize classical computation. See quantum circuit for a more precise formulation.)

Finally, upon termination of the algorithm, the result needs to be read off. In the case of a classical computer, we *sample* from the probability distribution on the three-bit register to obtain one definite three-bit string, say 000. Quantum mechanically, we *measure* the three-qubit state, which is equivalent to collapsing the quantum state down to a classical distribution (with the coefficients in the classical state being the squared magnitudes of the coefficients for the quantum state, as described above) followed by sampling from that distribution. Note that this destroys the original quantum state. Many algorithms will only give the correct answer with a certain probability, however by repeatedly initializing, running and measuring the quantum computer, the probability of getting the correct answer can be increased.

For more details on the sequences of operations used for various quantum algorithms, see universal quantum computer, Shor's algorithm, Grover's algorithm, Deutsch-Jozsa algorithm, amplitude amplification, quantum Fourier transform, quantum gate, quantum adiabatic algorithm and quantum error correction.

Potential

Integer factorization is believed to be computationally infeasible with an ordinary computer for large integers if they are the product of few prime numbers (e.g., products of two 300-digit primes).^[7] By comparison, a quantum computer could efficiently solve this problem using Shor's algorithm to find its factors. This ability would allow a quantum computer to decrypt many of the cryptographic systems in use today, in the sense that there would be a polynomial time (in the number of digits of the integer) algorithm for solving the problem. In particular, most of the popular public key ciphers are based on the difficulty of factoring integers (or the related discrete logarithm problem which can also be solved by Shor's algorithm), including forms of RSA. These are used to protect secure Web pages, encrypted email, and many other types of data. Breaking these would have significant ramifications for electronic privacy and security.

However, other existing cryptographic algorithms do not appear to be broken by these algorithms.^{[8] [9]} Some public-key algorithms are based on problems other than the integer factorization and discrete logarithm problems to which Shor's algorithm applies, like the McEliece cryptosystem based on a problem in coding theory.^{[8] [10]} Lattice based cryptosystems are also not known to be broken by quantum computers, and finding a polynomial time algorithm for solving the dihedral hidden subgroup problem, which would break many lattice based cryptosystems, is a well-studied open problem.^[11] It has been proven that applying Grover's algorithm to break a symmetric (secret key) algorithm by brute force requires roughly $2^{n/2}$ invocations of the underlying cryptographic algorithm, compared with roughly 2^n in the classical case,^[12] meaning that symmetric key lengths are effectively halved: AES-256 would have the same security against an attack using Grover's algorithm that AES-128 has against classical brute-force search (see Key size). Quantum cryptography could potentially fulfill some of the functions of public key cryptography.

Besides factorization and discrete logarithms, quantum algorithms offering a more than polynomial speedup over the best known classical algorithm have been found for several problems,^[13] including the simulation of quantum physical processes from chemistry and solid state physics, the approximation of Jones polynomials, and solving Pell's equation. No mathematical proof has been found that shows that an equally fast classical algorithm cannot be discovered, although this is considered unlikely. For some problems, quantum computers offer a polynomial speedup. The most well-known example of this is *quantum database search*, which can be solved by Grover's algorithm using quadratically fewer queries to the database than are required by classical algorithms. In this case the advantage is provable. Several other examples of provable quantum speedups for query problems have subsequently been discovered, such as for finding collisions in two-to-one functions and evaluating NAND trees.

Consider a problem that has these four properties:

1. The only way to solve it is to guess answers repeatedly and check them,
2. The number of possible answers to check is the same as the number of inputs,
3. Every possible answer takes the same amount of time to check, and
4. There are no clues about which answers might be better: generating possibilities randomly is just as good as checking them in some special order.

An example of this is a password cracker that attempts to guess the password for an encrypted file (assuming that the password has a maximum possible length).

For problems with all four properties, the time for a quantum computer to solve this will be proportional to the square root of the number of inputs. That can be a very large speedup, reducing some problems from years to seconds. It can be used to attack symmetric ciphers such as Triple DES and AES by attempting to guess the secret key.

Grover's algorithm can also be used to obtain a quadratic speed-up over a brute-force search for a class of problems known as NP-complete.

Since chemistry and nanotechnology rely on understanding quantum systems, and such systems are impossible to simulate in an efficient manner classically, many believe quantum simulation will be one of the most important applications of quantum computing.^[14]

There are a number of practical difficulties in building a quantum computer, and thus far quantum computers have only solved trivial problems. David DiVincenzo, of IBM, listed the following requirements for a practical quantum computer:^[15]

- scalable physically to increase the number of qubits;
- qubits can be initialized to arbitrary values;
- quantum gates faster than decoherence time;
- universal gate set;
- qubits can be read easily.

Quantum decoherence

One of the greatest challenges is controlling or removing quantum decoherence. This usually means isolating the system from its environment as the slightest interaction with the external world would cause the system to decohere. This effect is irreversible, as it is non-unitary, and is usually something that should be highly controlled, if not avoided. Decoherence times for candidate systems, in particular the transverse relaxation time T_2 (for NMR and MRI technology, also called the *dephasing time*), typically range between nanoseconds and seconds at low temperature.^[6]

These issues are more difficult for optical approaches as the timescales are orders of magnitude shorter and an often-cited approach to overcoming them is optical pulse shaping. Error rates are typically proportional to the ratio of operating time to decoherence time, hence any operation must be completed much more quickly than the decoherence time.

If the error rate is small enough, it is thought to be possible to use quantum error correction, which corrects errors due to decoherence, thereby allowing the total calculation time to be longer than the decoherence time. An often cited figure for required error rate in each gate is 10^{-4} . This implies that each gate must be able to perform its task in one 10,000th of the decoherence time of the system.

Meeting this scalability condition is possible for a wide range of systems. However, the use of error correction brings with it the cost of a greatly increased number of required qubits. The number required to factor integers using Shor's algorithm is still polynomial, and thought to be between L and L^2 , where L is the number of bits in the number to be factored; error correction algorithms would inflate this figure by an additional factor of L . For a 1000-bit number, this implies a need for about 10^4 qubits without error correction.^[16] With error correction, the figure would rise to about 10^7 qubits. Note that computation time is about L^2 or about 10^7 steps and on 1 MHz, about 10 seconds.

A very different approach to the stability-decoherence problem is to create a topological quantum computer with anyons, quasi-particles used as threads and relying on braid theory to form stable logic gates.^{[17] [18]}

Developments

There are a number of quantum computing *models*, distinguished by the basic elements in which the computation is decomposed. The four main models of practical importance are

- the *quantum gate array* (computation decomposed into sequence of few-qubit quantum gates),
- the *one-way quantum computer* (computation decomposed into sequence of one-qubit measurements applied to a highly entangled initial state (cluster state)),
- the *adiabatic quantum computer* (computation decomposed into a slow continuous transformation of an initial Hamiltonian into a final Hamiltonian, whose ground states contains the solution),
- and the topological quantum computer^[19] (computation decomposed into the braiding of anyons in a 2D lattice)

The *Quantum Turing machine* is theoretically important but direct implementation of this model is not pursued. All four models of computation have been shown to be equivalent to each other in the sense that each can simulate the other with no more than polynomial overhead.

For physically implementing a quantum computer, many different candidates are being pursued, among them (distinguished by the physical system used to realize the qubits):

- Superconductor-based quantum computers (including SQUID-based quantum computers)^{[20] [21]} (qubit implemented by the state of small superconducting circuits (Josephson junctions))
- Trapped ion quantum computer (qubit implemented by the internal state of trapped ions)
- Optical lattices (qubit implemented by internal states of neutral atoms trapped in an optical lattice)
- electrically-defined or self-assembled quantum dots (e.g. the Loss-DiVincenzo quantum computer or^[22]) (qubit given by the spin states of an electron trapped in the quantum dot)
- Quantum dot charge based semiconductor quantum computer (qubit is the position of an electron inside a double quantum dot)^[23]
- Nuclear magnetic resonance on molecules in solution (liquid-state NMR) (qubit provided by nuclear spins within the dissolved molecule)
- Solid-state NMR Kane quantum computers (qubit realized by the nuclear spin state of phosphorus donors in silicon)
- Electrons-on-helium quantum computers (qubit is the electron spin)
- Cavity quantum electrodynamics (CQED) (qubit provided by the internal state of atoms trapped in and coupled to high-finesse cavities)
- Molecular magnet
- Fullerene-based ESR quantum computer (qubit based on the electronic spin of atoms or molecules encased in fullerene structures)

- Optics-based quantum computer (Quantum optics) (qubits realized by appropriate states of different modes of the electromagnetic field, e.g.^[24])
- Diamond-based quantum computer^{[25] [26] [27]} (qubit realized by the electronic or nuclear spin of Nitrogen-vacancy centers in diamond)
- Bose–Einstein condensate-based quantum computer^[28]
- Transistor-based quantum computer - string quantum computers with entrainment of positive holes using an electrostatic trap
- Rare-earth-metal-ion-doped inorganic crystal based quantum computers^{[29] [30]} (qubit realized by the internal electronic state of dopants in optical fibers)

The large number of candidates demonstrates that the topic, in spite of rapid progress, is still in its infancy. But at the same time, there is also a vast amount of flexibility.

In 2005, researchers at the University of Michigan built a semiconductor chip which functioned as an ion trap. Such devices, produced by standard lithography techniques, may point the way to scalable quantum computing tools.^[31] An improved version was made in 2006.

In 2009, researchers at Yale University created the first rudimentary solid-state quantum processor. The two-qubit superconducting chip was able to run elementary algorithms. Each of the two artificial atoms (or qubits) were made up of a billion aluminum atoms but they acted like a single one that could occupy two different energy states.^{[32] [33]}

Another team, working at the University of Bristol, also created a silicon-based quantum computing chip, based on quantum optics. The team was able to run Shor's algorithm on the chip.^[34] The latest developments [for 2010] can be found in.^[35] Springer publish a Journal devoted to the subject [36].

A team of scientists from Australia and Japan have finally made a breakthrough in quantum teleportation. They have successfully transferred a complex set of quantum data with full transmission integrity achieved. Also the qubits being destroyed in one place but instantaneously resurrected in another, without affecting their superpositions.^[37]

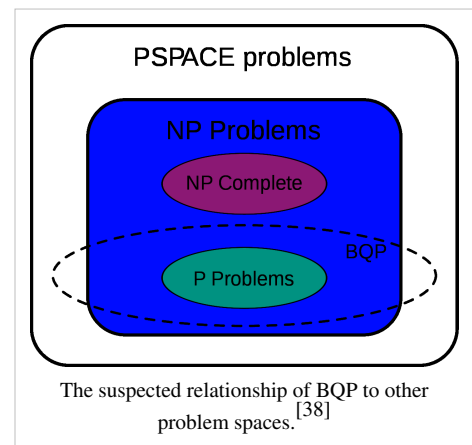
Relation to computational complexity theory

The class of problems that can be efficiently solved by quantum computers is called BQP, for "bounded error, quantum, polynomial time". Quantum computers only run probabilistic algorithms, so BQP on quantum computers is the counterpart of BPP ("bounded error, probabilistic, polynomial time") on classical computers. It is defined as the set of problems solvable with a polynomial-time algorithm, whose probability of error is bounded away from one half.^[39] A quantum computer is said to "solve" a problem if, for every instance, its answer will be right with high probability. If that solution runs in polynomial time, then that problem is in BQP.

BQP is contained in the complexity class $\#P$ (or more precisely in the associated class of decision problems $P^{\#P}$),^[40] which is a subclass of PSPACE.

BQP is suspected to be disjoint from NP-complete and a strict superset of P, but that is not known. Both integer factorization and discrete log are in BQP. Both of these problems are NP problems suspected to be outside BPP, and hence outside P. Both are suspected to not be NP-complete. There is a common misconception that quantum computers can solve NP-complete problems in polynomial time. That is not known to be true, and is generally suspected to be false.^[40]

Possibilities of the quantum computer to accelerate classical algorithms has rigid limits — upper bounds of quantum computation's complexity. The overwhelming part of classical calculations cannot be accelerated on the quantum



computer.^[41] A similar fact takes place for particular computational tasks, like the search problem, for which Grover's algorithm is optimal.^[42]

Although quantum computers may be faster than classical computers, those described above can't solve any problems that classical computers can't solve, given enough time and memory (however, those amounts might be practically infeasible). A Turing machine can simulate these quantum computers, so such a quantum computer could never solve an undecidable problem like the halting problem. The existence of "standard" quantum computers does not disprove the Church–Turing thesis.^[43] It has been speculated that theories of quantum gravity, such as M-theory or loop quantum gravity, may allow even faster computers to be built. Currently, it's an open problem to even *define* computation in such theories due to the *problem of time*, i.e. there's no obvious way to describe what it means for an observer to submit input to a computer and later receive output.^[44]

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Topological Quantum Computers

A **topological quantum computer** is a theoretical quantum computer that employs two-dimensional quasiparticles called anyons, whose world lines cross over one another to form braids in a three-dimensional spacetime (i.e., one temporal plus two spatial dimensions). These braids form the logic gates that make up the computer. The advantage of a quantum computer based on quantum braids over using trapped quantum particles is that the former is much more stable. The smallest perturbations can cause a quantum particle to decohere and introduce errors in the computation, but such small perturbations do not change the topological properties of the braids. This is like the effort required to cut a string and reattach the ends to form a different braid, as opposed to a ball (representing an ordinary quantum particle in four-dimensional spacetime) simply bumping into a wall. While the elements of a topological quantum computer originate in a purely mathematical realm, recent experiments indicate these elements can be created in the real world using semiconductors made of gallium arsenide near absolute zero and subjected to strong magnetic fields.

Introduction

Anyons are quasiparticles in a two-dimensional space. Anyons are not strictly fermions or bosons, but do share the characteristic of fermions in that they cannot occupy the same state. Thus, the world lines of two anyons cannot cross or merge. This allows braids to be made that make up a particular circuit. In the real world, anyons form from the excitations in an electron gas in a very strong magnetic field, and carry fractional units of magnetic flux in a particle-like manner. This phenomenon is called the fractional quantum Hall effect. The electron "gas" is sandwiched between two flat plates of gallium arsenide, which create the two-dimensional space required for anyons, and is cooled and subjected to intense transverse magnetic fields.

When anyons are braided, the transformation of the quantum state of the system depends only on the topological class of the anyons' trajectories (which are classified according to the braid group). Therefore, the quantum information which is stored in the state of the system is impervious to small errors in the trajectories. In 2005, Sankar Das Sarma, Michael Freedman, and Chetan Nayak proposed a quantum Hall device which would realize a

topological qubit. In a key development for topological quantum computers, in 2005 Vladimir J. Goldman, Fernando E. Camino, and Wei Zhou were said to have created the first experimental evidence for using fractional quantum Hall effect to create actual anyons, although others have suggested their results could be the product of phenomena not involving anyons. It should also be noted that nonabelian anyons, a species required for topological quantum computers, have yet to be experimentally confirmed.

The original proposal for topological quantum computation is due to Alexei Kitaev in 1997.

Topological vs. standard quantum computer

Topological quantum computers are equivalent in computational power to other standard models of quantum computation, in particular to the quantum circuit model and to the quantum Turing machine model. That is, any of these models can efficiently simulate any of the others. Nonetheless, certain algorithms may be a more natural fit to the topological quantum computer model. For example, algorithms for evaluating the Jones polynomial were first developed in the topological model, and only later converted and extended in the standard quantum circuit model.

Computations

To live up to its name, a topological quantum computer must provide the unique computation properties promised by a conventional quantum computer design, which uses trapped quantum particles. Fortunately in 2002, Michael H. Freedman along with Zhenghan Wang, both with Microsoft, and Michael Larsen of Indiana University proved that a topological quantum computer can, in principle, perform any computation that a conventional quantum computer can do.

They found that conventional quantum computer device, given a flawless (error-free) operation of its logic circuits, will give a solution with an absolute level of accuracy, whereas a topological quantum computing device with flawless operation will give the solution with only a finite level of accuracy. However, any level of precision for the answer can be obtained by adding more braid twists (logic circuits) to the topological quantum computer, in a simple linear relationship. In other words, a reasonable increase in elements (braid twists) can achieve a high degree of accuracy in the answer. Actual computation [gates] are done by edge states of fractional quantum Hall effect. This makes models one dimensional anyons important. In one space dimension anyons are defined algebraically.

Error correction and control

Even though quantum braids are inherently more stable than trapped quantum particles, there is still a need to control for error inducing thermal fluctuations, which produce random stray pairs of anyons which interfere with adjoining braids. Controlling these errors is simply a matter of separating the anyons to a distance where the rate of interfering strays drops to near zero. It has been estimated that the error rate for a logical NOT operation of a qubit state could be as low as 10^{-30} or less. Although this number has been criticized as being strongly overstated, there is nonetheless good reason to believe that topologically protected systems will be particularly immune to many sources of error that plague other schemes for quantum information processing.

Simulating the dynamics of a topological quantum computer may be a promising method of implementing fault-tolerant quantum computation even with a standard quantum information processing scheme. Raussendorf, Harrington, and Goyal have studied one model, with promising simulation results.

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Categorical and Topological Dynamics.

Category Theory and Categorical Dynamics

Concepts

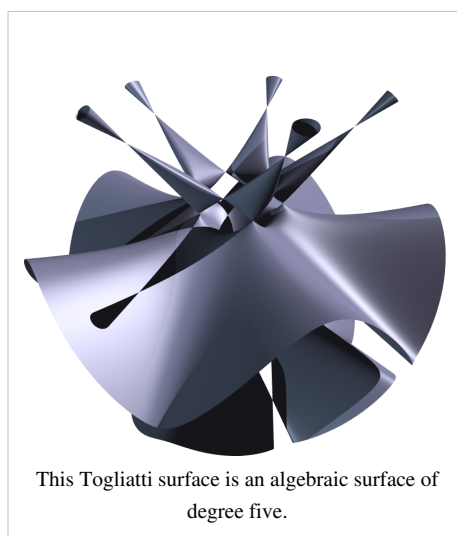
Algebraic Geometry

Algebraic geometry is a branch of mathematics which combines techniques of abstract algebra, especially commutative algebra, with the language and the problems of geometry. It occupies a central place in modern mathematics and has multiple conceptual connections with such diverse fields as complex analysis, topology and number theory. Initially a study of systems of polynomial equations in several variables, the subject of algebraic geometry starts where equation solving leaves off, and it becomes even more important to understand the intrinsic properties of the totality of solutions of a system of equations, than to find some solution; this leads into some of the deepest waters in the whole of mathematics, both conceptually and in terms of technique.

The fundamental objects of study in algebraic geometry are **algebraic varieties**, geometric manifestations of solutions of systems of polynomial equations. Plane algebraic curves, which include lines, circles, parabolas, lemniscates, and Cassini ovals, form one of the best studied classes of algebraic varieties. A point of the plane belongs to an algebraic curve if its coordinates satisfy a given polynomial equation. Basic questions involve relative position of different curves and relations between the curves given by different equations.

Descartes's idea of coordinates is central to algebraic geometry, but it has undergone a series of remarkable transformations beginning in the early 19th century. Before then, the coordinates were assumed to be tuples of real numbers, but this changed when first complex numbers, and then elements of an arbitrary field became acceptable. Homogeneous coordinates of projective geometry offered an extension of the notion of coordinate system in a different direction, and enriched the scope of algebraic geometry. Much of the development of algebraic geometry in the 20th century occurred within an abstract algebraic framework, with increasing emphasis being placed on 'intrinsic' properties of algebraic varieties not dependent on any particular way of embedding the variety in an ambient coordinate space; this parallels developments in topology and complex geometry.

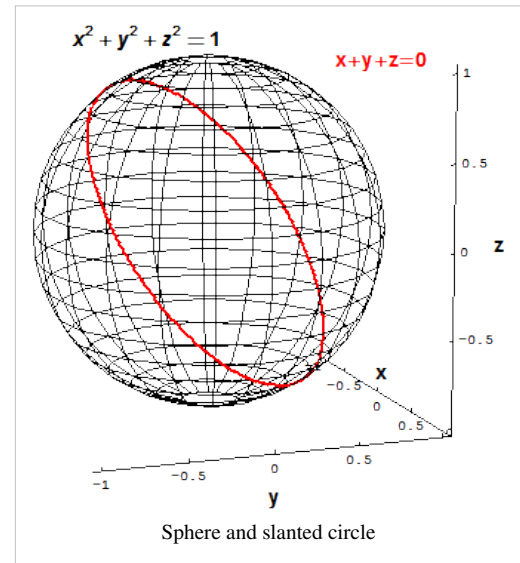
One key distinction between classical projective geometry of 19th century and modern algebraic geometry, in the form given to it by Grothendieck and Serre, is that the former is concerned with the more geometric notion of a point, while the latter emphasizes the more analytic concepts of a regular function and a regular map and extensively draws on sheaf theory. Another important difference lies in the scope of the subject. Grothendieck's idea of **scheme** provides the language and the tools for geometric treatment of arbitrary commutative rings and, in particular, bridges algebraic geometry with algebraic number theory. Andrew Wiles's celebrated proof of Fermat's last theorem is a vivid testament to the power of this approach. André Weil, Grothendieck, and Deligne also demonstrated that the fundamental ideas of topology of manifolds have deep analogues in algebraic geometry over finite fields.



This Togliatti surface is an algebraic surface of degree five.

Zeros of simultaneous polynomials

In classical algebraic geometry, the main objects of interest are the vanishing sets of collections of polynomials, meaning the set of all points that simultaneously satisfy one or more polynomial equations. For instance, the two-dimensional sphere in three-dimensional Euclidean space \mathbf{R}^3 could be defined as the set of all points (x,y,z) with



$$x^2 + y^2 + z^2 - 1 = 0.$$

A "slanted" circle in \mathbf{R}^3 can be defined as the set of all points (x,y,z) which satisfy the two polynomial equations

$$\begin{aligned} x^2 + y^2 + z^2 - 1 &= 0, \\ x + y + z &= 0. \end{aligned}$$

Affine varieties

First we start with a field k . In classical algebraic geometry, this field was always the complex numbers \mathbf{C} , but many of the same results are true if we assume only that k is algebraically closed. We define $\mathbf{A}^n(k)$ (or more simply \mathbf{A}^n , when k is clear from the context), called the **affine n -space over k** , to be k^n . The purpose of this apparently superfluous notation is to emphasize that one 'forgets' the vector space structure that k^n carries. Abstractly speaking, \mathbf{A}^n is, for the moment, just a collection of points.

A function $f: \mathbf{A}^n \rightarrow \mathbf{A}^1$ is said to be **regular** if it can be written as a polynomial, that is, if there is a polynomial p in $k[x_1, \dots, x_n]$ such that $f(t_1, \dots, t_n) = p(t_1, \dots, t_n)$ for every point (t_1, \dots, t_n) of \mathbf{A}^n .

Regular functions on affine n -space are thus exactly the same as polynomials over k in n variables. We will refer to the set of all regular functions on \mathbf{A}^n as $k[\mathbf{A}^n]$.

We say that a polynomial *vanishes* at a point if evaluating it at that point gives zero. Let S be a set of polynomials in $k[\mathbf{A}^n]$. The *vanishing set of S* (or *vanishing locus*) is the set $V(S)$ of all points in \mathbf{A}^n where every polynomial in S vanishes. In other words,

$$V(S) = \{(t_1, \dots, t_n) \mid \forall p \in S, p(t_1, \dots, t_n) = 0\}.$$

A subset of \mathbf{A}^n which is $V(S)$, for some S , is called an **algebraic set**. The V stands for *variety* (a specific type of algebraic set to be defined below).

Given a subset U of \mathbf{A}^n , can one recover the set of polynomials which generate it? If U is *any* subset of \mathbf{A}^n , define $I(U)$ to be the set of all polynomials whose vanishing set contains U . The I stands for *ideal*: if two polynomials f and g both vanish on U , then $f+g$ vanishes on U , and if h is any polynomial, then hf vanishes on U , so $I(U)$ is always an ideal of $k[\mathbf{A}^n]$.

Two natural questions to ask are:

- Given a subset U of \mathbf{A}^n , when is $U = V(I(U))$?

- Given a set S of polynomials, when is $S = I(V(S))$?

The answer to the first question is provided by introducing the Zariski topology, a topology on \mathbf{A}^n which directly reflects the algebraic structure of $k[\mathbf{A}^n]$. Then $U = V(I(U))$ if and only if U is a Zariski-closed set. The answer to the second question is given by Hilbert's Nullstellensatz. In one of its forms, it says that $I(V(S))$ is the prime radical of the ideal generated by S . In more abstract language, there is a Galois connection, giving rise to two closure operators; they can be identified, and naturally play a basic role in the theory; the example is elaborated at Galois connection.

For various reasons we may not always want to work with the entire ideal corresponding to an algebraic set U . Hilbert's basis theorem implies that ideals in $k[\mathbf{A}^n]$ are always finitely generated.

An algebraic set is called **irreducible** if it cannot be written as the union of two smaller algebraic sets. An irreducible algebraic set is also called a **variety**. It turns out that an algebraic set is a variety if and only if the polynomials defining it generate a prime ideal of the polynomial ring.

Regular functions

Just as continuous functions are the natural maps on topological spaces and smooth functions are the natural maps on differentiable manifolds, there is a natural class of functions on an algebraic set, called regular functions. A **regular function** on an algebraic set V contained in \mathbf{A}^n is defined to be the restriction of a regular function on \mathbf{A}^n , in the sense we defined above.

It may seem unnaturally restrictive to require that a regular function always extend to the ambient space, but it is very similar to the situation in a normal topological space, where the Tietze extension theorem guarantees that a continuous function on a closed subset always extends to the ambient topological space.

Just as with the regular functions on affine space, the regular functions on V form a ring, which we denote by $k[V]$. This ring is called the **coordinate ring of V** .

Since regular functions on V come from regular functions on \mathbf{A}^n , there should be a relationship between their coordinate rings. Specifically, to get a function in $k[V]$ we took a function in $k[\mathbf{A}^n]$, and we said that it was the same as another function if they gave the same values when evaluated on V . This is the same as saying that their difference is zero on V . From this we can see that $k[V]$ is the quotient $k[\mathbf{A}^n]/I(V)$.

The category of affine varieties

Using regular functions from an affine variety to \mathbf{A}^1 , we can define regular functions from one affine variety to another. First we will define a regular function from a variety into affine space: Let V be a variety contained in \mathbf{A}^n . Choose m regular functions on V , and call them f_1, \dots, f_m . We define a **regular function** f from V to \mathbf{A}^m by letting $f(t_1, \dots, t_n) = (f_1, \dots, f_m)$. In other words, each f_i determines one coordinate of the range of f .

If V is a variety contained in \mathbf{A}^m , we say that f is a **regular function** from V to V if the range of f is contained in V .

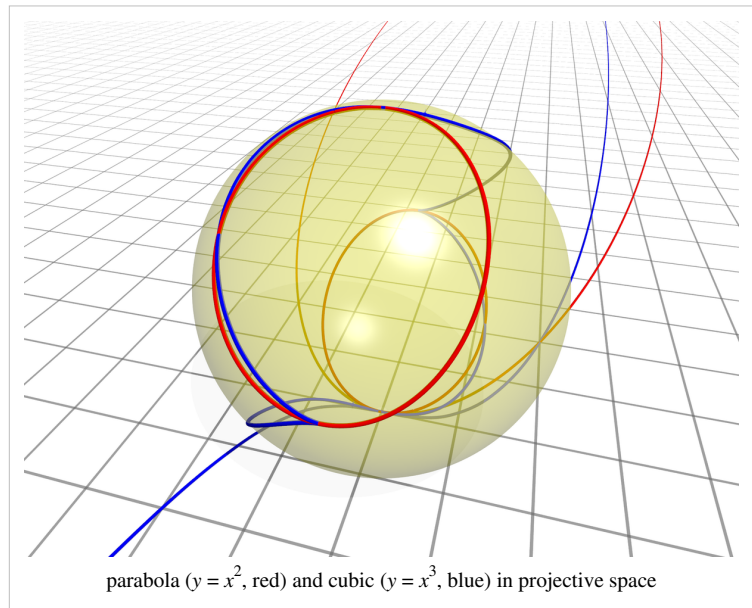
This makes the collection of all affine varieties into a category, where the objects are affine varieties and the morphisms are regular maps. The following theorem characterizes the category of affine varieties:

The category of affine varieties is the opposite category to the category of finitely generated integral k -algebras and their homomorphisms.

Projective space

Consider the variety $V(y - x^2)$. If we draw it, we get a parabola. As x increases, the slope of the line from the origin to the point (x, x^2) becomes larger and larger. As x decreases, the slope of the same line becomes smaller and smaller.

Compare this to the variety $V(y - x^3)$. This is a cubic equation. As x increases, the slope of the line from the origin to the point (x, x^3) becomes larger and larger just as before. But unlike before, as x decreases, the slope of the same line again becomes larger and larger. So the behavior "at infinity" of $V(y - x^3)$ is different from the behavior "at infinity" of $V(y - x^2)$. It is, however, difficult to make the concept of "at infinity" meaningful, if we restrict to working in affine space.



The remedy to this is to work in projective space. Projective space has properties analogous to those of a compact Hausdorff space. Among other things, it lets us make precise the notion of "at infinity" by including extra points. The behavior of a variety at those extra points then gives us more information about it. As it turns out, $V(y - x^3)$ has a singularity at one of those extra points, but $V(y - x^2)$ is smooth.

While projective geometry was originally established on a synthetic foundation, the use of homogeneous coordinates allowed the introduction of algebraic techniques. Furthermore, the introduction of projective techniques made many theorems in algebraic geometry simpler and sharper: For example, Bézout's theorem on the number of intersection points between two varieties can be stated in its sharpest form only in projective space. For this reason, projective space plays a fundamental role in algebraic geometry.

The modern viewpoint

The modern approaches to algebraic geometry redefine and effectively extend the range of basic objects in various levels of generality to schemes, formal schemes, ind-schemes, algebraic spaces, algebraic stacks and so on. The need for this arises already from the useful ideas within theory of varieties, e.g. the formal functions of Zariski can be accommodated by introducing nilpotent elements in structure rings; considering spaces of loops and arcs, constructing quotients by group actions and developing formal grounds for natural intersection theory and deformation theory lead to some of the further extensions.

Most remarkably, in late 1950s, algebraic varieties were subsumed into Alexander Grothendieck's concept of a scheme. Their local objects are affine schemes or prime spectra which are locally ringed spaces which form a category which is antiequivalent to the category of commutative unital rings, extending the duality between the category of affine algebraic varieties over a field k , and the category of finitely generated reduced k -algebras. The gluing is along Zariski topology; one can glue within the category of locally ringed spaces, but also, using the Yoneda embedding, within the more abstract category of presheaves of sets over the category of affine schemes. The Zariski topology in the set theoretic sense is then replaced by a Zariski topology in the sense of Grothendieck topology. Grothendieck introduced Grothendieck topologies having in mind more exotic but geometrically finer and more sensitive examples than the crude Zariski topology, namely the étale topology, and the two flat Grothendieck topologies: fppf and fpqc; nowadays some other examples became prominent including Nisnevich topology. Sheaves

can be furthermore generalized to stacks in the sense of Grothendieck, usually with some additional representability conditions leading to Artin stacks and, even finer, Deligne-Mumford stacks, both often called algebraic stacks.

Sometimes other algebraic sites replace the category of affine schemes. For example, Nikolai Durov has introduced commutative algebraic monads as a generalization of local objects in a generalized algebraic geometry. Versions of a tropical geometry, of an absolute geometry over a field of one element and an algebraic analogue of Arakelov's geometry were realized in this setup.

Another formal generalization is possible to Universal algebraic geometry in which every variety of algebra has its own algebraic geometry. The term *variety of algebra* should not be confused with *algebraic variety*.

The language of schemes, stacks and generalizations has proved to be a valuable way of dealing with geometric concepts and became cornerstones of modern algebraic geometry.

Algebraic stacks can be further generalized and for many practical questions like deformation theory and intersection theory, this is often the most natural approach. One can extend the Grothendieck site of affine schemes to a higher categorical site of derived affine schemes, by replacing the commutative rings with an infinity category of differential graded commutative algebras, or of simplicial commutative rings or a similar category with an appropriate variant of a Grothendieck topology. One can also replace presheaves of sets by presheaves of simplicial sets (or of infinity groupoids). Then, in presence of an appropriate homotopic machinery one can develop a notion of derived stack as such a presheaf on the infinity category of derived affine schemes, which is satisfying certain infinite categorical version of a sheaf axiom (and to be algebraic, inductively a sequence of representability conditions). Quillen model categories, Segal categories and quascategories are some of the most often used tools to formalize this yielding the **derived algebraic geometry**, introduced by the school of Carlos Simpson, including Andre Hirschowitz, Bertrand Toën, Gabrielle Vezzosi, Michel Vaquié and others; and developed further by Jacob Lurie, Bertrand Toën, and Gabrielle Vezzosi. Another (noncommutative) version of derived algebraic geometry, using A-infinity categories has been developed from early 1990-s by Maxim Kontsevich and followers.

History

Prehistory: Before the 19th century

Some of the roots of algebraic geometry date back to the work of the Hellenistic Greeks from the 5th century BC. The Delian problem, for instance, was to construct a length x so that the cube of side x contained the same volume as the rectangular box a^2b for given sides a and b . Menechmus (circa 350 BC) considered the problem geometrically by intersecting the pair of plane conics $ay = x^2$ and $xy = ab$.^[1] The later work, in the 3rd century BC, of Archimedes and Apollonius studied more systematically problems on conic sections,^[2] and also involved the use of coordinates.^[1] The Arab mathematicians were able to solve by purely algebraic means certain cubic equations, and then to interpret the results geometrically. This was done, for instance, by Ibn al-Haytham in the 10th century AD.^[3] Subsequently, Persian mathematician Omar Khayyám (born 1048 A.D.) discovered the general method of solving cubic equations by intersecting a parabola with a circle.^[4] Each of these early developments in algebraic geometry dealt with questions of finding and describing the intersections of algebraic curves.

Such techniques of applying geometrical constructions to algebraic problems were also adopted by a number of Renaissance mathematicians such as Gerolamo Cardano and Niccolò Fontana "Tartaglia" on their studies of the cubic equation. The geometrical approach to construction problems, rather than the algebraic one, was favored by most 16th and 17th century mathematicians, notably Blaise Pascal who argued against the use of algebraic and analytical methods in geometry.^[5] The French mathematicians Franciscus Vieta and later René Descartes and Pierre de Fermat revolutionized the conventional way of thinking about construction problems through the introduction of coordinate geometry. They were interested primarily in the properties of *algebraic curves*, such as those defined by Diophantine equations (in the case of Fermat), and the algebraic reformulation of the classical Greek works on conics and cubics (in the case of Descartes).

During the same period, Blaise Pascal and Gérard Desargues approached geometry from a different perspective, developing the synthetic notions of projective geometry. Pascal and Desargues also studied curves, but from the purely geometrical point of view: the analog of the Greek *ruler and compass construction*. Ultimately, the analytic geometry of Descartes and Fermat won out, for it supplied the 18th century mathematicians with concrete quantitative tools needed to study physical problems using the new calculus of Newton and Leibniz. However, by the end of the 18th century, most of the algebraic character of coordinate geometry was subsumed by the *calculus of infinitesimals* of Lagrange and Euler.

Nineteenth and early 20th century

It took the simultaneous 19th century developments of non-Euclidean geometry and Abelian integrals in order to bring the old algebraic ideas back into the geometrical fold. The first of these new developments was seized up by Edmond Laguerre and Arthur Cayley, who attempted to ascertain the generalized metric properties of projective space. Cayley introduced the idea of *homogeneous polynomial forms*, and more specifically quadratic forms, on projective space. Subsequently, Felix Klein studied projective geometry (along with other sorts of geometry) from the viewpoint that the geometry on a space is encoded in a certain class of transformations on the space. By the end of the 19th century, projective geometers were studying more general kinds of transformations on figures in projective space. Rather than the projective linear transformations which were normally regarded as giving the fundamental Kleinian geometry on projective space, they concerned themselves also with the higher degree birational transformations. This weaker notion of congruence would later lead members of the 20th century Italian school of algebraic geometry to classify algebraic surfaces up to birational isomorphism.

The second early 19th century development, that of Abelian integrals, would lead Bernhard Riemann to the development of Riemann surfaces.

Twentieth century

B. L. van der Waerden, Oscar Zariski, André Weil and others attempted to develop a rigorous foundation for algebraic geometry based on contemporary commutative algebra, including valuation theory and the theory of ideals.

In the 1950s and 1960s Jean-Pierre Serre and Alexander Grothendieck recast the foundations making use of sheaf theory. Later, from about 1960, and largely spearheaded by Grothendieck, the idea of schemes was worked out, in conjunction with a very refined apparatus of homological techniques. After a decade of rapid development the field stabilized in the 1970s, and new applications were made, both to number theory and to more classical geometric questions on algebraic varieties, singularities and moduli.

An important class of varieties, not easily understood directly from their defining equations, are the abelian varieties, which are the projective varieties whose points form an abelian group. The prototypical examples are the elliptic curves, which have a rich theory. They were instrumental in the proof of Fermat's last theorem and are also used in elliptic curve cryptography.

While much of algebraic geometry is concerned with abstract and general statements about varieties, methods for effective computation with concretely-given polynomials have also been developed. The most important is the technique of Gröbner bases which is employed in all computer algebra systems. Based on these methods, several solvers may compute all the solutions of a system of polynomial equations whose associated variety has dimension zero and thus consists in a finite number of points.

Applications

Algebraic geometry now finds application in statistics,^[6] control theory,^[7] robotics,^[8] error-correcting codes,^[9] phylogenetics^[10] and geometric modelling.^[11] There are also connections to string theory,^[12] game theory,^[13] graph matchings,^[14] solitons^[15] and integer programming.^[16] Google Scholar lists hundreds of more studies on algebraic geometry in biology^[17], chemistry^[18], economics^[19], physics^[20] and of course other areas of mathematics^[21].

Notes

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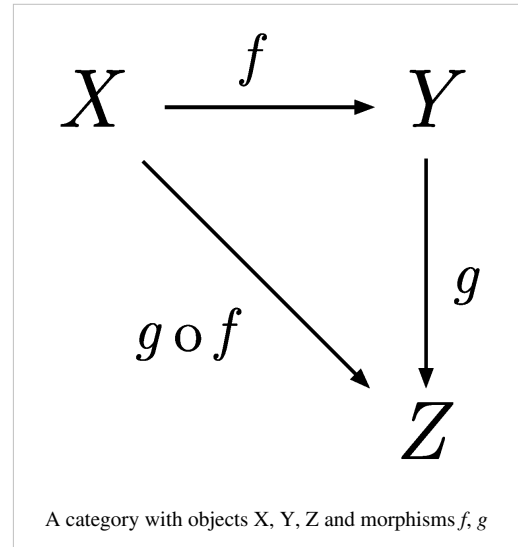
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Category theory

Category theory is an area of study in mathematics that examines in an abstract way the properties of particular mathematical concepts, by formalising them as collections of *objects* and *arrows* (also called morphisms, although this term also has a specific, non category-theoretical sense), where these collections satisfy certain basic conditions. Many significant areas of mathematics can be formalised as categories, and the use of category theory allows many intricate and subtle mathematical results in these fields to be stated, and proved, in a much simpler way than without the use of categories.

The most accessible example of a category is the category of sets, where the objects are sets and the arrows are functions from one set to another. However it is important to note that the objects of a category need not be sets nor the arrows functions; any way of formalising a mathematical concept such that it meets the basic conditions on the behaviour of objects and arrows is a valid category, and all the results of category theory will apply to it.



One of the simplest examples of a category (which is a very important concept in topology) is that of groupoid, defined as a category whose arrows or morphisms are all invertible. Categories now appear in most branches of mathematics, some areas of theoretical computer science where they correspond to types, and mathematical physics where they can be used to describe vector spaces. Categories were first introduced by Samuel Eilenberg and Saunders Mac Lane in 1942–45, in connection with algebraic topology.

Category theory has several faces known not just to specialists, but to other mathematicians. A term dating from the 1940s, "general abstract nonsense", refers to its high level of abstraction, compared to more classical branches of mathematics. Homological algebra is category theory in its aspect of organising and suggesting manipulations in abstract algebra. Diagram chasing is a visual method of arguing with abstract "arrows" joined in diagrams. Note that arrows between categories are called functors, subject to specific defining commutativity conditions; moreover, categorical diagrams and sequences can be defined as functors (viz. Mitchell, 1965). An arrow between two functors is a natural transformation when it is subject to certain naturality or commutativity conditions. Functors and natural transformations ('naturality') are the key concepts in category theory.^[1] Topos theory is a form of abstract sheaf theory, with geometric origins, and leads to ideas such as pointless topology. A topos can also be considered as a specific type of category with two additional topos axioms.

Background

The study of categories is an attempt to *axiomatically* capture what is commonly found in various classes of related *mathematical structures* by relating them to the *structure-preserving functions* between them. A systematic study of category theory then allows us to prove general results about any of these types of mathematical structures from the axioms of a category.

Consider the following example. The class **Grp** of groups consists of all objects having a "group structure". One can proceed to prove theorems about groups by making logical deductions from the set of axioms. For example, it is immediately proved from the axioms that the identity element of a group is unique.

Instead of focusing merely on the individual objects (e.g., groups) possessing a given structure, category theory emphasizes the morphisms — the structure-preserving mappings — *between* these objects; by studying these morphisms, we are able to learn more about the structure of the objects. In the case of groups, the morphisms are the

group homomorphisms. A group homomorphism between two groups "preserves the group structure" in a precise sense – it is a "process" taking one group to another, in a way that carries along information about the structure of the first group into the second group. The study of group homomorphisms then provides a tool for studying general properties of groups and consequences of the group axioms.

A similar type of investigation occurs in many mathematical theories, such as the study of continuous maps (morphisms) between topological spaces in topology (the associated category is called **Top**), and the study of smooth functions (morphisms) in manifold theory.

If one axiomatizes relations instead of functions, one obtains the theory of allegories.

Functors

Abstracting again, a category is *itself* a type of mathematical structure, so we can look for "processes" which preserve this structure in some sense; such a process is called a functor. A functor associates to every object of one category an object of another category, and to every morphism in the first category a morphism in the second.

In fact, what we have done is define a category *of categories and functors* – the objects are categories, and the morphisms (between categories) are functors.

By studying categories and functors, we are not just studying a class of mathematical structures and the morphisms between them; we are studying the *relationships between various classes of mathematical structures*. This is a fundamental idea, which first surfaced in algebraic topology. Difficult *topological* questions can be translated into *algebraic* questions which are often easier to solve. Basic constructions, such as the fundamental group or fundamental groupoid ^[2] of a topological space, can be expressed as fundamental functors ^[2] to the category of groupoids in this way, and the concept is pervasive in algebra and its applications.

Natural transformation

Abstracting yet again, constructions are often "naturally related" – a vague notion, at first sight. This leads to the clarifying concept of natural transformation, a way to "map" one functor to another. Many important constructions in mathematics can be studied in this context. "Naturality" is a principle, like general covariance in physics, that cuts deeper than is initially apparent.

Historical notes

In 1942–45, Samuel Eilenberg and Saunders Mac Lane introduced categories, functors, and natural transformations as part of their work in topology, especially algebraic topology. Their work was an important part of the transition from intuitive and geometric homology to axiomatic homology theory. Eilenberg and Mac Lane later wrote that their goal was to understand natural transformations; in order to do that, functors had to be defined, which required categories.

Stanisław Ulam, and some writing on his behalf, have claimed that related ideas were current in the late 1930s in Poland. Eilenberg was Polish, and studied mathematics in Poland in the 1930s. Category theory is also, in some sense, a continuation of the work of Emmy Noether (one of Mac Lane's teachers) in formalizing abstract processes; Noether realized that in order to understand a type of mathematical structure, one needs to understand the processes preserving that structure. In order to achieve this understanding, Eilenberg and Mac Lane proposed an axiomatic formalization of the relation between structures and the processes preserving them.

The subsequent development of category theory was powered first by the computational needs of homological algebra, and later by the axiomatic needs of algebraic geometry, the field most resistant to being grounded in either axiomatic set theory or the Russell-Whitehead view of united foundations. General category theory, an extension of universal algebra having many new features allowing for semantic flexibility and higher-order logic, came later; it is now applied throughout mathematics.

Certain categories called *topoi* (singular *topos*) can even serve as an alternative to axiomatic set theory as a foundation of mathematics. These foundational applications of category theory have been worked out in fair detail as a basis for, and justification of, constructive mathematics. More recent efforts to introduce undergraduates to categories as a foundation for mathematics include Lawvere and Rosebrugh (2003) and Lawvere and Schanuel (1997).

Categorical logic is now a well-defined field based on type theory for intuitionistic logics, with applications in functional programming and domain theory, where a cartesian closed category is taken as a non-syntactic description of a lambda calculus. At the very least, category theoretic language clarifies what exactly these related areas have in common (in some abstract sense).

Categories, objects, and morphisms

A category C consists of the following three mathematical entities:

- A class $\text{ob}(C)$, whose elements are called *objects*;
- A class $\text{hom}(C)$, whose elements are called morphisms or maps or *arrows*. Each morphism f has a unique *source object* a and *target object* b .

The expression $f: a \rightarrow b$, would be verbally stated as " f is a morphism from a to b ".

The expression $\mathbf{hom}(a, b)$ — alternatively expressed as $\mathbf{hom}(a, b)$, or $\mathbf{hom}_C(a, b)$, or $\mathbf{mor}(a, b)$, or $C(a, b)$ — denotes the *hom-class* of all morphisms from a to b .

- A binary operation \circ , called *composition of morphisms*, such that for any three objects a, b , and c , we have $\text{hom}(a, b) \times \text{hom}(b, c) \rightarrow \text{hom}(a, c)$. The composition of $f: a \rightarrow b$ and $g: b \rightarrow c$ is written as $g \circ f$ or gf ,^[3]

governed by two axioms:

- **Associativity:** If $f: a \rightarrow b$, $g: b \rightarrow c$ and $h: c \rightarrow d$ then $h \circ (g \circ f) = (h \circ g) \circ f$, and
- **Identity:** For every object x , there exists a morphism $1_x: x \rightarrow x$ called the *identity morphism* for x , such that for every morphism $f: a \rightarrow b$, we have $1_b \circ f = f = f \circ 1_a$.

From these axioms, it can be proved that there is exactly one identity morphism for every object. Some authors deviate from the definition just given by identifying each object with its identity morphism.

Relations among morphisms (such as $fg = h$) are often depicted using commutative diagrams, with "points" (corners) representing objects and "arrows" representing morphisms.

Properties of morphisms

Morphisms can have any of the following properties. A morphism $f: a \rightarrow b$ is a:

- **monomorphism** (or *monic*) if $fg_1 = fg_2$ implies $g_1 = g_2$ for all morphisms $g_1, g_2: x \rightarrow a$.
- **epimorphism** (or *epic*) if $g_1f = g_2f$ implies $g_1 = g_2$ for all morphisms $g_1, g_2: b \rightarrow x$.
- **bimorphism** if f is both epic and monic.
- **isomorphism** if there exists a morphism $g: b \rightarrow a$ such that $fg = 1_b$ and $gf = 1_a$.^[4]
- **endomorphism** if $a = b$. $\text{end}(a)$ denotes the class of endomorphisms of a .
- **automorphism** if f is both an endomorphism and an isomorphism. $\text{aut}(a)$ denotes the class of automorphisms of a .
- **retraction** if a right inverse of f exists, i.e. if there exists a morphism $g: b \rightarrow a$ with $fg = 1_b$.
- **section** if a left inverse of f exists, i.e. if there exists a morphism $g: b \rightarrow a$ with $gf = 1_a$.

Every retraction is an epimorphism, and every section is a monomorphism. Hence the following three statements are equivalent:

- f is a monomorphism and a retraction;
- f is an epimorphism and a section;
- f is an isomorphism.

Functors

Functors are structure-preserving maps between categories. They can be thought of as morphisms in the category of all (small) categories.

A (**covariant**) functor F from a category C to a category D , written $F:C \rightarrow D$, consists of:

- for each object x in C , an object $F(x)$ in D ; and
- for each morphism $f: x \rightarrow y$ in C , a morphism $F(f): F(x) \rightarrow F(y)$,

such that the following two properties hold:

- For every object x in C , $F(1_x) = 1_{F(x)}$;
- For all morphisms $f: x \rightarrow y$ and $g: y \rightarrow z$, $F(g \circ f) = F(g) \circ F(f)$.

A **contravariant** functor $F: C \rightarrow D$, is like a covariant functor, except that it "turns morphisms around" ("reverses all the arrows"). More specifically, every morphism $f: x \rightarrow y$ in C must be assigned to a morphism $F(f): F(y) \rightarrow F(x)$ in D . In other words, a contravariant functor is a covariant functor from the opposite category C^{op} to D .

Natural transformations and isomorphisms

A *natural transformation* is a relation between two functors. Functors often describe "natural constructions" and natural transformations then describe "natural homomorphisms" between two such constructions. Sometimes two quite different constructions yield "the same" result; this is expressed by a natural isomorphism between the two functors.

If F and G are (covariant) functors between the categories C and D , then a natural transformation η from F to G associates to every object X in C a morphism $\eta_X: F(X) \rightarrow G(X)$ in D such that for every morphism $f: X \rightarrow Y$ in C , we have $\eta_Y \circ F(f) = G(f) \circ \eta_X$; this means that the following diagram is commutative:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \eta_X \downarrow & & \downarrow \eta_Y \\ G(X) & \xrightarrow{G(f)} & G(Y) \end{array}$$

The two functors F and G are called *naturally isomorphic* if there exists a natural transformation from F to G such that η_X is an isomorphism for every object X in C .

Universal constructions, limits, and colimits

Using the language of category theory, many areas of mathematical study can be cast into appropriate categories, such as the categories of all sets, groups, topologies, and so on. These categories surely have some objects that are "special" in a certain way, such as the empty set or the product of two topologies, yet in the definition of a category, objects are considered to be atomic, i.e., we *do not know* whether an object A is a set, a topology, or any other abstract concept – hence, the challenge is to define special objects without referring to the internal structure of those objects. But how can we define the empty set without referring to elements, or the product topology without referring to open sets?

The solution is to characterize these objects in terms of their relations to other objects, as given by the morphisms of the respective categories. Thus, the task is to find *universal properties* that uniquely determine the objects of interest.

Indeed, it turns out that numerous important constructions can be described in a purely categorical way. The central concept which is needed for this purpose is called categorical *limit*, and can be dualized to yield the notion of a *colimit*.

Equivalent categories

It is a natural question to ask: under which conditions can two categories be considered to be "essentially the same", in the sense that theorems about one category can readily be transformed into theorems about the other category? The major tool one employs to describe such a situation is called *equivalence of categories*, which is given by appropriate functors between two categories. Categorical equivalence has found numerous applications in mathematics.

Further concepts and results

The definitions of categories and functors provide only the very basics of categorical algebra; additional important topics are listed below. Although there are strong interrelations between all of these topics, the given order can be considered as a guideline for further reading.

- The functor category D^C has as objects the functors from C to D and as morphisms the natural transformations of such functors. The Yoneda lemma is one of the most famous basic results of category theory; it describes representable functors in functor categories.
- Duality: Every statement, theorem, or definition in category theory has a *dual* which is essentially obtained by "reversing all the arrows". If one statement is true in a category C then its dual will be true in the dual category C^{op} . This duality, which is transparent at the level of category theory, is often obscured in applications and can lead to surprising relationships.
- Adjoint functors: A functor can be left (or right) adjoint to another functor that maps in the opposite direction. Such a pair of adjoint functors typically arises from a construction defined by a universal property; this can be seen as a more abstract and powerful view on universal properties.

Higher-dimensional categories

Many of the above concepts, especially equivalence of categories, adjoint functor pairs, and functor categories, can be situated into the context of *higher-dimensional categories*. Briefly, if we consider a morphism between two objects as a "process taking us from one object to another", then higher-dimensional categories allow us to profitably generalize this by considering "higher-dimensional processes".

For example, a (strict) 2-category is a category together with "morphisms between morphisms", i.e., processes which allow us to transform one morphism into another. We can then "compose" these "bimorphisms" both horizontally and vertically, and we require a 2-dimensional "exchange law" to hold, relating the two composition laws. In this context, the standard example is **Cat**, the 2-category of all (small) categories, and in this example, bimorphisms of morphisms are simply natural transformations of morphisms in the usual sense. Another basic example is to consider a 2-category with a single object; these are essentially monoidal categories. Bicategories are a weaker notion of 2-dimensional categories in which the composition of morphisms is not strictly associative, but only associative "up to" an isomorphism.

This process can be extended for all natural numbers n , and these are called n -categories. There is even a notion of ω -category corresponding to the ordinal number ω .

Higher-dimensional categories are part of the broader mathematical field of higher-dimensional algebra, a concept introduced by Ronald Brown. For a conversational introduction to these ideas, see John Baez, 'A Tale of n -categories' (1996).^[5]

Notes

- [1] *Categories for the Working Mathematician*, 2nd Edition, p 18: "As Eilenberg-Mac Lane first observed, 'category' has been defined in order to be able to define 'functor' and 'functor' has been defined in order to be able to define 'natural transformation'".
- [2] <http://planetphysics.org/encyclopedia/FundamentalGroupoidFunctor.html>
- [3] Some authors compose in the opposite order, writing fg or $f \circ g$ for $g \circ f$. Computer scientists using category theory very commonly write $f;g$ for $g \circ f$.
- [4] Note that a morphism that is both epic and monic is not necessarily an isomorphism! An elementary counterexample: in the category consisting of two objects A and B , the identity morphisms, and a single morphism f from A to B , f is both epic and monic but is not an isomorphism.
- [5] <http://math.ucr.edu/home/baez/week73.html>

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Higher-dimensional algebra

*This article is about **higher-dimensional algebra and supercategories** in generalized category theory, super-category theory, and also its extensions in nonabelian algebraic topology and metamathematics.^[1]*

Supercategories were first introduced in 1970,^[2] and were subsequently developed for applications in theoretical physics (especially quantum field theory and topological quantum field theory) and mathematical biology or mathematical biophysics.^[3]

Double groupoids, fundamental groupoids, 2-categories, categorical QFTs and TQFTs

In **higher-dimensional algebra (HDA)**, a double groupoid is a generalisation of a one-dimensional groupoid to two dimensions,^[4] and the latter groupoid can be considered as a special case of a category with all invertible arrows, or morphisms.

Double groupoids are often used to capture information about geometrical objects such as higher-dimensional manifolds (or n -dimensional manifolds).^[5] In general, an n -dimensional manifold is a space that locally looks like an n -dimensional Euclidean space, but whose global structure may be non-Euclidean. A first step towards defining higher dimensional algebras is the concept of 2-category of higher category theory, followed by the more 'geometric' concept of double category^{[6] [7] [8]} Other pathways in HDA involve: bicategories, homomorphisms of bicategories, variable categories (*aka*, indexed, or parametrized categories), topoi, effective descent, enriched and internal categories, as well as quantum categories^{[9] [10] [11]} and quantum double groupoids.^[12] In the latter case, by considering fundamental groupoids defined via a 2-functor allows one to think about the physically interesting case of quantum fundamental groupoids (QFGs) in terms of the bicategory **Span(Groupoids)**, and then constructing 2-Hilbert spaces and 2-linear maps for manifolds and cobordisms. At the next step, one obtains cobordisms with corners via natural transformations of such 2-functors. A claim was then made that, with the gauge group $SU(2)$, *“the extended TQFT, or ETQFT, gives a theory equivalent to the Ponzano-Regge model of quantum gravity”*;^[13] similarly, the Turaev-Viro model would be then obtained with representations of $SU_q(2)$. Therefore, according to the construction proposed by Jeffrey Morton, one can describe the state space of a gauge theory – or many kinds of quantum field theories (QFTs) and local quantum physics, in terms of the transformation groupoids given by symmetries, as for example in the case of a gauge theory, by the gauge transformations acting on states that are, in this case, connections. In the case of symmetries related to quantum groups, one would obtain structures that are representation categories of quantum groupoids,^[14] instead of the 2-vector spaces that are representation categories of groupoids.

Double categories, Category of categories and Supercategories

A higher level concept is thus defined as a category of categories, or **super-category**, which generalises to higher dimensions the notion of category – regarded as any structure which is an interpretation of Lawvere's axioms of the *elementary theory of abstract categories* (ETAC).^{[15] [16] [17] [18]} Thus, a supercategory and also a super-category, can be regarded as natural extensions of the concepts of meta-category,^[19] multicategory, and multi-graph, k -partite graph, or colored graph (see a color figure, and also its definition in graph theory).

Double groupoids were first introduced by Ronald Brown in 1976, in ref.^[20] and were further developed towards applications in nonabelian algebraic topology.^{[21] [22] [23] [24]} A related, 'dual' concept is that of a double algebroid, and the more general concept of R-algebroid.

Nonabelian algebraic topology

Many of the higher dimensional algebraic structures are noncommutative and, therefore, their study is a very significant part of nonabelian category theory, and also of Nonabelian Algebraic Topology (NAAT)^{[25] [26]} which generalises to higher dimensions ideas coming from the fundamental group.^[27] Such algebraic structures in dimensions greater than 1 develop the nonabelian character of the fundamental group, and they are in a precise sense '*more nonabelian than the groups*'.^{[28] [29]} These noncommutative, or more specifically, nonabelian structures reflect more accurately the geometrical complications of higher dimensions than the known homology and homotopy groups commonly encountered in classical algebraic topology. An important part of nonabelian algebraic topology is concerned with the properties and applications of homotopy groupoids and filtered spaces. Noncommutative double groupoids and double algebroids are only the first examples of such higher dimensional structures that are nonabelian. The new methods of Nonabelian Algebraic Topology (NAAT) '*can be applied to determine homotopy invariants of spaces, and homotopy classification of maps, in cases which include some classical results, and allow results not available by classical methods*'.^[30] Cubical omega-groupoids, higher homotopy groupoids, crossed modules, crossed complexes and Galois groupoids are key concepts in developing applications related to homotopy of filtered spaces, higher dimensional space structures, the construction of the fundamental groupoid of a topos \mathcal{E} in the general theory of topoi, and also in their physical applications in nonabelian quantum theories, and recent developments in quantum gravity, as well as categorical and topological dynamics.^[31] Further examples of such applications include the generalisations of noncommutative geometry formalizations of the noncommutative standard models *via* fundamental double groupoids and spacetime structures even more general than topoi or the lower-dimensional noncommutative spacetimes encountered in several topological quantum field theories and noncommutative geometry theories of quantum gravity.

A fundamental result in NAAT is the generalised, higher homotopy van Kampen theorem proven by R. Brown which states that '*the homotopy type of a topological space can be computed by a suitable colimit or homotopy colimit over homotopy types of its pieces*'. A related example is that of van Kampen theorems for categories of covering morphisms in lexensive categories.^[32] Other reports of generalisations of the van Kampen theorem include statements for 2-categories^[33] and a topos of topoi [34]. Important results in HDA are also the extensions of the Galois theory in categories and variable categories, or indexed/parametrized' categories.^{[35] [36]} The Joyal-Tierney representation theorem for topoi is also a generalisation of the Galois theory.^[37] Thus, indexing by bicategories in the sense of Benabou one also includes here the Joyal-Tierney theory.^[38]

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Higher category theory

Higher category theory is the part of category theory at a *higher order*, which means that some equalities are replaced by explicit arrows in order to be able to explicitly study the structure behind those equalities.

Strict higher categories

N-categories are defined inductively using the enriched category theory: 0-categories are sets, and $(n+1)$ -categories are categories enriched over the monoidal category of n-categories (with the monoidal structure given by finite products).^[1] This construction is well defined, as shown in the article on n-categories. This concept introduces higher arrows, higher compositions and higher identities, which must well behave together. For example, the category of small categories is in fact a 2-category, with natural transformations as second degree arrows. However this concept is too strict for some purposes (for example, homotopy theory), where "weak" structures arise in the form of higher categories.^[2]

Weak higher categories

In weak n-categories, the associativity and identity conditions are no longer strict (that is, they are not given by equalities), but rather are satisfied up to an isomorphism of the next level. An example in topology is the composition of paths, which is associative only up to homotopy. These isomorphisms must well behave between hom-sets and expressing this is the difficulty in the definition of weak n-categories. Weak 2-categories, also called bicategories, were the first to be defined explicitly. A particularity of these is that a bicategory with one object is exactly a monoidal category, so that bicategories can be said to be "monoidal categories with many objects." Weak 3-categories, also called tricategories, and higher-level generalizations are increasingly harder to define explicitly. Several definitions have been given, and telling when they are equivalent, and in what sense, has become a new object of study in category theory.

Quasicategories

Weak Kan complexes, or quasi-categories, are semisimplicial complexes satisfying a weak version of the Kan condition. Joyal showed that they are a good foundation for higher category theory. Recently the theory has been systematized further by Jacob Lurie who simply call them infinity categories, though the latter term is also a generic term for all models of (∞, k) categories for any k.

Simplicially enriched category

Simplicially enriched categories, or simplicial categories, are categories enriched over simplicial sets. However, when we look at them as a model for $(\infty, 1)$ -categories, then many categorical notions, say limits do not agree with the corresponding notions in the sense of enriched categories. The same for other enriched models like topologically enriched categories.

Topologically enriched categories

Topologically enriched categories (sometimes simply topological categories) are categories enriched over some convenient category of topological spaces, e.g. the category of compactly generated Hausdorff topological spaces.

Segal categories

These are models of higher categories introduced by Hirschowitz and Simpson in 1988^[3], partly inspired by results of Graeme Segal in 1974.

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- *nlab* (<http://ncatlab.org/nlab/show/HomePage>), the collective and open wiki notebook project on higher category theory and applications in physics, mathematics and philosophy
- Joyal's Catlab (<http://ncatlab.org/joyalscatlab/show/HomePage>), a wiki dedicated to polished expositions of categorical and higher categorical mathematics with proofs

External links

- John Baez Tale of n -Categories (<http://math.ucr.edu/home/baez/week73.html>)
- The n -Category Cafe (<http://golem.ph.utexas.edu/category/>) - a group blog devoted to higher category theory.

Algebraic topology

Algebraic topology is a branch of mathematics which uses tools from abstract algebra to study topological spaces. The basic goal is to find algebraic invariants that classify topological spaces up to homeomorphism, though usually most classify up to homotopy equivalence.

Although algebraic topology primarily uses algebra to study topological problems, using topology to solve algebraic problems is sometimes also possible. Algebraic topology, for example, allows for a convenient proof that any subgroup of a free group is again a free group.

The method of algebraic invariants

An older name for the subject was combinatorial topology, implying an emphasis on how a space X was constructed from simpler ones (the modern standard tool for such construction is the CW-complex). The basic method now applied in algebraic topology is to investigate spaces via algebraic invariants by mapping them, for example, to groups which have a great deal of manageable structure in a way that respects the relation of homeomorphism (or more general homotopy) of spaces. This allows one to recast statements about topological spaces into statements about groups, which are often easier to prove.

Two major ways in which this can be done are through fundamental groups, or more generally homotopy theory, and through homology and cohomology groups. The fundamental groups give us basic information about the structure of a topological space, but they are often nonabelian and can be difficult to work with. The fundamental group of a (finite) simplicial complex does have a finite presentation.

Homology and cohomology groups, on the other hand, are abelian and in many important cases finitely generated. Finitely generated abelian groups are completely classified and are particularly easy to work with.

Setting in category theory

In general, all constructions of algebraic topology are functorial; the notions of category, functor and natural transformation originated here. Fundamental groups and homology and cohomology groups are not only *invariants* of the underlying topological space, in the sense that two topological spaces which are homeomorphic have the same associated groups, but their associated morphisms also correspond — a continuous mapping of spaces induces a group homomorphism on the associated groups, and these homomorphisms can be used to show non-existence (or, much more deeply, existence) of mappings.

Results on homology

Several useful results follow immediately from working with finitely generated abelian groups. The free rank of the n -th homology group of a simplicial complex is equal to the n -th Betti number, so one can use the homology groups of a simplicial complex to calculate its Euler-Poincaré characteristic. As another example, the top-dimensional integral homology group of a closed manifold detects orientability: this group is isomorphic to either the integers or 0, according as the manifold is orientable or not. Thus, a great deal of topological information is encoded in the homology of a given topological space.

Beyond simplicial homology, which is defined only for simplicial complexes, one can use the differential structure of smooth manifolds via de Rham cohomology, or Čech or sheaf cohomology to investigate the solvability of differential equations defined on the manifold in question. De Rham showed that all of these approaches were interrelated and that, for a closed, oriented manifold, the Betti numbers derived through simplicial homology were the same Betti numbers as those derived through de Rham cohomology. This was extended in the 1950s, when Eilenberg and Steenrod generalized this approach. They defined homology and cohomology as functors equipped

with natural transformations subject to certain axioms (e.g., a weak equivalence of spaces passes to an isomorphism of homology groups), verified that all existing (co)homology theories satisfied these axioms, and then proved that such an axiomatization uniquely characterized the theory.

A new approach uses a functor from filtered spaces to crossed complexes defined directly and homotopically using relative homotopy groups; a higher homotopy van Kampen theorem proved for this functor enables basic results in algebraic topology, especially on the border between homology and homotopy, to be obtained without using singular homology or simplicial approximation. This approach is also called nonabelian algebraic topology, and generalises to higher dimensions ideas coming from the fundamental group.

Applications of algebraic topology

Classic applications of algebraic topology include:

- The Brouwer fixed point theorem: every continuous map from the unit n -disk to itself has a fixed point.
- The n -sphere admits a nowhere-vanishing continuous unit vector field if and only if n is odd. (For $n = 2$, this is sometimes called the "hairy ball theorem".)
- The Borsuk–Ulam theorem: any continuous map from the n -sphere to Euclidean n -space identifies at least one pair of antipodal points.
- Any subgroup of a free group is free. This result is quite interesting, because the statement is purely algebraic yet the simplest proof is topological. Namely, any free group G may be realized as the fundamental group of a graph X . The main theorem on covering spaces tells us that every subgroup H of G is the fundamental group of some covering space Y of X ; but every such Y is again a graph. Therefore its fundamental group H is free.

On the other hand this type of application is also handled by the use of covering morphisms of groupoids, and that technique has yielded subgroup theorems not yet proved by methods of algebraic topology (see the book by Higgins listed under groupoids).

- Topological combinatorics

Notable algebraic topologists

- Frank Adams
- Karol Borsuk
- Luitzen Egbertus Jan Brouwer
- William Browder
- Ronald Brown (mathematician)
- Henri Cartan
- Samuel Eilenberg
- Hans Freudenthal
- Peter Freyd
- Alexander Grothendieck
- Friedrich Hirzebruch
- Heinz Hopf
- Michael J. Hopkins
- Witold Hurewicz
- Egbert van Kampen
- Daniel Kan
- Hermann Künneth
- Solomon Lefschetz
- Jean Leray

- Saunders Mac Lane
- Mark Mahowald
- J. Peter May
- John Coleman Moore
- Jack Morava
- Goro Nishida
- Sergei Novikov
- Lev Pontryagin
- Mikhail Postnikov
- Daniel Quillen
- Jean-Pierre Serre
- Stephen Smale
- Norman Steenrod
- Dennis Sullivan
- René Thom
- Hiroshi Toda
- Leopold Vietoris
- Hassler Whitney
- J. H. C. Whitehead

Important theorems in algebraic topology

- Borsuk-Ulam theorem
- Brouwer fixed point theorem
- Cellular approximation theorem
- Eilenberg–Zilber theorem
- Freudenthal suspension theorem
- Hurewicz theorem
- Künneth theorem
- Poincaré duality theorem
- Universal coefficient theorem
- Van Kampen's theorem
- Generalized van Kampen's theorems^[1]
- Higher homotopy, generalized van Kampen's theorem^[2] ^[3]
- Whitehead's theorem

Notes

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Topological dynamics

In mathematics, **topological dynamics** is a branch of the theory of dynamical systems in which qualitative, asymptotic properties of dynamical systems are studied from the viewpoint of general topology.

Scope

The central object of study in topological dynamics is a **topological dynamical system**, i.e. a topological space, together with a continuous transformation, a continuous flow, or more generally, a semigroup of continuous transformations of that space. The origins of topological dynamics lie in the study of asymptotical properties of trajectories of systems of autonomous ordinary differential equations, in particular, the behavior of limit sets and various manifestations of "repetitiveness" of the motion, such as periodic trajectories, recurrence and minimality, stability, non-wandering points. George Birkhoff is considered to be the founder of the field. A structure theorem for minimal distal flows proved by Hillel Furstenberg in the early 1960s inspired much work on classification of minimal flows. A lot of research in the 1970s and 1980s was devoted to topological dynamics of one-dimensional maps, in particular, piecewise linear self-maps of the interval and the circle.

Unlike the theory of smooth dynamical systems, where the main object of study is a smooth manifold with a diffeomorphism or a smooth flow, phase spaces considered in topological dynamics are general metric spaces (usually, compact). This necessitates development of entirely different techniques but allows extra degree of flexibility even in the smooth setting, because invariant subsets of a manifold are frequently very complicated topologically (cf limit cycle, strange attractor); additionally, shift spaces arising via symbolic representations can be considered on an equal footing with more geometric actions. Topological dynamics has intimate connections with ergodic theory of dynamical systems, and many fundamental concepts of the latter have topological analogues (cf Kolmogorov–Sinai entropy and topological entropy).

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Graph dynamical system

In mathematics, the concept of **graph dynamical systems** can be used to capture a wide range of processes taking place on graphs or networks. A major theme in the mathematical and computational analysis of GDSs is to relate their structural properties (e.g. the network connectivity) and the global dynamics that result.

The work on GDSs considers finite graphs and finite state spaces. As such, the research typically involves techniques from, e.g., graph theory, combinatorics, algebra, and dynamical systems rather than differential geometry. In principle, one could define and study GDSs over an infinite graph (e.g. cellular automata over \mathbb{Z}^k or interacting particle systems), as well as GDSs with infinite state space (e.g. \mathbb{R} as in coupled map lattices); see, e.g., Wu.^[1] In the following everything is implicitly assumed to be finite unless stated otherwise.

Formal definition

A graph dynamical system is constructed from the following components:

- A finite *graph* Y with vertex set $v[Y] = \{1, 2, \dots, n\}$. Depending on the context the graph can be directed or undirected.
- A state x_v for each vertex v of Y taken from a finite set K . The *system state* is the n -tuple $x = (x_1, x_2, \dots, x_n)$, and $x[v]$ is the tuple consisting of the states associated to the vertices in the 1-neighborhood of v in Y (in some fixed order).
- A *vertex function* f_v for each vertex v . The vertex function maps the state of vertex v at time t to the vertex state at time $t + 1$ based on the states associated to the 1-neighborhood of v in Y .
- An *update scheme* specifying the mechanism by which the mapping of individual vertex states is carried out so as to induce a discrete dynamical system with map $F: K^n \rightarrow K^n$.

The *phase space* associated to a dynamical system with map $F: K^n \rightarrow K^n$ is the finite directed graph with vertex set K^n and directed edges $(x, F(x))$. The structure of the phase space is governed by the properties of the graph Y , the vertex functions (f_v) , and the update scheme. The research in this area seeks to infer phase space properties based on the structure of the system constituents. The analysis has a local-to-global character.

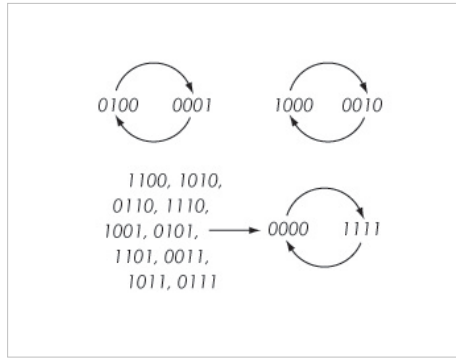
Generalized cellular automata (GCA)

If, for example, the update scheme consists of applying the vertex functions synchronously one obtains the class of *generalized cellular automata* (CA). In this case, the global map $F: K^n \rightarrow K^n$ is given by

$$F(x)_v = f_v(x[v]) .$$

This class is referred to as generalized cellular automata since the classical or standard cellular automata are typically defined and studied over regular graphs or grids, and the vertex functions are typically assumed to be identical.

Example: Let Y be the circle graph on vertices $\{1, 2, 3, 4\}$ with edges $\{1, 2\}$, $\{2, 3\}$, $\{3, 4\}$ and $\{1, 4\}$, denoted Circ_4 . Let $K = \{0, 1\}$ be the state space for each vertex and use the function $\text{nor}_3: K^3 \rightarrow K$ defined by $\text{nor}_3(x, y, z) = (1 + x)(1 + y)(1 + z)$ with arithmetic modulo 2 for all vertex functions. Then for example the system state $(0, 1, 0, 0)$ is mapped to $(0, 0, 0, 1)$ using a synchronous update. All the transitions are shown in the phase space below.



Sequential dynamical systems (SDS)

If the vertex functions are applied asynchronously in the sequence specified by a word $w = (w_1, w_2, \dots, w_m)$ or permutation $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ of $v[Y]$ one obtains the class of *Sequential dynamical systems* (SDS).^[2] In this case it is convenient to introduce the Y -local maps F_i constructed from the vertex functions by

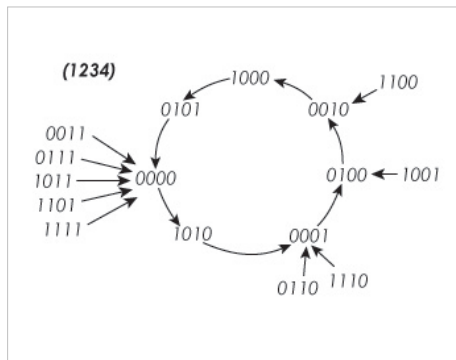
$$F_i(x) = (x_1, x_2, \dots, x_{i-1}, f_i(x[i]), x_{i+1}, \dots, x_n) .$$

The SDS map $F = [F_Y, w] : K^n \rightarrow K^n$ is the function composition

$$[F_Y, w] = F_{w(m)} \circ F_{w(m-1)} \circ \dots \circ F_{w(2)} \circ F_{w(1)} .$$

If the update sequence is a permutation one frequently speaks of a *permutation SDS* to emphasize this point.

Example: Let Y be the circle graph on vertices $\{1,2,3,4\}$ with edges $\{1,2\}$, $\{2,3\}$, $\{3,4\}$ and $\{1,4\}$, denoted Circ_4 . Let $K=\{0,1\}$ be the state space for each vertex and use the function $\text{nor}_3 : K^3 \rightarrow K$ defined by $\text{nor}_3(x, y, z) = (1+x)(1+y)(1+z)$ with arithmetic modulo 2 for all vertex functions. Using the update sequence $(1,2,3,4)$ then the system state $(0, 1, 0, 0)$ is mapped to $(0, 0, 1, 0)$. All the system state transitions for this sequential dynamical system are shown in the phase space below.



Stochastic graph dynamical systems

From, e.g., the point of view of applications it is interesting to consider the case where one or more of the components of a GDS contains stochastic elements. Motivating applications could include processes that are not fully understood (e.g. dynamics within a cell) and where certain aspects for all practical purposes seem to behave according to some probability distribution. There are also applications governed by deterministic principles whose description is so complex or unwieldy that it makes sense to consider probabilistic approximations.

Every element of a graph dynamical system can be made stochastic in several ways. For example, in a sequential dynamical system the update sequence can be made stochastic. At each iteration step one may choose the update sequence w at random from a given distribution of update sequences with corresponding probabilities. The matching probability space of update sequences induces a probability space of SDS maps. A natural object to study in this regard is the Markov chain on state space induced by this collection of SDS maps. This case is referred to as *update*

sequence stochastic GDS and is motivated by, e.g., processes where "events" occur at random according to certain rates (e.g. chemical reactions), synchronization in parallel computation/discrete event simulations, and in computational paradigms described later.

This specific example with stochastic update sequence illustrates two general facts for such systems: when passing to a stochastic graph dynamical system one is generally led to (1) a study of Markov chains (with specific structure governed by the constituents of the GDS), and (2) the resulting Markov chains tend to be large having an exponential number of states. A central goal in the study of stochastic GDS is to be able to derive reduced models.

One may also consider the case where the vertex functions are stochastic, i.e., *function stochastic GDS*. For example, Random Boolean networks are examples of function stochastic GDS using a synchronous update scheme and where the state space is $K = \{0, 1\}$. Finite probabilistic cellular automata (PCA) is another example of function stochastic GDS. In principle the class of Interacting particle systems (IPS) covers finite and infinite PCA, but in practice the work on IPS is largely concerned with the infinite case since this allows one to introduce more interesting topologies on state space.

Applications

Graph dynamical systems constitute a natural framework for capturing distributed systems such as biological networks and epidemics over social networks, many of which are frequently referred to as complex systems.

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External links

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Analysis of Systems

System analysis in the field of electrical engineering characterizes electrical systems and their properties. System Analysis can be used to represent almost anything from population growth to audio speakers, electrical engineers often use it because of its direct relevance to many areas of their discipline, most notably signal processing and communication systems.

Characterization of systems

A system is characterized by how it responds to input signals. In general, a system has one or more input signals and one or more output signals. Therefore, one natural characterization of systems is by how many inputs and outputs they have:

- *SISO* (Single Input, Single Output)
- *SIMO* (Single Input, Multiple Outputs)
- *MISO* (Multiple Inputs, Single Output)
- *MIMO* (Multiple Inputs, Multiple Outputs)

It is often useful (or necessary) to break up a system into smaller pieces for analysis. Therefore, we can regard a SIMO system as multiple SISO systems (one for each output), and similarly for a MIMO system. By far, the greatest amount of work in system analysis has been with SISO systems, although many parts inside SISO systems have multiple inputs (such as adders).

Signals can be continuous or discrete in time, as well as continuous or discrete in the values they take at any given time:

- Signals that are continuous in time and continuous in value are known as *analog signals*.
- Signals that are discrete in time and discrete in value are known as *digital signals*.
- Signals that are discrete in time and continuous in value are called *discrete-time signals*. While important mathematically, systems that process discrete time signals are difficult to physically realize. The methods developed for analyzing discrete time signals and systems are usually applied to digital and analog signals and systems.
- Signals that are continuous in time and discrete in value are sometimes seen in the timing analysis of logic circuits, but have little to no use in system analysis.

With this categorization of signals, a system can then be characterized as to which type of signals it deals with:

- A system that has analog input and analog output is known as an *analog system*.
- A system that has digital input and digital output is known as a *digital system*.
- Systems with analog input and digital output or digital input and analog output are possible. However, it is usually easiest to break these systems up for analysis into their analog and digital parts, as well as the necessary analog to digital or digital to analog converter.

Another way to characterize systems is by whether their output at any given time depends only on the input at that time or perhaps on the input at some time in the past (or in the future!).

- *Memoryless* systems do not depend on any past input.
- Systems *with memory* do depend on past input.
- *Causal* systems do not depend on any future input.
- *Non-causal* or *anticipatory* systems do depend on future input.

Note: It is not possible to physically realize a non-causal system operating in "real time". However, from the standpoint of analysis, they are important for two reasons. First, the ideal system for a given application is often a noncausal system, which although not physically possible can give insight into the design of a

derivated causal system to accomplish a similar purpose. Second, there are instances when a system does not operate in "real time" but is rather simulated "off-line" by a computer, such as post-processing an audio or video recording.

Further, some non-causal systems can operate in pseudo-real time by introducing lag: if a system depends on input for 1 second in future, it can process in real time with 1 second lag.

Analog systems with memory may be further classified as *lumped* or *distributed*. The difference can be explained by considering the meaning of memory in a system. Future output of a system with memory depends on future input and a number of state variables, such as values of the input or output at various times in the past. If the number of state variables necessary to describe future output is finite, the system is lumped; if it is infinite, the system is distributed.

Finally, systems may be characterized by certain properties which facilitate their analysis:

- A system is *linear* if it has the superposition and scaling properties.
- A system that is not linear is *non-linear*.
- If the output of a system does not depend explicitly on time, the system is said to be time-invariant; otherwise it is time-variant
- A system that will always produce the same output for a given input is said to be deterministic.
- A system that will produce different outputs for a given input is said to be stochastic.

There are many methods of analysis developed specifically for linear time-invariant (*LTI*) deterministic systems. Unfortunately, in the case of analog systems, none of these properties are ever perfectly achieved. Linearity implies that operation of a system can be scaled to arbitrarily large magnitudes, which is not possible. Time-invariance is violated by aging effects that can change the outputs of analog systems over time (usually years or even decades). Thermal noise and other random phenomena ensure that the operation of any analog system will have some degree of stochastic behavior. Despite these limitations, however, it is usually reasonable to assume that deviations from these ideals will be small.

LTI Systems

As mentioned above, there are many methods of analysis developed specifically for LTI systems. This is due to their simplicity of specification. An LTI system is completely specified by its transfer function (which is a rational function for digital and lumped analog LTI systems). Alternatively, we can think of an LTI system being completely specified by its frequency response. A third way to specify an LTI system is by its characteristic linear differential equation (for analog systems) or linear difference equation (for digital systems). Which description is most useful depends on the application.

The distinction between lumped and distributed LTI systems is important. A lumped LTI system is specified by a finite number of parameters, be it the zeros and poles of its transfer function, or the coefficients of its differential equation, whereas specification of a distributed LTI system requires a complete function

Dynamic Bayesian network

A **dynamic Bayesian network** is a Bayesian network that represents sequences of variables. These sequences are often time-series (for example, in speech recognition) or sequences of symbols (for example, protein sequences). The hidden Markov model and the Kalman filter can be considered as the most simple dynamic Bayesian networks.

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Dynamic network analysis

Dynamic network analysis (DNA) is an emergent scientific field that brings together traditional social network analysis (SNA), link analysis (LA) and multi-agent systems (MAS) within network science and network theory. There are two aspects of this field. The first is the statistical analysis of DNA data. The second is the utilization of simulation to address issues of network dynamics. DNA networks vary from traditional social networks in that they are larger, dynamic, multi-mode, multi-plex networks, and may contain varying levels of uncertainty. The main difference of DNA to SNA is DNA taken the domain of time into account. One of the most notable and earliest case of the use of DNA is in Sampson's monastery study, where he took snapshots of the same network from different intervals and observed and analyzed the evolution of the network. ^[1]

DNA statistical tools are generally optimized for large-scale networks and admit the analysis of multiple networks simultaneously in which, there are multiple types of nodes (multi-node) and multiple types of links (multi-plex). In contrast, SNA statistical tools focus on single or at most two mode data and facilitate the analysis of only one type of link at a time.

DNA statistical tools tend to provide more measures to the user, because they have measures that use data drawn from multiple networks simultaneously. From a computer simulation perspective, nodes in DNA are like atoms in quantum theory, nodes can be, though need not be, treated as probabilistic. Whereas nodes in a traditional SNA model are static, nodes in a DNA model have the ability to learn. Properties change over time; nodes can adapt: A company's employees can learn new skills and increase their value to the network; Or, capture one terrorist and three more are forced to improvise. Change propagates from one node to the next and so on. DNA adds the element of a network's evolution and considers the circumstances under which change is likely to occur.

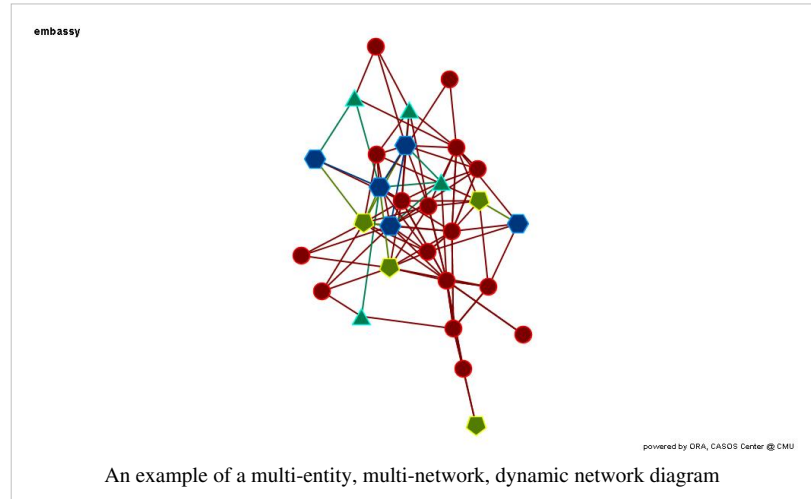
Illustrative problems that people in the DNA area work on

- Developing metrics and statistics to assess and identify change within and across networks.
- Developing and validating simulations to study network change, evolution, adaptation, decay. See Computer simulation and organizational studies
- Developing and testing theory of network change, evolution, adaptation, decay
- Developing and validating formal models of network generation and evolution
- Developing techniques to visualize network change overall or at the node or group level
- Developing statistical techniques to see whether differences observed over time in networks are due to simply different samples from a distribution of links and nodes or changes over time in the underlying distribution of links and nodes
- Developing control processes for networks over time
- Developing algorithms to change distributions of links in networks over time
- Developing algorithms to track groups in networks over time
- Developing tools to extract or locate networks from various data sources such as texts
- Developing statistically valid measurements on networks over time
- Examining the robustness of network metrics under various types of missing data
- Empirical studies of multi-mode multi-link multi-time period networks
- Examining networks as probabilistic time-variant phenomena
- Forecasting change in existing networks
- Identifying trails through time given a sequence of networks
- Identifying changes in node criticality given a sequence of networks anything else related to multi-mode multi-link multi-time period networks

Kathleen Carley, of Carnegie Mellon University, is a leading authority in this field.

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- Kathleen M. Carley, Jana Diesner, Jeffrey Reminga, Maksim Tsvetovat, 2008, Toward an Interoperable Dynamic Network Analysis Toolkit, DSS Special Issue on Cyberinfrastructure for Homeland Security: Advances in Information Sharing, Data Mining, and Collaboration Systems. Decision Support Systems ^[2] 43(4):1324-1347



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External links

- Radcliffe Exploratory Seminar on Dynamic Networks (<http://www.eecs.harvard.edu/~parkes/RadcliffeSeminar.htm>)
- Center for Computational Analysis of Social and Organizational Systems (CASOS) (<http://www.casos.cs.cmu.edu/>)

Dynamic circuit network

A **dynamic circuit network (DCN)** is an advanced computer networking technology that combines traditional packet-switched communication based on the Internet Protocol, as used in the Internet, with circuit-switched technologies that are characteristic of traditional telephone network systems. This combination allows user-initiated ad-hoc dedicated allocation of network bandwidth for high-demand, real-time applications and network services, delivered over an optical fiber infrastructure.^[1]

Implementation

Dynamic circuit networks were pioneered by the Internet2 advanced networking consortium.^[2] The experimental Internet2 HOPI infrastructure, decommissioned in 2007, was a forerunner to the current SONET-based Ciena Network underlying the Internet2 DCN. The Internet2 DCN began operation in late 2007 as part of the larger Internet2 network.^[3] It provides advanced networking capabilities and resources to the scientific and research communities, such as the Large Hadron Collider (LHC) project.^[4]

The Internet2 DCN is based on open-source, standards-based software, the Inter-domain Controller (IDC) protocol, developed in cooperation with ESnet^[5] and GÉANT2.^[3] The entire software set is known as the Dynamic Circuit Network Software Suite (DCN SS).

Inter-domain Controller protocol

The Inter-domain Controller protocol manages the dynamic provisioning of network resources participating in a dynamic circuit network across multiple administrative domain boundaries.^[6] It is a SOAP-based XML messaging protocol, secured by Web Services Security (v1.1) using the XML Digital Signature standard. It is transported over HTTP Secure (HTTPS) connections.

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External links

- Internet2 Website (<http://www.internet2.edu/>)
- Dynamic Circuit Network Suite (<https://wiki.internet2.edu/confluence/display/DCNSS/Home>)

Tensor product network

A **tensor product network**, in neural networks, is a network that exploits the properties of tensors to model associative concepts such as variable assignment. Orthonormal vectors are chosen to model the ideas (such as variable names and target assignments), and the tensor product of these vectors construct a network whose mathematical properties allow the user to easily extract the association from it.

Cybernetics

Cybernetics is the interdisciplinary study of the structure of regulatory systems. Cybernetics is closely related to control theory and systems theory. Both in its origins and in its evolution in the second half of the 20th century, cybernetics is equally applicable to physical and social (that is, language-based) systems.

Overview

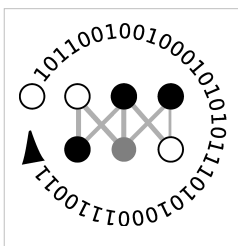
Cybernetics is most applicable when the system being analysed is involved in a closed signal loop; that is, where action by the system causes some change in its environment and that change is fed to the system via information (feedback) that causes the system to adapt to these new conditions: the system's changes affect its behavior. This "circular causal" relationship is necessary and sufficient for a cybernetic perspective. System Dynamics, a related field, originated with applications of electrical engineering control theory to other kinds of simulation models (especially business systems) by Jay Forrester at MIT in the 1950s. Convenient GUI

system dynamics software developed into user friendly versions by the 1990s and have been applied to diverse systems. SD models solve the problem of simultaneity (mutual causation) by updating all variables in small time increments with positive and negative feedbacks and time delays structuring the interactions and control. The best known SD model is probably the 1972 *The Limits to Growth*. This model forecast that exponential growth would lead to economic collapse during the 21st century under a wide variety of growth scenarios.

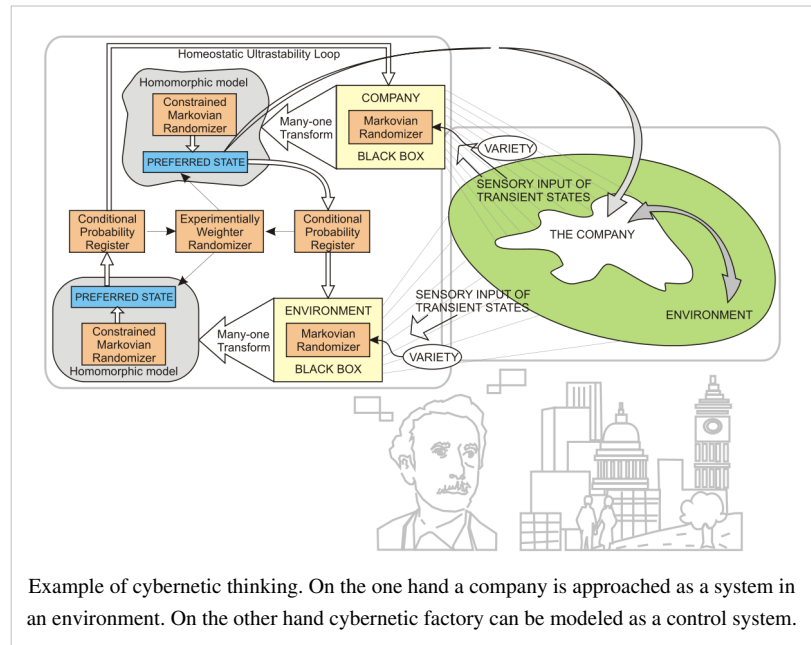
Contemporary cybernetics began as an interdisciplinary study connecting the fields of control systems, electrical network theory, mechanical engineering, logic modeling, evolutionary biology, neuroscience, anthropology, and psychology in the 1940s, often attributed to the Macy Conferences.

Other fields of study which have influenced or been influenced by cybernetics include game theory, system theory (a mathematical counterpart to cybernetics), perceptual control theory, sociology, psychology (especially neuropsychology, behavioral psychology, cognitive psychology), philosophy, and architecture and organizational theory.^[1]

Definition



The term *cybernetics* stems from the Greek κυβερνήτης (*kybernētēs*, steersman, governor, pilot, or rudder — the same root as government). Cybernetics is a broad field of study, but the essential goal of cybernetics is to understand and define the functions and processes of systems that have goals and that participate in circular, causal chains that move from action to sensing to comparison with desired goal, and again to action. Studies in cybernetics



provide a means for examining the design and function of any system, including social systems such as business management and organizational learning, including for the purpose of making them more efficient and effective.

Cybernetics was defined by Norbert Wiener, in his book of that title, as the study of control and communication in the animal and the machine. Stafford Beer called it the science of effective organization and Gordon Pask extended it to include information flows "in all media" from stars to brains. It includes the study of feedback, black boxes and derived concepts such as communication and control in living organisms, machines and organizations including self-organization. Its focus is how anything (digital, mechanical or biological) processes information, reacts to information, and changes or can be changed to better accomplish the first two tasks ^[2]. A more philosophical definition, suggested in 1956 by Louis Couffignal, one of the pioneers of cybernetics, characterizes cybernetics as "the art of ensuring the efficacy of action" ^[3]. The most recent definition has been proposed by Louis Kauffman, President of the American Society for Cybernetics, "Cybernetics is the study of systems and processes that interact with themselves and produce themselves from themselves" ^[4].

Concepts studied by cyberneticists (or, as some prefer, cyberneticians) include, but are not limited to: learning, cognition, adaption, social control, emergence, communication, efficiency, efficacy and interconnectivity. These concepts are studied by other subjects such as engineering and biology, but in cybernetics these are removed from the context of the individual organism or device.

Other fields of study which have influenced or been influenced by cybernetics include game theory; system theory (a mathematical counterpart to cybernetics); psychology, especially neuropsychology, behavioral psychology and cognitive psychology; philosophy; anthropology; and even theology, ^[5] telematic art, and architecture. ^[6]

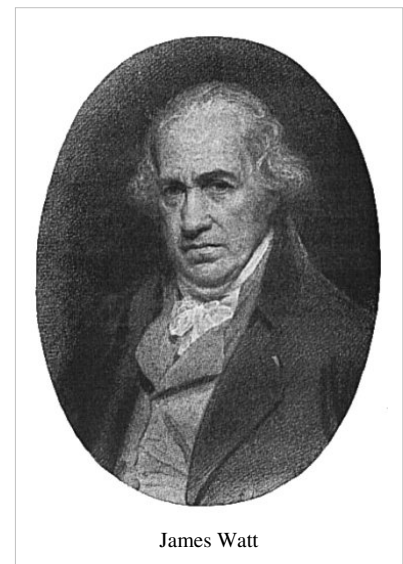
History

The roots of cybernetic theory

The word *cybernetics* was first used in the context of "the study of self-governance" by Plato in *The Laws* to signify the governance of people. The word 'cybernétique' was also used in 1834 by the physicist André-Marie Ampère (1775–1836) to denote the sciences of government in his classification system of human knowledge.

The first artificial automatic regulatory system, a water clock, was invented by the mechanician Ktesibios. In his water clocks, water flowed from a source such as a holding tank into a reservoir, then from the reservoir to the mechanisms of the clock. Ktesibios's device used a cone-shaped float to monitor the level of the water in its reservoir and adjust the rate of flow of the water accordingly to maintain a constant level of water in the reservoir, so that it neither overflowed nor was allowed to run dry. This was the first artificial truly automatic self-regulatory device that required no outside intervention between the feedback and the controls of the mechanism. Although they did not refer to this concept by the name of Cybernetics (they considered it a field of engineering), Ktesibios and others such as Heron and Su Song are considered to be some of the first to study cybernetic principles.

The study of *teleological mechanisms* (from the Greek τέλος or *telos* for *end*, *goal*, or *purpose*) in machines with *corrective feedback* dates from as far back as the late 18th century when James Watt's steam engine was equipped with a governor, a centrifugal feedback valve for controlling the speed of the engine. Alfred Russel Wallace identified this as the principle of evolution in his famous 1858 paper. In 1868 James Clerk Maxwell published a theoretical article on governors, one of the first to discuss and refine the principles of self-regulating devices. Jakob von Uexküll



James Watt

applied the feedback mechanism via his model of functional cycle (*Funktionskreis*) in order to explain animal behaviour and the origins of meaning in general.

The early 20th century

Contemporary cybernetics began as an interdisciplinary study connecting the fields of control systems, electrical network theory, mechanical engineering, logic modeling, evolutionary biology and neuroscience in the 1940s. Electronic control systems originated with the 1927 work of Bell Telephone Laboratories engineer Harold S. Black on using negative feedback to control amplifiers. The ideas are also related to the biological work of Ludwig von Bertalanffy in General Systems Theory.

Early applications of negative feedback in electronic circuits included the control of gun mounts and radar antenna during World War II. Jay Forrester, a graduate student at the Servomechanisms Laboratory at MIT during WWII working with Gordon S. Brown to develop electronic control systems for the U.S. Navy, later applied these ideas to social organizations such as corporations and cities as an original organizer of the MIT School of Industrial Management at the MIT Sloan School of Management. Forrester is known as the founder of System Dynamics.

W. Edwards Deming, the Total Quality Management guru for whom Japan named its top post-WWII industrial prize, was an intern at Bell Telephone Labs in 1927 and may have been influenced by network theory. Deming made "Understanding Systems" one of the four pillars of what he described as "Profound Knowledge" in his book "The New Economics."

Numerous papers spearheaded the coalescing of the field. In 1935 Russian physiologist P.K. Anokhin published a book in which the concept of feedback ("back afferentation") was studied. The study and mathematical modelling of regulatory processes became a continuing research effort and two key articles were published in 1943. These papers were "Behavior, Purpose and Teleology" by Arturo Rosenblueth, Norbert Wiener, and Julian Bigelow; and the paper "A Logical Calculus of the Ideas Immanent in Nervous Activity" by Warren McCulloch and Walter Pitts.

Cybernetics as a discipline was firmly established by Wiener, McCulloch and others, such as W. Ross Ashby and W. Grey Walter.

Walter was one of the first to build autonomous robots as an aid to the study of animal behaviour. Together with the US and UK, an important geographical locus of early cybernetics was France.

In the spring of 1947, Wiener was invited to a congress on harmonic analysis, held in Nancy, France. The event was organized by the Bourbaki, a French scientific society, and mathematician Szolem Mandelbrojt (1899–1983), uncle of the world-famous mathematician Benoît Mandelbrot.

During this stay in France, Wiener received the offer to write a manuscript on the unifying character of this part of applied mathematics, which is found in the study of Brownian motion and in telecommunication engineering. The following summer, back in the United States, Wiener decided to introduce the neologism cybernetics into his scientific theory. The name *cybernetics* was coined to denote the study of "teleological mechanisms" and was popularized through his book *Cybernetics, or Control and Communication in the Animal and Machine* (Hermann & Cie, Paris, 1948). In the UK this became the focus for the Ratio Club.

In the early 1940s John von Neumann, although better known for his work in mathematics and computer science, did contribute a unique and unusual addition to the world of cybernetics: Von Neumann cellular automata, and their logical follow up the Von Neumann Universal Constructor. The result of these deceptively simple thought-experiments was the concept of self replication which cybernetics adopted as a core concept. The concept that the same properties of genetic reproduction applied to social memes, living cells, and even computer viruses is further proof of the somewhat surprising universality of cybernetic study.



John von Neumann

Wiener popularized the social implications of cybernetics, drawing analogies between automatic systems (such as a regulated steam engine) and human institutions in his best-selling *The Human Use of Human Beings : Cybernetics and Society* (Houghton-Mifflin, 1950).

While not the only instance of a research organization focused on cybernetics, the Biological Computer Lab ^[7] at the University of Illinois, Urbana/Champaign, under the direction of Heinz von Foerster, was a major center of cybernetic research ^[8] for almost 20 years, beginning in 1958.

The fall and rebirth of cybernetics

For a time during the past 30 years, the field of cybernetics followed a boom-bust cycle of becoming more and more dominated by the subfields of artificial intelligence and machine-biological interfaces (i.e. cyborgs) and when this research fell out of favor, the field as a whole fell from grace.

In the 1970s new cyberneticians emerged in multiple fields, but especially in biology. The ideas of Maturana, Varela and Atlan, according to Dupuy (1986) "realized that the cybernetic metaphors of the program upon which molecular biology had been based rendered a conception of the autonomy of the living being impossible. Consequently, these thinkers were led to invent a new cybernetics, one more suited to the organizations which mankind discovers in nature - organizations he has not himself invented"^[9]. However, during the 1980s the question of whether the features of this new cybernetics could be applied to social forms of organization remained open to debate.^[9]

In political science, Project Cybersyn attempted to introduce a cybernetically controlled economy during the early 1970s. In the 1980s, according to Harries-Jones (1988) "unlike its predecessor, the new cybernetics concerns itself with the interaction of autonomous political actors and subgroups, and the practical and reflexive consciousness of the subjects who produce and reproduce the structure of a political community. A dominant consideration is that of recursiveness, or self-reference of political action both with regards to the expression of political consciousness and with the ways in which systems build upon themselves".^[10]

One characteristic of the emerging new cybernetics considered in that time by Geyer and van der Zouwen, according to Bailey (1994), was "that it views information as constructed and reconstructed by an individual interacting with the environment. This provides an epistemological foundation of science, by viewing it as observer-dependent.

Another characteristic of the new cybernetics is its contribution towards bridging the "micro-macro gap". That is, it links the individual with the society"^[11] Another characteristic noted was the "transition from classical cybernetics to the new cybernetics [that] involves a transition from classical problems to new problems. These shifts in thinking involve, among others, (a) a change from emphasis on the system being steered to the system doing the steering, and the factor which guides the steering decisions.; and (b) new emphasis on communication between several systems which are trying to steer each other"^[11]. The work of Gregory Bateson was also strongly influenced by cybernetics.

Recent endeavors into the true focus of cybernetics, systems of control and emergent behavior, by such related fields as game theory (the analysis of group interaction), systems of feedback in evolution, and metamaterials (the study of materials with properties beyond the Newtonian properties of their constituent atoms), have led to a revived interest in this increasingly relevant field.^[2]

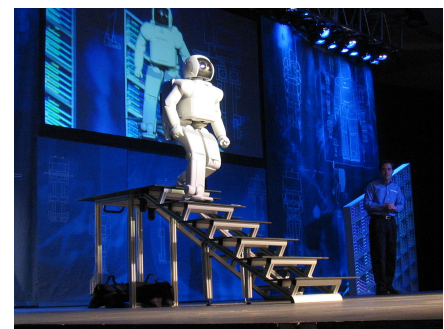
Subdivisions of the field

Cybernetics is an earlier but still-used generic term for many types of subject matter. These subjects also extend into many others areas of science, but are united in their study of control of systems.

Basic cybernetics

Cybernetics studies systems of control as a concept, attempting to discover the basic principles underlying such things as

- Artificial intelligence
- Robotics
- Computer Vision
- Control systems
- Emergence
- Learning organization
- New Cybernetics
- Second-order cybernetics
- Interactions of Actors Theory
- Conversation Theory



ASIMO uses sensors and intelligent algorithms to avoid obstacles and navigate stairs.

In biology

Cybernetics in biology is the study of cybernetic systems present in biological organisms, primarily focusing on how animals adapt to their environment, and how information in the form of genes is passed from generation to generation^[12]. There is also a secondary focus on combining artificial systems with biological systems.

- Bioengineering
- Biocybernetics
- Bionics
- Homeostasis
- Medical cybernetics
- Synthetic Biology
- Systems Biology

In computer science

Computer science directly applies the concepts of cybernetics to the control of devices and the analysis of information.

- Robotics
- Decision support system
- Cellular automaton
- Simulation
- Technology

In engineering

Cybernetics in engineering is used to analyze cascading failures and System Accidents, in which the small errors and imperfections in a system can generate disasters. Other topics studied include:

- Adaptive systems
- Engineering cybernetics
- Ergonomics
- Biomedical engineering
- Systems engineering



An artificial heart, a product of biomedical engineering.

In management

- Entrepreneurial cybernetics
- Management cybernetics
- Organizational cybernetics
- Operations research
- Systems engineering

In mathematics

Mathematical Cybernetics focuses on the factors of information, interaction of parts in systems, and the structure of systems.

- Dynamical system
- Information theory
- Systems theory

In psychology

- Homunculus
- Psycho-Cybernetics
- Systems psychology
- Perceptual Control Theory

In sociology

By examining group behavior through the lens of cybernetics, sociology seeks the reasons for such spontaneous events as smart mobs and riots, as well as how communities develop rules, such as etiquette, by consensus without formal discussion. Affect Control Theory explains role behavior, emotions, and labeling theory in terms of homeostatic maintenance of sentiments associated with cultural categories. The most comprehensive attempt ever made in the social sciences to increase cybernetics in a generalized theory of society was made by Talcott Parsons.

In this way, cybernetics establish the basic hierarchy in Parsons' AGIL paradigm, which is the ording system-dimension of his action theory. These and other cybernetic models in sociology are reviewed in a book edited by McClelland and Fararo^[13].

- Affect Control Theory
- Memetics
- Sociocybernetics

In art

The artist Roy Ascott theorised the cybernetics of art in "Behaviourist Art and the Cybernetic Vision". *Cybernetica*, Journal of the International Association for Cybernetics (Namur), 1967.

- Telematic art
- Interactive Art
- Systems art

Related fields

Complexity science

Complexity science attempts to understand the nature of complex systems.

- Complex Adaptive System
- Complex systems
- Complexity theory

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External links

General

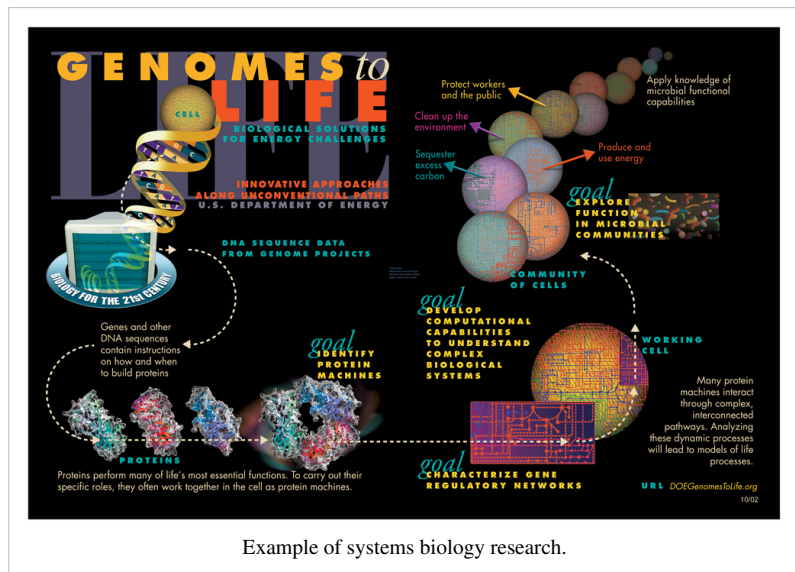
- Norbert Wiener and Stefan Odobleja - A Comparative Analysis (<http://www.bu.edu/wcp/Papers/Comp/CompJurc.htm>)
- Reading List for Cybernetics (<http://www.cscs.umich.edu/~crshalizi/notabene/cybernetics.html>)
- *Principia Cybernetica Web* (<http://pespmc1.vub.ac.be/DEFAULT.html>)
- Web Dictionary of Cybernetics and Systems (<http://pespmc1.vub.ac.be/ASC/indexASC.html>)
- Glossary Slideshow (136 slides) (<http://www.gwu.edu/~asc/slide/s1.html>)
- Basics of Cybernetics (<http://www.smithsrisca.demon.co.uk/cybernetics.html>)
- What is Cybernetics? (http://www.youtube.com/watch?v=_hjAXkNbPfk) Livas short introductory videos on YouTube
- A History of Systemic and Cybernetic Thought. From Homeostasis to the Teardrop (http://www.pclibya.com/cybernetic_teardrop.pdf)

Societies

- American Society for Cybernetics (<http://www.asc-cybernetics.org/>)
- IEEE Systems, Man, & Cybernetics Society (<http://www.ieeesmc.org/>)
- The Cybernetics Society (<http://www.cybsoc.org>)

Systems Biology

Systems biology is a term used to describe a number of trends in bioscience research, and a movement which draws on those trends. Proponents describe systems biology as a biology-based inter-disciplinary study field that focuses on complex interactions in biological systems, claiming that it uses a new perspective (holism instead of reduction). Particularly from year 2000 onwards, the term is used widely in the biosciences, and in a variety of contexts. An often stated ambition of systems biology is the modeling and discovery of emergent properties, properties of a system whose theoretical description is only possible using techniques which fall under the remit of systems biology.



Example of systems biology research.

Overview

Systems biology can be considered from a number of different aspects:

- As a **field of study**, particularly, the study of the interactions between the components of *biological systems*, and how these interactions give rise to the function and behavior of that system (for example, the enzymes and metabolites in a metabolic pathway).^{[1] [2]}
- As a **paradigm**, usually defined in antithesis to the so-called reductionist paradigm (biological organisation), although fully consistent with the scientific method. The distinction between the two paradigms is referred to in these quotations:

"The reductionist approach has successfully identified most of the components and many of the interactions but, unfortunately, offers no convincing concepts or methods to understand how system properties emerge...the pluralism of causes and effects in biological networks is better addressed by observing, through quantitative measures, multiple components simultaneously and by rigorous data integration with mathematical models" Science^[3]

"Systems biology...is about putting together rather than taking apart, integration rather than reduction. It requires that we develop ways of thinking about integration that are as rigorous as our reductionist programmes, but different....It means changing our philosophy, in the full sense of the term" Denis Noble^[4]

- As a series of **operational protocols used for performing research**, namely a cycle composed of theory, analytic or computational modelling to propose specific testable hypotheses about a biological system, experimental validation, and then using the newly acquired quantitative description of cells or cell processes to refine the computational model or theory.^{[5] [6]} Since the objective is a model of the interactions in a system, the experimental techniques that most suit systems biology are those that are system-wide and attempt to be as complete as possible. Therefore, transcriptomics, metabolomics, proteomics and high-throughput techniques are used to collect quantitative data for the construction and validation of models.
- As the application of dynamical systems theory to molecular biology.
- As a **socioscientific phenomenon** defined by the strategy of pursuing integration of complex data about the interactions in biological systems from diverse experimental sources using interdisciplinary tools and personnel.

This variety of viewpoints is illustrative of the fact that systems biology refers to a cluster of peripherally overlapping concepts rather than a single well-delineated field. However the term has widespread currency and popularity as of 2007, with chairs and institutes of systems biology proliferating worldwide.

History

Systems biology finds its roots in:

- the quantitative modeling of enzyme kinetics, a discipline that flourished between 1900 and 1970,
- the mathematical modeling of population growth,
- the simulations developed to study neurophysiology, and
- control theory and cybernetics.

One of the theorists who can be seen as one of the precursors of systems biology is Ludwig von Bertalanffy with his general systems theory.^[7] One of the first numerical simulations in biology was published in 1952 by the British neurophysiologists and Nobel prize winners Alan Lloyd Hodgkin and Andrew Fielding Huxley, who constructed a mathematical model that explained the action potential propagating along the axon of a neuronal cell.^[8] Their model described a cellular function emerging from the interaction between two different molecular components, a potassium and a sodium channels, and can therefore be seen as the beginning of computational systems biology.^[9] In 1960, Denis Noble developed the first computer model of the heart pacemaker.^[10]

The formal study of systems biology, as a distinct discipline, was launched by systems theorist Mihajlo Mesarovic in 1966 with an international symposium at the Case Institute of Technology in Cleveland, Ohio entitled "Systems Theory and Biology."^{[11] [12]}

The 1960s and 1970s saw the development of several approaches to study complex molecular systems, such as the Metabolic Control Analysis and the biochemical systems theory. The successes of molecular biology throughout the 1980s, coupled with a skepticism toward theoretical biology, that then promised more than it achieved, caused the quantitative modelling of biological processes to become a somewhat minor field.

However the birth of functional genomics in the 1990s meant that large quantities of high quality data became available, while the computing power exploded, making more realistic models possible. In 1997, the group of Masaru Tomita published the first quantitative model of the metabolism of a whole (hypothetical) cell.

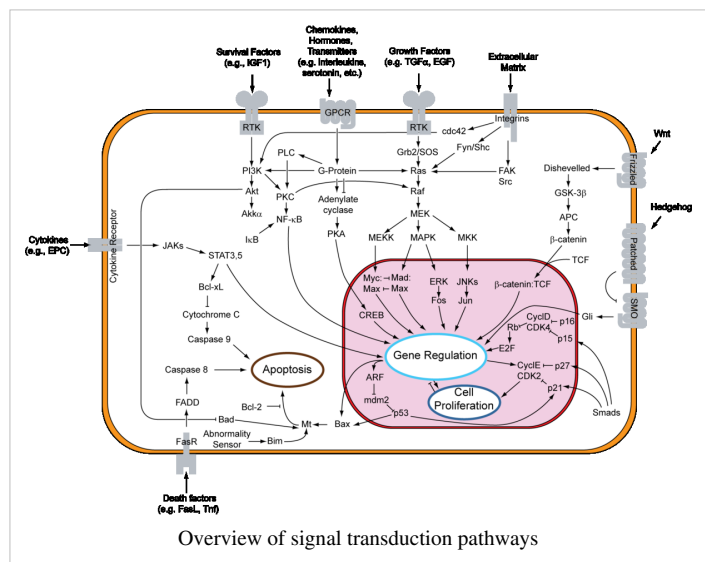
Around the year 2000, after Institutes of Systems Biology were established in Seattle and Tokyo, systems biology emerged as a movement in its own right, spurred on by the completion of various genome projects, the large increase in data from the omics (e.g. genomics and proteomics) and the accompanying advances in high-throughput experiments and bioinformatics. Since then, various research institutes dedicated to systems biology have been developed. As of summer 2006, due to a shortage of people in systems biology^[13] several doctoral training centres in systems biology have been established in many parts of the world.

Disciplines associated with systems biology

According to the interpretation of Systems Biology as the ability to obtain, integrate and analyze complex data from multiple experimental sources using interdisciplinary tools, some typical technology platforms are:

- **Phenomics:** Organismal variation in phenotype as it changes during its life span.
- **Genomics:** Organismal deoxyribonucleic acid (DNA) sequence, including intra-organisam cell specific variation. (i.e. Telomere length variation etc.).
- **Epigenomics / Epigenetics:** Organismal and corresponding cell specific transcriptomic regulating factors not empirically coded in the genomic sequence. (i.e. DNA methylation, Histone Acetelation etc.).
- **Transcriptomics:** Organismal, tissue or whole cell gene expression measurements by DNA microarrays or serial analysis of gene expression
- **Interferomics:** Organismal, tissue, or cell level transcript correcting factors (i.e. RNA interference)
- **Translatomics / Proteomics:** Organismal, tissue, or cell level measurements of proteins and peptides via two-dimensional gel electrophoresis, mass spectrometry or multi-dimensional protein identification techniques (advanced HPLC systems coupled with mass spectrometry). Sub disciplines include phosphoproteomics, glycoproteomics and other methods to detect chemically modified proteins.
- **Metabolomics:** Organismal, tissue, or cell level measurements of all small-molecules known as metabolites.
- **Glycomics:** Organismal, tissue, or cell level measurements of carbohydrates.
- **Lipidomics:** Organismal, tissue, or cell level measurements of lipids.

In addition to the identification and quantification of the above given molecules further techniques analyze the dynamics and interactions within a cell. This includes:



- Interactomics: Organismal, tissue, or cell level study of interactions between molecules. Currently the authoritative molecular discipline in this field of study is protein-protein interactions (PPI), although the working definition does not pre-clude inclusion of other molecular disciplines such as those defined here.
- Fluxomics: Organismal, tissue, or cell level measurements of molecular dynamic changes over time.
- Biomics: systems analysis of the biome.

The investigations are frequently combined with large scale perturbation methods, including gene-based (RNAi, mis-expression of wild type and mutant genes) and chemical approaches using small molecule libraries. Robots and automated sensors enable such large-scale experimentation and data acquisition. These technologies are still emerging and many face problems that the larger the quantity of data produced, the lower the quality. A wide variety of quantitative scientists (computational biologists, statisticians, mathematicians, computer scientists, engineers, and physicists) are working to improve the quality of these approaches and to create, refine, and retest the models to accurately reflect observations.

The systems biology approach often involves the development of mechanistic models, such as the reconstruction of dynamic systems from the quantitative properties of their elementary building blocks.^{[14] [15]} For instance, a cellular network can be modelled mathematically using methods coming from chemical kinetics and control theory. Due to the large number of parameters, variables and constraints in cellular networks, numerical and computational techniques are often used.

Other aspects of computer science and informatics are also used in systems biology. These include:

- New forms of computational model, such as the use of process calculi to model biological processes (notable approaches include stochastic π -calculus, BioAmbients, Beta Binders, BioPEPA and Brane calculus) and constraint-based modeling.
- Integration of information from the literature, using techniques of information extraction and text mining.
- Development of online databases and repositories for sharing data and models, approaches to database integration and software interoperability via loose coupling of software, websites and databases, or commercial suits.
- Development of syntactically and semantically sound ways of representing biological models.

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- WIREs Systems Biology and Medicine (<http://wires.wiley.com/WileyCDA/WiresJournal/wisId-WSBM.html>) - open access review journal on systems biology and medicine
- EURASIP Journal on Bioinformatics and Systems Biology (<http://www.hindawi.com/journals/bsb/>)
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External links

- Institute for Systems Biology: SBI (<http://www.systemsbiology.org>)
- Applied BioDynamics Laboratory: Boston University (<http://www.bu.edu/abl/>)
- Institute for Research in Immunology and Cancer (IRIC): Université de Montréal (<http://www.irc.ca>)
- Systems Biology - BioChemWeb.org (<http://www.biochemweb.org/systems.shtml>)
- Systems Biology Portal (<http://www.systems-biology.org/>) - administered by the Systems Biology Institute
- Semantic Systems Biology (<http://www.semantic-systems-biology.org>)
- SystemsX.ch (<http://www.systemsx.ch/>) - The Swiss Initiative in Systems Biology
- Systems Biology at the Pacific Northwest National Laboratory (<http://www.sysbio.org/>)
- Institute of Bioinformatics and Systems Biology (<http://bioinfo.nctu.edu.tw/>), National Chiao Tung University, Taiwan

Neurosciences

Neuroscience is the scientific study of the nervous system.^[1] Traditionally, neuroscience has been seen as a branch of biology. However, it is currently an interdisciplinary science that collaborates with other fields such as chemistry, computer science, engineering, mathematics, medicine, philosophy, physics, and psychology. The term neurobiology is usually used interchangeably with the term neuroscience, although the former refers specifically to the biology of the nervous system, whereas the latter refers to the entire science of the nervous system.

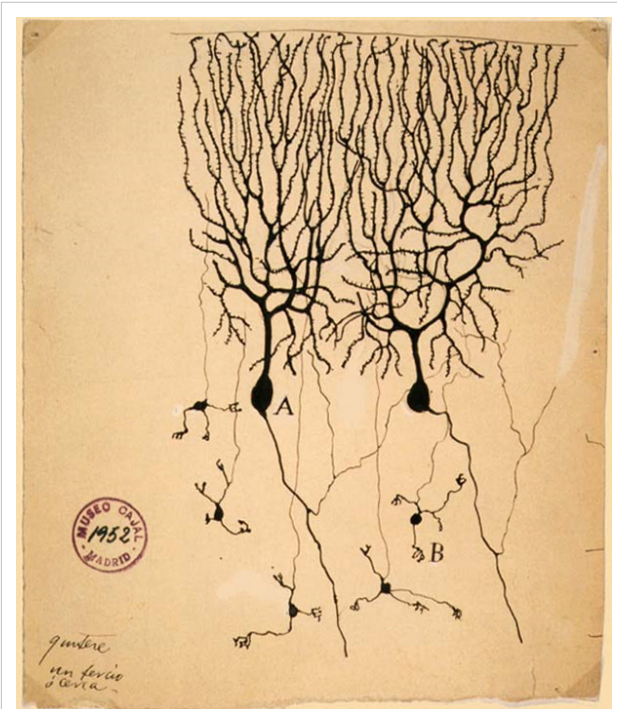
The scope of neuroscience has broadened to include different approaches used to study the molecular, cellular, developmental, structural, functional, evolutionary, computational, and medical aspects of the nervous system. The techniques used by neuroscientists have also expanded enormously, from molecular and cellular studies of individual nerve cells to imaging of sensory and motor tasks in the brain. Recent theoretical advances in neuroscience have also been aided by the study of neural networks.

Given the increasing number of scientists who study the nervous system, several prominent neuroscience organizations have been formed to provide a forum to all neuroscientists and educators. For example, the International Brain Research Organization was founded in 1960,^[2] the European Brain and Behaviour Society in 1968,^[3] and the Society for Neuroscience in 1969.^[4]

History

The study of the nervous system dates back to ancient Egypt. Evidence of trepanation, the surgical practice of either drilling or scraping a hole into the skull with the purpose of curing headaches or mental disorders or relieving cranial pressure, being performed on patients dates back to Neolithic times and has been found in various cultures throughout the world. Manuscripts dating back to 1700BC indicated that the Egyptians had some knowledge about symptoms of brain damage.^[5]

Early views on the function of the brain regarded it to be a "cranial stuffing" of sorts. In Egypt, from the late Middle Kingdom onwards, the brain was regularly removed in preparation for mummification. It was



Drawing by Santiago Ramón y Cajal (1899) of neurons in the pigeon cerebellum

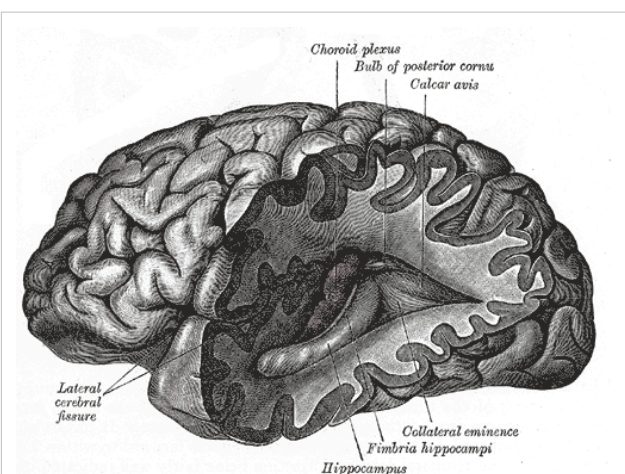


Illustration from Gray's Anatomy (1918) of a lateral view of the human brain, featuring the hippocampus among other neuroanatomical features

believed at the time that the heart was the seat of intelligence. According to Herodotus, the first step of mummification is to "take a crooked piece of iron, and with it draw out the brain through the nostrils, thus getting rid of a portion, while the skull is cleared of the rest by rinsing with drugs."^[6]

The view that the heart was the source of consciousness was not challenged until the time of Hippocrates. He believed that the brain was not only involved with sensation—since most specialized organs (e.g., eyes, ears, tongue) are located in the head near the brain—but was also the seat of intelligence. Plato also speculated that the brain was the seat of the rational part of the soul.^[7] Aristotle, however, believed the heart was the center of intelligence and that the brain served to cool the blood. This view was generally accepted until the Roman physician Galen, a follower of Hippocrates and physician to Roman gladiators, observed that his patients lost their mental faculties when they had sustained damage to their brains.

In al-Andalus, Abulcasis, the father of modern surgery, developed material and technical designs which are still used today in neurosurgery. Averroes suggested the existence of Parkinson's disease and attributed photoreceptor properties to the retina. Avenzoar described meningitis, intracranial thrombophlebitis, mediastinal tumours and made contributions to modern neuropharmacology. Maimonides wrote about neuropsychiatric disorders and described rabies and belladonna intoxication.^[8] Elsewhere in medieval Europe, Vesalius (1514–1564) and René Descartes (1596–1650) also made several contributions to neuroscience.

Studies of the brain became more sophisticated after the invention of the microscope and the development of a staining procedure by Camillo Golgi during the late 1890s. The procedure used a silver chromate salt to reveal the intricate structures of individual neurons. His technique was used by Santiago Ramón y Cajal and led to the formation of the neuron doctrine, the hypothesis that the functional unit of the brain is the neuron. Golgi and Ramón y Cajal shared the Nobel Prize in Physiology or Medicine in 1906 for their extensive observations, descriptions, and categorizations of neurons throughout the brain. The neuron doctrine was supported by experiments following Luigi Galvani's pioneering work in the electrical excitability of muscles and neurons. In the late 19th century, Emil du Bois-Reymond, Johannes Peter Müller, and Hermann von Helmholtz demonstrated that neurons were electrically excitable and that their activity predictably affected the electrical state of adjacent neurons.

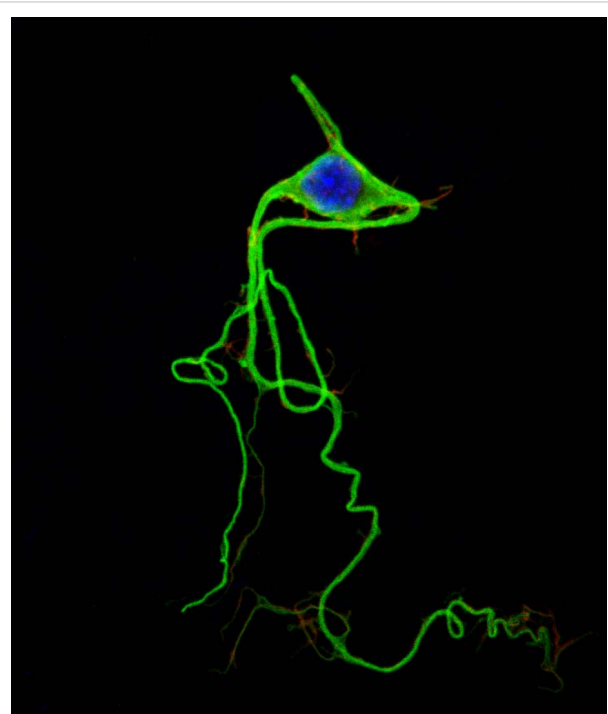
In parallel with this research, work with brain-damaged patients by Paul Broca suggested that certain regions of the brain were responsible for certain functions. At the time, Broca's findings were seen as a confirmation of Franz Joseph Gall's theory that language was localized and certain psychological functions were localized in the cerebral cortex.^[9] ^[10] The localization of function hypothesis was supported by observations of epileptic patients conducted by John Hughlings Jackson, who correctly inferred the organization of the motor cortex by watching the progression of seizures through the body. Carl Wernicke further developed the theory of the specialization of specific brain structures in language comprehension and production. Modern research still uses the Brodmann cerebral cytoarchitectonic map (referring to study of cell structure) anatomical definitions from this era in continuing to show that distinct areas of the cortex are activated in the execution of specific tasks.^[11]

In 1952, Alan Lloyd Hodgkin and Andrew Huxley presented a mathematical model for transmission of electrical signals in neurons of the giant axon of a squid, action potentials, and how they are initiated and propagated, known as the Hodgkin-Huxley model. In 1961-2, Richard FitzHugh and J. Nagumo simplified Hodgkin-Huxley, in what is called the FitzHugh–Nagumo model. In 1962, Bernard Katz modeled neurotransmission across the space between neurons known as synapses. In 1981 Catherine Morris and Harold Lecar combined these models in the Morris-Lecar model. In 1984, J. L. Hindmarsh and R.M. Rose further modeled neurotransmission.

Beginning in 1966, Eric Kandel and James Schwartz examined the biochemical analysis of changes in neurons associated with learning and memory storage.

Foundations of modern neuroscience

The scientific study of the nervous system increased significantly during the second half of the twentieth century, principally due to revolutions in molecular biology, electrophysiology, and computational neuroscience. It has become possible to understand, in much detail, the complex processes occurring within a single neuron. However, how networks of neurons produce complex cognitions and behaviors is still poorly understood.



Photograph of a stained neuron in a chicken embryo

“The task of neural science is to explain behavior in terms of the activities of the brain. How does the brain marshal its millions of individual nerve cells to produce behavior, and how are these cells influenced by the environment...? The last frontier of the biological sciences—their ultimate challenge—is to understand the biological basis of consciousness and the mental processes by which we perceive, act, learn, and remember. — Eric Kandel, *Principles of Neural Science*, 4th ed.”

The nervous system is composed of a network of neurons along with other, supportive, cells (e.g., glial cells). Neurons form functional circuits, each responsible for specific functions of behavior at the organismal level. Thus, neuroscience can be studied at many different levels, ranging from the molecular and cellular levels to the systems and cognitive levels.

At the molecular level, the basic questions addressed in molecular neuroscience include the mechanisms by which neurons express and respond to molecular signals and how axons form complex connectivity patterns. At this level, tools from molecular biology and genetics are used to understand how neurons develop and how genetic changes affect biological functions. The morphology, molecular identity, and physiological characteristics of neurons and how they relate to different types of behavior are also of considerable interest.

At the cellular level, the fundamental questions addressed in cellular neuroscience include the mechanisms of how neurons process signals physiologically and electrochemically. They address how signals are processed by dendrites, somas and axons, and how neurotransmitters and electrical signals are used to process signals in a neuron. Another major area of neuroscience is directed at investigations of the development of the nervous system. These questions include the patterning and regionalization of the nervous system, neural stem cells, differentiation of neurons and glia, neuronal migration, axonal and dendritic development, trophic interactions, and synapse formation.

At the systems level, the questions addressed in systems neuroscience include how neural circuits are formed and used anatomically and physiologically to produce functions such as reflexes, sensory integration, motor coordination, circadian rhythms, emotional responses, learning, and memory. In other words, they address how these neural circuits function and the mechanisms through which behaviors are generated. For example, systems level analysis addresses questions concerning specific sensory and motor modalities: how does vision work? How do

songbirds learn new songs and bats localize with ultrasound? How does the somatosensory system process tactile information? The related fields of neuroethology and neuropsychology address the question of how neural substrates underlie specific animal and human behaviors. Neuroendocrinology and psychoneuroimmunology examine interactions between the nervous system and the endocrine and immune systems, respectively.

At the cognitive level, cognitive neuroscience addresses the questions of how psychological functions are produced by neural circuitry. The emergence of powerful new measurement techniques such as neuroimaging (e.g., fMRI, PET, SPECT), electrophysiology, and human genetic analysis combined with sophisticated experimental techniques from cognitive psychology allows neuroscientists and psychologists to address abstract questions such as how human cognition and emotion are mapped to specific neural substrates.

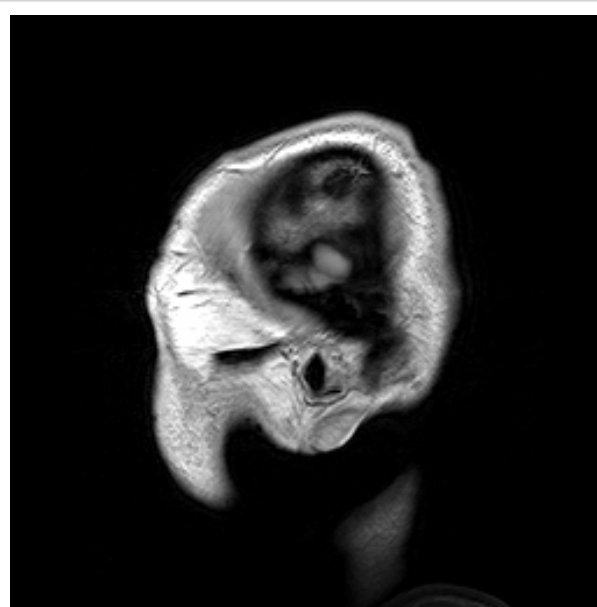
Neuroscience is also allied with the social and behavioral sciences as well as nascent interdisciplinary fields such as neuroeconomics, decision theory, and social neuroscience to address complex questions about interactions of the brain with its environment.

Neuroscience and medicine

Neurology, psychiatry, neurosurgery, psychosurgery, neuropathology, neuroradiology, clinical neurophysiology and addiction medicine are medical specialties that specifically address the diseases of the nervous system. These terms also refer to clinical disciplines involving diagnosis and treatment of these diseases. Neurology works with diseases of the central and peripheral nervous systems, such as amyotrophic lateral sclerosis (ALS) and stroke, and their medical treatment while psychiatry focuses on affective, behavioral, cognitive, and perceptual disorders. Neuropathology focuses upon the classification and underlying pathogenic mechanisms of central and peripheral nervous system and muscle diseases, with an emphasis on morphologic, microscopic, and chemically observable alterations. Neurosurgery and psychosurgery work primarily with surgical treatment of diseases of the central and peripheral nervous

systems. The boundaries between these specialties have been blurring recently as they are all influenced by basic research in neuroscience. Brain imaging also enables objective, biological insights into mental illness, which can lead to faster diagnosis, more accurate prognosis, and help assess patient progress over time.^[12]

Integrative neuroscience makes connections across these specialized areas of focus.



Parasagittal MRI of the head of a patient with benign familial macrocephaly

Major branches

Modern neuroscience education and research activities can be very roughly categorized into the following major branches, based on the subject and scale of the system in examination as well as distinct experimental or curricular approaches. Individual neuroscientists, however, often work on questions that span several distinct subfields.

Branch	Description
Behavioral neuroscience	Behavioral neuroscience (also known as biological psychology, biopsychology, or psychobiology) is the application of the principles of biology (viz., neurobiology) to the study of genetic, physiological, and developmental mechanisms of behavior in humans and non-human animals.
Cellular neuroscience	Cellular neuroscience is the study of neurons at a cellular level including morphology and physiological properties.
Cognitive neuroscience	Cognitive neuroscience is the study of biological substrates underlying cognition with a specific focus on the neural substrates of mental processes.
Computational neuroscience	Computational neuroscience is the study of brain function in terms of the information processing properties of the structures that make up the nervous system. Computational neuroscience can also refer to the use of computer simulations and theoretical models to study the function of the nervous system.
Cultural neuroscience	Cultural neuroscience is the study of how cultural values, practices and beliefs shape and are shaped by the mind, brain and genes across multiple timescales. ^[13]
Developmental neuroscience	Developmental neuroscience studies the processes that generate, shape, and reshape the nervous system and seeks to describe the cellular basis of neural development to address underlying mechanisms.
Molecular neuroscience	Molecular Neuroscience is a branch of neuroscience that examines the biology of the nervous system with molecular biology, molecular genetics, protein chemistry, and related methodologies.
Neuroengineering	Neuroengineering is a discipline within biomedical engineering that uses engineering techniques to understand, repair, replace, or enhance neural systems.
Neuroimaging	Neuroimaging includes the use of various techniques to either directly or indirectly image the structure and function of the brain.
Neuroinformatics	Neuroinformatics is a discipline within bioinformatics that conducts the organization of neuroscience data and application of computational models and analytical tools.
Neurolinguistics	Neurolinguistics is the study of the neural mechanisms in the human brain that control the comprehension, production, and acquisition of language.
Neurology and Psychiatry	Neurology is the medical specialty that works with disorders of the nervous system. Psychiatry is the medical specialty that works with the disorders of the mind—which include various affective, behavioral, cognitive, and perceptual disorders. (Also see note below.)
Social neuroscience	Social neuroscience is an interdisciplinary field devoted to understanding how biological systems implement social processes and behavior, and to using biological concepts and methods to inform and refine theories of social processes and behavior.
Systems neuroscience	Systems neuroscience is the study the function of neural circuits and systems.

In 1990s, neuroscientist Jaak Panksepp coined the term "affective neuroscience" to emphasize that research of emotion should be a branch of the neurosciences, distinguishable from the nearby fields of cognitive neuroscience or behavioral neuroscience.^[14] More recently, the social aspect of the emotional brain has been integrated in what is called "social-affective neuroscience" or simply social neuroscience.

Future directions

At this time in neuroscience research, several major questions remained unsolved, especially in cognitive neuroscience. For example, neuroscientists have yet to fully explain the neural basis of consciousness, learning, memory, perception, sensation, and sleep. Several questions regarding the development and evolution of the brain remain unsolved. Researchers have also yet to fully delineate the neural bases of mental disorders such as addiction, Alzheimer's disease, Parkinson's disease, and psychotic disorders (e.g., schizophrenia). Neuroscientific research on free will is also in the early stages of understanding.^[15] Thus, neuroscientists are continuously collaborating with other scientists and researchers to address many of these unresolved problems.^[16] Finally, proponents of the science of morality, such as the neuroscientist and writer Sam Harris, maintain that neuroscience will play an important role in the search for optimal moral systems.^[17]

Public education and outreach

In addition to conducting traditional research in laboratory settings, neuroscientists have also been involved in the promotion of awareness and knowledge about the nervous system among the general public and government officials. Such promotions have been done by both individual neuroscientists and large organizations. For example, individual neuroscientists have promoted neuroscience education among young students by organizing the International Brain Bee (IBB), which is an academic competition for high school or secondary school students worldwide.^[18] In the United States, large organizations such as the Society for Neuroscience have promoted neuroscience education by developing a primer called Brain Facts,^[19] collaborating with members of public education to develop Neuroscience Core Concepts for K-12 teachers and students,^[20] and cosponsoring a campaign with the Dana Foundation called Brain Awareness Week to increase public awareness about the progress and benefits of brain research.^[21]

Finally, neuroscientists have also collaborated with other education experts to study and refine educational techniques to optimize learning among students, an emerging field called educational neuroscience.^[22] Federal Agencies in the United States, such as the National Institute of Health (NIH) and National Science Foundation (NSF), have also funded research that pertain to best practices in teaching and learning of neuroscience concepts.

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- Sternberg, E. (2007) *Are You a Machine? The Brain, the Mind and What it Means to be Human*. Amherst, NY: Prometheus Books.

External links

- Neuroscience (<http://www.bbc.co.uk/programmes/b00fbd26>) on In Our Time at the BBC. (listen now (http://www.bbc.co.uk/iplayer/console/b00fbd26/In_Our_Time_Neuroscience))
- Neuroscience Information Framework (NIF) (<http://www.neuinfo.org>)
- Neurobiology (<http://www.dmoz.org/Science/Biology/Neurobiology/>) at the Open Directory Project
- IBRO (International Brain Research Organization) (<http://www.ibro.info>)
- Society for Neuroscience (SFN) (<http://www.sfn.org>)
- American Society for Neurochemistry (<http://www.asneurochem.org>)
- Neuroscience Online (electronic neuroscience textbook) (<http://neuroscience.uth.tmc.edu/>)
- Faculty for Undergraduate Neuroscience (FUN) (<http://www.funfaculty.org/drupal/>)
- Neuroscience for Kids (<http://faculty.washington.edu/chudler/neurok.html>)
- Neuroscience Discussion Group (<https://www.researchgate.net/group/Neuroscience>) in ResearchGate
- Neuroscience Discussion Forum (<http://www.neuroscienceforums.com>)
- HHMI Neuroscience lecture series - *Making Your Mind: Molecules, Motion, and Memory* (<http://www.hhmi.org/biointeractive/neuroscience/lectures.html>)
- British Neuroscience Association (<http://www.bna.org.uk>)

Biocybernetics

Biocybernetics is the application of cybernetics to biological science, composed of biological disciplines that benefit from the application of cybernetics: neurology, multicellular systems and others. Biocybernetics plays a major role in systems biology, seeking to integrate different levels of information to understand how biological systems function.

Biocybernetics as an abstract science is a part of theoretical biology, and based upon the principles of systemics.

Terminology

Biocybernetics is a cojoined word from bio (Greek: βίο / life) and cybernetics (Greek: κυβερνητική / controlling-governing). It is sometimes written together or with a blank or written fully as biological cybernetics, whilst the same rules apply. Most write it together though, as Google statistics show. The same applies to neuro cybernetics which should also be looked up as neurological, when doing extensive research.

Early fathers of biocybernetics

Ross Ashby, 1956 ^[1]

Hans Drischel, 1972 ^[2]

Norbert Wiener, 1948 ^[3]

Same or familiar fields

As those disciplines are dealing on theoretical/abstract foundations and are in accordance with the popularity of computers. Thus papers and research is in greater numbers going on under different names: e.g. molecular cybernetics -> molecular computational systems OR molecular systems theory OR molecular systemics OR molecular information/informational systems

Please heed this when you engage in an extensive search for information to assure access to a broad range of papers.

Categories

- biocybernetics - the study of an entire living organism
 - neurocybernetics - cybernetics dealing with neurological models. (Psycho-Cybernetics was the title of a self-help book, and is not a scientific discipline)
 - molecular cybernetics - cybernetics dealing with molecular systems (e.g. molecular biology cybernetics)
 - cellular cybernetics - cybernetics dealing with cellular systems (e.g. information technology/cell phones,... or biological cells)
 - evolutionary cybernetics - study of the evolution of informational systems (See also evolutionary programming, evolutionary algorithm)
 - any distinct informational system within the realm of biology
-

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External links

- Journal "Biological Cybernetics" (<http://www.springerlink.com/link.asp?id=100465>)
- Scientific portal on biological cybernetics (<http://www.biological-cybernetics.de>)
- UCLA Biocybernetics Laboratory (<http://biocyb.cs.ucla.edu/research.html>)

Computational neuroscience

Computational neuroscience is the study of brain function in terms of the information processing properties of the structures that make up the nervous system.^[1] It is an interdisciplinary science that links the diverse fields of neuroscience, cognitive science and psychology with electrical engineering, computer science, mathematics and physics.

Computational neuroscience is somewhat distinct from psychological connectionism and theories of learning from disciplines such as machine learning, neural networks and statistical learning theory in that it emphasizes descriptions of functional and biologically realistic neurons (and neural systems) and their physiology and dynamics. These models capture the essential features of the biological system at multiple spatial-temporal scales, from membrane currents, protein and chemical coupling to network oscillations, columnar and topographic architecture and learning and memory. These computational models are used to frame hypotheses that can be directly tested by current or future biological and/or psychological experiments.

History

The term "computational neuroscience" was introduced by Eric L. Schwartz, who organized a conference, held in 1985 in Carmel, California at the request of the Systems Development Foundation, to provide a summary of the current status of a field which until that point was referred to by a variety of names, such as neural modeling, brain theory and neural networks. The proceedings of this definitional meeting were later published as the book "Computational Neuroscience" (1990).^[2]

The early historical roots of the field can be traced to the work of people such as Louis Lapicque, Hodgkin & Huxley, Hubel & Wiesel, and David Marr, to name but a few. Lapicque introduced the integrate and fire model of the neuron in a seminal article published in 1907;^[3] this model is still one of the most popular models in computational neuroscience for both cellular and neural networks studies, as well as in mathematical neuroscience because of its simplicity (see the recent review article ^[4] published recently for the centenary of the original Lapicque's 1907 paper - this review also contains an English translation of the original paper). About 40 years later, Hodgkin & Huxley developed the voltage clamp and created the first biophysical model of the action potential. Hubel & Wiesel discovered that neurons in primary visual cortex, the first cortical area to process information coming from the retina, have oriented receptive fields and are organized in columns.^[5] David Marr's work focused on the interactions between neurons, suggesting computational approaches to the study of how functional groups of neurons within the hippocampus and neocortex interact, store, process, and transmit information. Computational modeling of biophysically realistic neurons and dendrites began with the work of Wilfrid Rall, with the first multicompartmental model using cable theory.

Organizations

The Organization for Computational Neuroscience^[6] is a non-profit organization one of whose tasks is to organize the annual international Computational Neuroscience meeting^[6].

Major topics

Research in computational neuroscience can be roughly categorized into several lines of inquiries. Most computational neuroscientists collaborate closely with experimentalists in analyzing novel data and synthesizing new models of biological phenomena.

Single-neuron modeling

Even single neurons have complex biophysical characteristics. Hodgkin and Huxley's original model only employed two voltage-sensitive currents, the fast-acting sodium and the inward-rectifying potassium. Though successful in predicting the timing and qualitative features of the action potential, it nevertheless failed to predict a number of important features such as adaptation and shunting. Scientists now believe that there are a wide variety of voltage-sensitive currents, and the implications of the differing dynamics, modulations and sensitivity of these currents is an important topic of computational neuroscience.^[7]

The computational functions of complex dendrites are also under intense investigation. There is a large body of literature regarding how different currents interact with geometric properties of neurons.^[8]

Some models are also tracking biochemical pathways at very small scales such as spines or synaptic clefts.

There are many software packages, such as GENESIS and NEURON, that allow rapid and systematic *in silico* modeling of realistic neurons. Blue Brain, a project founded by Henry Markram from the École Polytechnique Fédérale de Lausanne, aims to construct a biophysically detailed simulation of a cortical column on the Blue Gene supercomputer.

Development, axonal patterning and guidance

How do axons and dendrites form during development? How do axons know where to target and how to reach these targets? How do neurons migrate to the proper position in the central and peripheral systems? How do synapses form? We know from molecular biology that distinct parts of the nervous system release distinct chemical cues, from growth factors to hormones that modulate and influence the growth and development of functional connections between neurons.

Theoretical investigations into the formation and patterning of synaptic connection and morphology are still nascent. One hypothesis that has recently garnered some attention is the *minimal wiring hypothesis*, which postulates that the formation of axons and dendrites effectively minimizes resource allocation while maintaining maximal information storage.^[9]

Sensory processing

Early models of sensory processing understood within a theoretical framework is credited to Horace Barlow. Somewhat similar to the minimal wiring hypothesis described in the preceding section, Barlow understood the processing of the early sensory systems to be a form of efficient coding, where the neurons encoded information which minimized the number of spikes. Experimental and computational work have since supported this hypothesis in one form or another.

Current research in sensory processing is divided among biophysical modelling of different subsystems and more theoretical modelling function of perception. Current models of perception have suggested that the brain performs some form of Bayesian inference and integration of different sensory information in generating our perception of the physical world.

Memory and synaptic plasticity

Earlier models of memory are primarily based on the postulates of Hebbian learning. Biologically relevant models such as Hopfield net have been developed to address the properties of associative, rather than content-addressable style of memory that occur in biological systems. These attempts are primarily focusing on the formation of medium-term and long-term memory, localizing in the hippocampus. Models of working memory, relying on theories of network oscillations and persistent activity, have been built to capture some features of the prefrontal cortex in context-related memory.^[10]

One of the major problems in neurophysiological memory is how it is maintained and changed through multiple time scales. Unstable synapses are easy to train but also prone to stochastic disruption. Stable synapses forget less easily, but they are also harder to consolidate. One recent computational hypothesis involves cascades of plasticity^[11] that allow synapses to function at multiple time scales. Stereochemically detailed models of the acetylcholine receptor-based synapse with Monte Carlo method, working at the time scale of microseconds, have been built.^[12] It is likely that computational tools will contribute greatly to our understanding of how synapses function and change in relation to external stimulus in the coming decades.

Behaviors of networks

Biological neurons are connected to each other in a complex, recurrent fashion. These connections are, unlike most artificial neural networks, sparse and most likely, specific. It is not known how information is transmitted through such sparsely connected networks. It is also unknown what the computational functions, if any, of these specific connectivity patterns are.

The interactions of neurons in a small network can be often reduced to simple models such as the Ising model. The statistical mechanics of such simple systems are well-characterized theoretically. There has been some recent evidence that suggests that dynamics of arbitrary neuronal networks can be reduced to pairwise interactions.(Schneidman et al., 2006; Shlens et al., 2006.)^[13] It's unknown, however, whether such descriptive dynamics impart any important computational function. With the emergence of two-photon microscopy and calcium imaging, we now have powerful experimental methods with which to test the new theories regarding neuronal networks.

In some cases the complex interactions between *inhibitory* and *excitatory* neurons can be simplified using mean field theory that gives rise to population model of neural networks. While many neuro-theorists prefer such models with reduced complexity, others argue that uncovering structure function relations depends on including as much neuronal and network structure as possible. Models of this type are typically built in large simulations platforms like GENESIS or Neuron. There have been some attempts to provide unified methods that bridge, and integrate, these levels of complexity.^[14]

Cognition, discrimination and learning

Computational modeling of higher cognitive functions has only begun recently. Experimental data comes primarily from single-unit recording in primates. The frontal lobe and parietal lobe function as integrators of information from multiple sensory modalities. There are some tentative ideas regarding how simple mutually inhibitory functional circuits in these areas may carry out biologically relevant computation.^[15]

The brain seems to be able to discriminate and adapt particularly well in certain contexts. For instance, human beings seem to have an enormous capacity for memorizing and recognizing faces. One of the key goals of computational neuroscience is to dissect how biological systems carry out these complex computations efficiently and potentially replicate these processes in building intelligent machines.

The brain's large-scale organizational principles are illuminated by many fields, including biology, psychology, and clinical practice. Integrative neuroscience attempts to consolidate these observations through unified descriptive models, and databases of behavioral measures and recordings. These are the basis for some quantitative modeling of large-scale brain activity.^[16]

Consciousness

One of the ultimate goals of psychology/neuroscience is to be able to explain the everyday experience of conscious life. Francis Crick and Christof Koch made some attempts in formulating a consistent framework for future work in neural correlates of consciousness (NCC), though much of the work in this field remains speculative.^[17]

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External links

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- Network: Computation in Neural Systems (<http://www.informaworld.com/network>)
- Biological Cybernetics (<http://www.springerlink.com/openurl.asp?genre=journal&issn=0340-1200>)
- Journal of Computational Neuroscience (<http://www.springer.com/10827>)
- Neural Computation (<http://www.mitpressjournals.org/loi/neco>)
- Neural Networks (<http://www.sciencedirect.com/science/journal/08936080>)
- Neurocomputing (<http://www.elsevier.com/locate/neucom>)
- Cognitive Neurodynamics (<http://www.springerlink.com/content/1871-4099/>)
- Frontiers in Computational Neuroscience (<http://frontiersin.org/neuroscience/computationalneuroscience/>)
- PLoS Computational Biology (<http://www.ploscompbiol.org/home.action>)
- Frontiers in Neuroinformatics (<http://www.frontiersin.org/Journal/specialty.aspx?s=752&name=neuroinformatics&x=y>)

Software

- Brian (<http://www.briansimulator.org/>), a simulator for spiking neural networks.
- Nengo (<http://nengo.ca/>), a GUI or script driven large-scale spiking neural network simulator
- Emergent, neural simulation software.
- Genesis (<http://www.genesis-sim.org/GENESIS/>), a general neural simulation system.
- HHsim (<http://www-2.cs.cmu.edu/~dst/HHsim/>), a neuronal membrane simulator.
- HNeT (http://www.andcorporation.com/index.html?frame_hnet.html), Holographic Neural Technology.
- LENS (<http://tedlab.mit.edu/~dr/Lens/>), The Light, Efficient Network Simulator.
- MCell (<http://www.mcell.cnl.salk.edu/>), A Monte Carlo Simulator of Cellular Microphysiology.
- STEPS (<http://steps.sourceforge.net/>), A Gillespie SSA engine for mesoscopic pathway simulations in complex 3D geometries.
- ModelDB (<http://senselab.med.yale.edu/modeldb>), a large open-access database of program codes of published computational neuroscience models.
- NEST (<http://www.nest-initiative.org>), a simulation tool for large neuronal systems.
- Neuroconstruct (<http://www.neuroconstruct.org>), software for developing biologically realistic 3D neural networks.
- Neurofitter (<http://neurofitter.sourceforge.net>), a parameter tuning package for electrophysiological neuron models.
- Neurojet (<http://www.neurojet.net>), a neural network simulator specialized for the hippocampus.
- NEURON (<http://www.neuron.yale.edu/>), a neuron simulator also useful to simulate neural networks.
- Neurospaces (<http://www.neurospaces.org/>), an efficient neural simulation system that uses software engineering principles from the industry.
- Neuroscience related Python tools (<http://neuralensemble.org/>)
 - pyNN (<http://neuralensemble.org/trac/PyNN>)
- PyDSTool (<http://pydstool.sourceforge.net>), a simulator and dynamical systems analysis tool with biophysical neuron and network model specification/construction and data analysis toolboxes.
- SNNAP (<http://snnap.uth.tmc.edu/>), a single neuron and neural network simulator tool.
- Topographica (<http://topographica.org/>), a software package for computational modelling of neural maps.
- AnimatLab, A neuromechanical simulator that combines biomechanical and biologically realistic neural network simulation. It allows the user to test hypotheses on the neural basis of behavior in a physically accurate 3-D virtual environment. <http://www.animatlab.com/>

Conferences

- Computational and Systems Neuroscience (COSYNE) (<http://www.cosyne.org>)— a computational neuroscience meeting with a systems neuroscience focus.
- Annual Computational Neuroscience Meeting (CNS) (<http://www.cnsorg.org>)— a yearly computational neuroscience meeting.
- Neural Information Processing Systems (NIPS) (<http://www.nips.cc>)— a leading annual conference covering other machine learning topics as well.
- Computational Cognitive Neuroscience Conference (CCNC) (<http://www.ccnconference.org>)— a yearly conference.
- International Conference on Cognitive Neurodynamics (ICCN) (<http://www.iccn2007.org/>)— a yearly conference.
- UK Mathematical Neurosciences Meeting (<http://www.icms.org.uk/workshops/mathneuro>)— a new yearly conference, focused on mathematical aspects.

- The NeuroComp Conference (<http://www.neurocomp.fr/index.php?page=welcome>)— a yearly computational neuroscience conference (France).
- Bernstein Conference on Computational Neuroscience (BCCN) (http://www.nncn.de/Aktuelles-en/bernsteinsymposium/Symposium/view?set_language=en)— a yearly conference in Germany, organized by the Bernstein Network for Computational Neuroscience (http://www.nncn.de/willkommen-en/view?set_language=en).

Websites

- Perlewitz's computational neuroscience on the web (<http://home.earthlink.net/~perlewitz/>)
- compneuro.org (<http://www.compneuro.org>), books and programs for neural modeling
- Encyclopedia of Computational Neuroscience (http://www.scholarpedia.org/article/Encyclopedia_of_Computational_Neuroscience), part of Scholarpedia, an online expert curated encyclopedia on computational neuroscience, dynamical systems and machine intelligence
- NeuroWiki (<http://purl.net/net/neurowiki>), a wiki discussion forum about neuroscience research, especially systems, theoretical/computational, and cognitive neuroscience

Courses

- NeuroWiki:CompNeuroCourses (<http://purl.net/net/neurowiki/CompNeuroCourses>), a list of comp neuro courses with material available online
- Methods in Computational Neuroscience (http://www.mbl.edu/education/courses/special_topics/mcn.html) Summer course at the MBL, which features major figures in the field (Abbott, Bialek, Sejnowski, et al.) as guest faculty.
- Okinawa Computational Neuroscience Course (<http://www.irp.oist.jp/ocnc/index.html>) Summer course at OIST with international guest faculty and competitively selected international students.

Research groups

- Center for Theoretical Neuroscience at Columbia University (<http://neurotheory.columbia.edu>)
- Redwood Center for Theoretical Neuroscience at University of California, Berkeley (<https://redwood.berkeley.edu>)
- Bernstein Network for Computational Neuroscience (http://www.nncn.de/willkommen-en/view?set_language=en), containing the Bernstein Centers (Berlin (<http://www.bccn-berlin.de/>), Freiburg (<http://www.bcf.uni-freiburg.de/>), Goettingen (<http://www.bccn-goettingen.de/>), Munich (<http://www.bccn-munich.de/>))
- BM-Science— Brain & Mind Technologies Research Centre, Finland (<http://www.bm-science.com>)
- Committee on Computational Neuroscience at The University of Chicago (<http://cns.bsd.uchicago.edu>)
- Neuroengineering Laboratory at the University of Pennsylvania (<http://www.neuroengineering.upenn.edu>)
- Computational Neuroscience Group at the KFKI RIPNP of the Hungarian Academy of Sciences (<http://cneuro.rmki.kfki.hu>)
- Computational Neuroscience Laboratory, CNRS, Gif sur Yvette, France (<http://cns.iaf.cnrs-gif.fr/Main.html>)
- Computational Neurobiology Laboratory at the Salk Institute (CNL) (<http://www.cnl.salk.edu>)
- Centre for Theoretical Neuroscience (CTN) at the University of Waterloo (<http://ctn.uwaterloo.ca>)
- MIT Media Lab, Synthetic Neurobiology Group (<http://neuro.media.mit.edu/>)
- Institute for Theoretical Biology, Humboldt-Universitaet zu Berlin (<http://itb.biologie.hu-berlin.de/>)
- Computational Neuroscience Group at King's College London (<http://www.mth.kcl.ac.uk/research/cns/cns>)
- Computational Neuroscience Group (<http://neuro.fi.isc.cnr.it>) at Istituto dei Sistemi Complessi (<http://www.fi.isc.cnr.it>), Florence, Italy

- The Laboratory for Neuroengineering at the Georgia Institute of Technology (<http://www.neuro.gatech.edu/>)
- Boston University Department of Cognitive and Neural Systems(CNS) (<http://cns-web.bu.edu/>)
- MIT Center for Biological & Computational Learning (CBCL) (<http://cbcl.mit.edu/cbcl/index.html>)
- NYU Center for Theoretical Visual Neuroscience (<http://www.cns.nyu.edu/sloan-swartz.php>)
- Center for Theoretical Neuroscience at Columbia University (<http://neurotheory.columbia.edu>)
- Center for the Neural Basis of Cognition at Carnegie Mellon University/University of Pittsburgh (<http://www.cnbc.cmu.edu>)
- Integrative and Computational Neuroscience Research Unit (UNIC), CNRS, Gif sur Yvette, France (<http://www.unic.cnrs-gif.fr>)
- Interdisciplinary Center for Neural Computation at Hebrew University (<http://icnc.huji.ac.il/>)
- Computational Neuroscience Group at the Norwegian University of Life Sciences (<http://compneuro.umb.no/>)
- Gatsby Computational Neuroscience Unit at University College London (<http://www.gatsby.ucl.ac.uk/>)
- Computational Neuroscience Group, University of Hertfordshire (<http://homepages.stca.herts.ac.uk/~comqvs>)
- Martinos Computational Neuroscience Center (<http://www.martinos.org/compneuro>) for integrating neuroimaging and computational neuroscience
- Georgetown Laboratory for Computational Cognitive Neuroscience (<http://riesenhuberlab.neuro.georgetown.edu>)
- Hertie Center for Clinical Brain Research, Laboratory for Action Representation and Learning (<http://www.uni-tuebingen.de/uni/knv/ar1>)
- Computational Neuroscience Lab, University of Queensland (<http://cns.qbi.uq.edu.au>)
- Computational Cognitive Neuroscience Lab, University of Colorado at Boulder (<http://ccnlab.colorado.edu>)
- Theoretical Neuroscience Group, Florida Atlantic University (<http://tng.ccs.fau.edu>)
- Centre for Cognitive Neuroscience and Cognitive Systems at the University of Kent (<http://www.cs.kent.ac.uk/projects/cncs/index.html>)
- Computational Neuroscience Engineering Lab, University of Florida (<http://www.cnel.ufl.edu>)
- Institute for Adaptive and Neural Computation, University of Edinburgh (<http://www.anc.ed.ac.uk/people/>)
- Centre for Robotics and Neural Systems, University of Plymouth (<http://www.tech.plym.ac.uk/SoCCE/CRNS/>)
- Theoretical Neurobiology Lab, University of Antwerp (<http://www.tnb.ua.ac.be>)
- Computational Neuroscience Unit, Okinawa Institute of Science and Technology (<http://www.irp.oist.jp/cns/>)
- Omneuron 3T MRI Research Center, California (PI: [[Christopher deCharms (<http://www.omneuron.com/>)]])
- Group for Neural Theory, Ecole normale superieure, Paris (<http://www.gnt.ens.fr/>)
- Computational Biology and Neurocomputing, Stockholm (<http://www.csc.kth.se/forskning/cb/cbn/>)
- Centre for Neural Dynamics, University of Ottawa (<http://www.neurodynamic.uottawa.ca/>)
- Computational Neuroscience Research Group, Tampere University of Technology (<http://www.cs.tut.fi/sgn/cns/>)
- Computational Neuroscience Group at FGU, Czech Academy of Sciences, Prague (<http://comput.biomed.cas.cz/>)
- Systems Neuroscience Group, Australia (<http://sites.google.com/site/systemsneurosciencegroup/>)

Papers

- Review ([http://papers.cnl.salk.edu/PDFs/Computational Neuroscience 1988-3883.pdf](http://papers.cnl.salk.edu/PDFs/Computational%20Neuroscience%201988-3883.pdf))— Sejnowski TJ, Koch C, Churchland PS (September 1988). "Computational neuroscience" (<http://www.sciencemag.org/cgi/pmidlookup?view=long&pmid=3045969>). *Science* **241** (4871): 1299–306. doi:10.1126/science.3045969. PMID 3045969.
- A Theory of Object Recognition: Computations and Circuits in the Feedforward Path of the Ventral Stream in Primate Visual Cortex (<http://cbcl.mit.edu/projects/cbcl/publications/ai-publications/2005/AIM-2005-036.pdf>)— Biologically-based vision algorithm

Molecular Neurosciences and Molecular Medicine

Molecular neuroscience is a branch of neuroscience that examines the biology of the nervous system with molecular biology, molecular genetics, protein chemistry and related methodologies.^[1] Molecular biology studies how deoxyribonucleic acid (DNA) forms ribonucleic acid (RNA) which makes protein. When molecular biology is studied to gain understanding of the nervous system, then this is the basis of molecular neuroscience. Molecular neuroscience studies ion channels, receptors, enzymes to understand neural function. Ionotropic receptors, metabotropic receptors, molecular anatomy, nervous system, neurodegenerative disease and molecular mechanisms neurotransmitter release, receptor cloning, signal transduction mechanisms, synaptic plasticity response, and voltage gated ion channels are a few of the fields studied by molecular neuroscientists.^[2]

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- [2] Revest, Patricia (1998) (digitised by google books online). *Molecular Neuroscience* ([http://books.google.com/?id=Ek4Gq5jmQM4C&pg=RA1-PA21&lpg=RA1-PA21&dq="Molecular+neuroscience"+definition](http://books.google.com/?id=Ek4Gq5jmQM4C&pg=RA1-PA21&lpg=RA1-PA21&dq=)). Taylor & Francis. ISBN 1859962505, 9781859962503. . Retrieved 2008-12-26.

Complex Systems

*This article largely discusses complex systems as **a subject of mathematics** and the attempts to emulate physical complex systems with emergent properties. For other scientific and professional disciplines addressing complexity in their fields see the complex systems article and references.*

A **complex system** is a system composed of interconnected parts that as a whole exhibit one or more properties (behavior among the possible properties) not obvious from the properties of the individual parts.^[1]

A system's complexity may be of one of two forms: disorganized complexity and organized complexity.^[2] In essence, disorganized complexity is a matter of a very large number of parts, and organized complexity is a matter of the subject system (quite possibly with only a limited number of parts) exhibiting emergent properties.

Examples of complex systems that complexity models are developed for include ant colonies, human economies and social structures, climate, nervous systems, cells and living things, including human beings, as well as modern energy or telecommunication infrastructures. Indeed, many systems of interest to humans are complex systems.

Complex systems are studied by many areas of natural science, mathematics, and social science. Fields that specialize in the interdisciplinary study of complex systems include systems theory, complexity theory, systems ecology, and cybernetics.

Overview

A complex system is a network of heterogeneous components that interact nonlinearly, to give rise to emergent behavior.^[3] The term *complex systems* has multiple meanings depending on its scope:

- A specific kind of systems which are complex
- A field of science studying these systems; *see further complex systems*
- A paradigm that complex systems have to be studied with non-linear dynamics; *see further complexity*

Various informal descriptions of complex systems have been put forward, and these may give some insight into their properties. A special edition of *Science* about complex systems^[4] highlighted several of these:

- A complex system is a highly structured system, which shows structure with variations (N. Goldenfeld and Kadanoff)
 - A complex system is one whose evolution is very sensitive to initial conditions or to small perturbations, one in which the number of independent interacting components is large, or one in which there are multiple pathways by which the system can evolve (Whitesides and Ismagilov)
 - A complex system is one that by design or function or both is difficult to understand and verify (Weng, Bhalla and Iyengar)
 - A complex system is one in which there are multiple interactions between many different components (D. Rind)
 - Complex systems are systems in process that constantly evolve and unfold over time (W. Brian Arthur).
-

History

Although one can argue that humans have been studying complex systems for thousands of years, the modern scientific study of complex systems is relatively young when compared to conventional science areas with simple system assumption such as physics and chemistry. The history of the scientific study of these systems follows several different research trends.

In the area of mathematics, arguably the largest contribution to the study of complex systems was the discovery of chaos in deterministic systems, a feature of certain dynamical systems that is strongly related to nonlinearity.^[5] The study of neural networks was also integral in advancing the mathematics needed to study complex systems.

The notion of self-organizing systems is tied up to work in nonequilibrium thermodynamics, including that pioneered by chemist and Nobel laureate Ilya Prigogine in his study of dissipative structures.

Types of complex systems

Chaotic systems

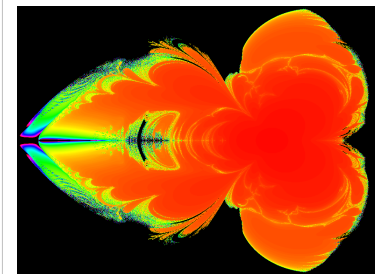
For a dynamical system to be classified as chaotic, it must have the following properties:^[6]

1. it must be sensitive to initial conditions,
2. it must be topologically mixing, and
3. its periodic orbits must be dense.

Sensitivity to initial conditions means that each point in such a system is arbitrarily closely approximated by other points with significantly different future trajectories. Thus, an arbitrarily small perturbation of the current trajectory may lead to significantly different future behavior.

Complex adaptive systems

Complex adaptive systems (CAS) are special cases of complex systems. They are complex in that they are diverse and made up of multiple interconnected elements and adaptive in that they have the capacity to change and learn from experience. Examples of complex adaptive systems include the stock market, social insect and ant colonies, the biosphere and the ecosystem, the brain and the immune system, the cell and the developing embryo, manufacturing businesses and any human social group-based endeavor in a cultural and social system such as political parties or communities. This includes some large-scale online systems, such as collaborative tagging or social bookmarking systems.



Assign z to z^2 minus the conjugate of z , plus the original value of the pixel for each pixel, then count how many cycles it took when the absolute value of z exceeds two; inversion (borders are inner set), so that you can see that it threatens to fail that third condition, even if it meets condition two.

Nonlinear system

The behavior of nonlinear systems is not subject to the principle of superposition while that of Linear systems is subject to superposition. Thus, a nonlinear system is one whose behavior can't be expressed as a sum of the behaviors of its parts (or of their multiples).

Topics on complex systems

Features of complex systems

Complex systems may have the following features:

Difficult to determine boundaries

It can be difficult to determine the boundaries of a complex system. The decision is ultimately made by the observer.

Complex systems may be open

Complex systems are usually open systems — that is, they exist in a thermodynamic gradient and dissipate energy. In other words, complex systems are frequently far from energetic equilibrium: but despite this flux, there may be pattern stability, see synergetics.

Complex systems may have a memory

The history of a complex system may be important. Because complex systems are dynamical systems they change over time, and prior states may have an influence on present states. More formally, complex systems often exhibit hysteresis.

Complex systems may be nested

The components of a complex system may themselves be complex systems. For example, an economy is made up of organisations, which are made up of people, which are made up of cells - all of which are complex systems.

Dynamic network of multiplicity

As well as coupling rules, the dynamic network of a complex system is important. Small-world or scale-free networks which have many local interactions and a smaller number of inter-area connections are often employed. Natural complex systems often exhibit such topologies. In the human cortex for example, we see dense local connectivity and a few very long axon projections between regions inside the cortex and to other brain regions.

May produce emergent phenomena

Complex systems may exhibit behaviors that are emergent, which is to say that while the results may be sufficiently determined by the activity of the systems' basic constituents, they may have properties that can only be studied at a higher level. For example, the termites in a mound have physiology, biochemistry and biological development that are at one level of analysis, but their social behavior and mound building is a property that emerges from the collection of termites and needs to be analysed at a different level.

Relationships are non-linear

In practical terms, this means a small perturbation may cause a large effect (see butterfly effect), a proportional effect, or even no effect at all. In linear systems, effect is *always* directly proportional to cause. See nonlinearity.

Relationships contain feedback loops

Both negative (damping) and positive (amplifying) feedback are always found in complex systems. The effects of an element's behaviour are fed back to in such a way that the element itself is altered.

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- [3] Rocha, Luis M. (1999). "Complex Systems Modeling: Using Metaphors From Nature in Simulation and Scientific Models (<http://informatics.indiana.edu/rocha/complex/csm.html>)". BITS: Computer and Communications News. Computing, Information, and Communications Division. Los Alamos National Laboratory. November 1999.
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Further reading

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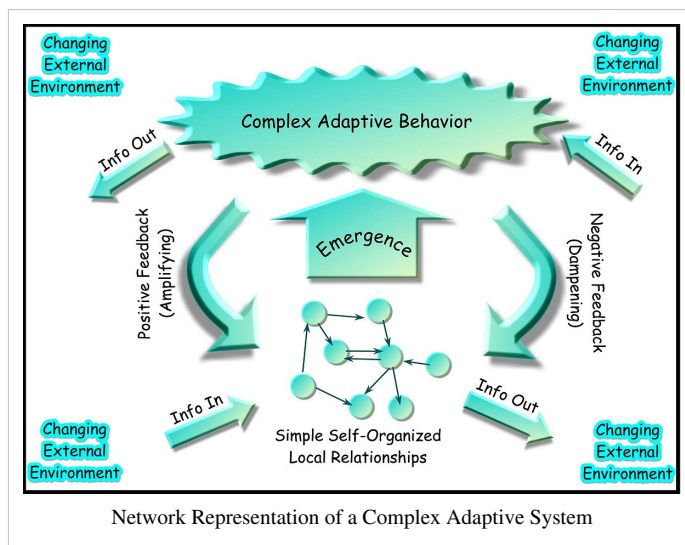
External links

Articles/General Information

- Complex systems (http://www.scholarpedia.org/article/Complex_Systems) in scholarpedia.
- (European) Complex Systems Society (<http://cssociety.org>)
- (Australian) Complex systems research network. (<http://www.complexsystems.net.au/>)
- Complex Systems Modeling (<http://informatics.indiana.edu/rocha/complex/csm.html>) based on Luis M. Rocha, 1999.
- CRM Complex systems research group (<http://www.crm.cat/HarmonicAnalysis/defaultHarmonicAnalysis.htm>)
- Center for Complex Systems Research, Univ. of Illinois (<http://www.ccsr.uiuc.edu/>)

Complex Systems Biology

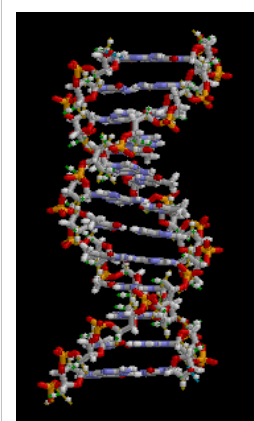
Complex systems biology (CSB) is a branch or subfield of mathematical and theoretical biology concerned with complexity of both structure and function in biological organisms, as well as the emergence and evolution of organisms and species, with emphasis being placed on the complex interactions of, and within, bionetworks^[1], and on the fundamental relations and relational patterns that are essential to life^{[2] [3] [4] [5] [6]}. **CSB** is thus a field of theoretical sciences aimed at discovering and modeling the relational patterns essential to life that has only a partial overlap with complex systems theory^[7], and also with the systems approach to biology called systems biology; this is because the latter is restricted primarily to simplified models of biological organization and organisms, as well as to only a general consideration of philosophical or semantic questions related to complexity in biology^[8]. Moreover, a wide range of abstract theoretical complex systems are studied as a field of applied mathematics, with or without relevance to biology, chemistry or physics.



Topics in complex systems biology

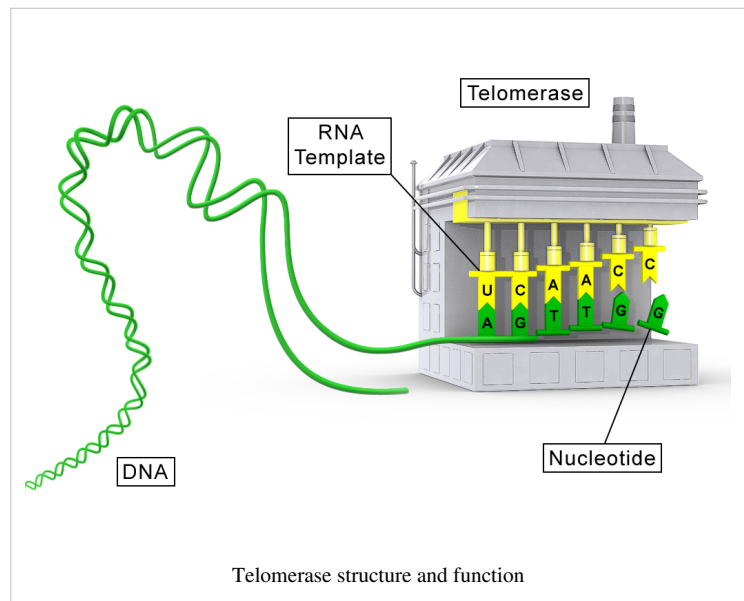
The following is only a partial list of topics covered in complex systems biology:

- Evolution theories and population genetics
 - Population genetics models
 - Molecular evolution theories
- Quantum biocomputation
- Quantum genetics^[9]
- Relational biology^{[10] [11] [12] [13]}
- Self-reproduction^[14] (also called self-replication in a more general context)
- Computational gene models
 - DNA topology
 - DNA sequencing theory
- Evolutionary developmental biology
- Autopoiesis
- Protein folding



Animated Molecular
Model of a DNA double
helix

- Telomerase conformations and functions *in vivo*
- Epigenetics
- Interactomics^[15] [16]
- Cell signaling
- Signal transduction networks
- Complex neural nets
- Genetic networks
- Morphogenesis
- Digital morphogenesis
- Complex adaptive systems
- Topological models of morphogenesis
- Population dynamics of fisheries
- Epidemiology



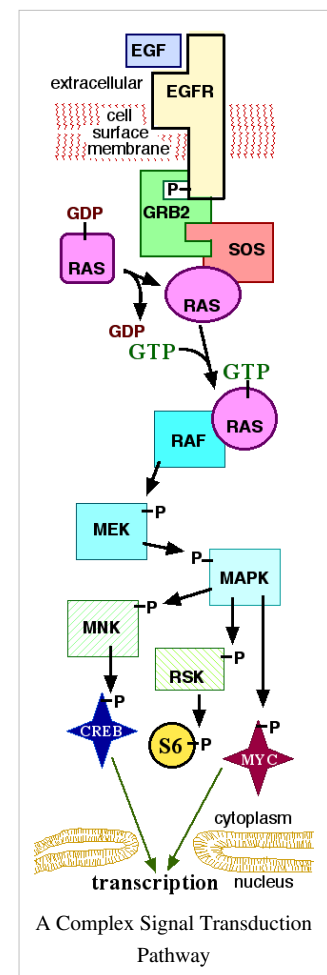
Related journals

- Acta Biotheoretica^[17]
- Bioinformatics^[18]
- Biological Theory^[19]
- BioSystems^[20]
- Bulletin of Mathematical Biology^[21]
- Ecological Modelling^[22]
- Journal of Mathematical Biology^[23]
- Journal of Theoretical Biology^[24]
- Mathematical Biosciences^[25]
- Medical Hypotheses^[26]
- Theoretical and Applied Genetics^[27]
- Theoretical Biology and Medical Modelling^[28]
- Theoretical Population Biology^[29]
- Theory in Biosciences^[30] (formerly: Biologisches Zentralblatt)

CBS societies and institutes

- Society for Mathematical Biology
- ESMTB: European Society for Mathematical and Theoretical Biology^[31]
- Division of Mathematical Biology at NIMR^[32]
- The Israeli Society for Theoretical and Mathematical Biology^[33]
- Société Francophone de Biologie Théorique^[34]

- International Society for Biosemiotic Studies^[35]



Biographies

- Charles Darwin
- D'Arcy Thompson
- William Ross Ashby
- Ludwig von Bertalanffy
- Ronald Brown
- Joseph Fourier
- Brian Goodwin
- George Karreman
- Charles S. Peskin
- Nicolas Rashevsky ^[36]
- Robert Rosen
- Anatol Rapoport
- Rosalind Franklin
- Francis Crick
- René Thom
- Vito Volterra
- Norbert Wiener

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Further reading

- A general list of Theoretical biology/Mathematical biology references, including an updated list of actively contributing authors ^[37].
- A list of references for applications of category theory in relational biology ^[38].
- An updated list of publications of theoretical biologist Robert Rosen ^[39]
- Theory of Biological Anthropology (Documents No. 9 and 10 in English) ^[40]
- Drawing the Line Between Theoretical and Basic Biology (a forum article by Isidro T. Savillo) ^[41]
- Semantic Systems Biology ^[42]
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External links

- Center for Complex Systems and Brain Sciences at Florida Atlantic (<http://www.ccs.fau.edu/>)
- Santa Fe Institute
- Bulletin of Mathematical Biology (<http://www.springerlink.com/content/119979/>)
- European Society for Mathematical and Theoretical Biology (<http://www.esmtb.org/>)
- Journal of Mathematical Biology (<http://www.springerlink.com/content/100436/>)
- Biomathematics Research Centre at University of Canterbury (<http://www.math.canterbury.ac.nz/bio/>)
- Centre for Mathematical Biology at Oxford University (<http://www.maths.ox.ac.uk/cmb/>)
- Mathematical Biology at the National Institute for Medical Research (<http://mathbio.nimr.mrc.ac.uk/>)
- Institute for Medical BioMathematics (<http://www.imbm.org/>)
- *Mathematical Biology Systems of Differential Equations* (<http://eqworld.ipmnet.ru/en/solutions/syspde/spde-toc2.pdf>) from EqWorld: The World of Mathematical Equations
- Systems Biology Workbench - a set of tools for modelling biochemical networks (<http://sbw.kgi.edu>)
- The Collection of Biostatistics Research Archive (<http://www.biostatsresearch.com/repository/>)
- Statistical Applications in Genetics and Molecular Biology (<http://www.bepress.com/sagmb/>)
- The International Journal of Biostatistics (<http://www.bepress.com/ijb/>)
- Theoretical Modeling of Cellular Physiology at Ecole Normale Supérieure, Paris (<http://www.biologie.ens.fr/bcsmcbs/>)
- Theoretical and mathematical biology website (<http://www.kli.ac.at/theorylab/index.html>)
- Complexity Discussion Group (<http://www.complex.vcu.edu/>)
- UCLA Biocybernetics Laboratory (<http://biocyb.cs.ucla.edu/research.html>)
- TUCS Computational Biomodelling Laboratory (<http://www.tucs.fi/research/labs/combio.php>)
- Nagoya University Division of Biomodeling (<http://www.agr.nagoya-u.ac.jp/english/e3senko-1.html>)

- Technische Universiteit Biomodeling and Informatics (<http://www.bmi2.bmt.tue.nl/Biomedinf/>)
- New England Complex Systems Institute
- Northwestern Institute on Complex Systems (NICO) (<http://www.northwestern.edu/nico/>)
- Complexity Digest (<http://comdig.unam.mx/>)
- Centro de Ciencias de la Complejidad (<http://c3.fisica.unam.mx/>), UNAM
- Complexity Complex at the University of Warwick (<http://go.warwick.ac.uk/complexity/>)
- Southampton Institute for Complex Systems Simulation (<http://www.icss.soton.ac.uk/>)
- Center for the Study of Complex Systems at the University of Michigan (<http://www.cscs.umich.edu/>)
- ARC Centre for Complex Systems, Australia
- (European) Complex Systems Society (<http://cssociety.org>)
- (Australian) Complex systems research network. (<http://www.complexsystems.net.au/>)
- Complex Systems Modeling (<http://informatics.indiana.edu/rocha/complex/csm.html>) based on Luis M. Rocha, 1999.
- CRM Complex systems research group (<http://www.crm.cat/HarmonicAnalysis/defaultHarmonicAnalysis.htm>)

Mathematical, Relational and Theoretical Biology

Mathematical and theoretical biology is an interdisciplinary scientific research field with a range of applications in biology, medicine and biotechnology.^[1] The field may be referred to as **mathematical biology** or **biomathematics** to stress the mathematical side, or as **theoretical biology** to stress the biological side.^[2] It includes at least four major subfields: *biological mathematical modeling*, *relational biology/complex systems biology (CSB)*, *bioinformatics* and *computational biomodeling/biocomputing*.^[3] ^[4] Mathematical biology aims at the mathematical representation, treatment and modeling of biological processes, using a variety of applied mathematical techniques and tools. It has both theoretical and practical applications in biological, biomedical and biotechnology research. For example, in cell biology, protein interactions are often represented as "cartoon" models, which, although easy to visualize, do not accurately describe the systems studied. In order to do this, precise mathematical models are required. By describing the systems in a quantitative manner, their behavior can be better simulated, and hence properties can be predicted that might not be evident to the experimenter.

Such mathematical areas as calculus, probability theory, statistics, linear algebra, abstract algebra, graph theory, combinatorics, algebraic geometry, topology, dynamical systems, differential equations and coding theory are now being applied in biology.^[5]

Importance

Applying mathematics to biology has a long history, but only recently has there been an explosion of interest in the field. Some reasons for this include:

- the explosion of data-rich information sets, due to the genomics revolution, which are difficult to understand without the use of analytical tools,
- recent development of mathematical tools such as chaos theory to help understand complex, nonlinear mechanisms in biology,
- an increase in computing power which enables calculations and simulations to be performed that were not previously possible, and
- an increasing interest in *in silico* experimentation due to ethical considerations, risk, unreliability and other complications involved in human and animal research.

Areas of research

Several areas of specialized research in mathematical and theoretical biology^{[6] [7] [8] [9] [10]} as well as external links to related projects in various universities are concisely presented in the following subsections, including also a large number of appropriate validating references from a list of several thousands of published authors contributing to this field. Many of the included examples are characterised by highly complex, nonlinear, and supercomplex mechanisms, as it is being increasingly recognised that the result of such interactions may only be understood through a combination of mathematical, logical, physical/chemical, molecular and computational models. Due to the wide diversity of specific knowledge involved, biomathematical research is often done in collaboration between mathematicians, biomathematicians, theoretical biologists, physicists, biophysicists, biochemists, bioengineers, engineers, biologists, physiologists, research physicians, biomedical researchers, oncologists, molecular biologists, geneticists, embryologists, zoologists, chemists, etc.

Computer models and automata theory

A monograph on this topic summarizes an extensive amount of published research in this area up to 1987,^[11] including subsections in the following areas: computer modeling in biology and medicine, arterial system models, neuron models, biochemical and oscillation networks, quantum automata^[12], quantum computers in molecular biology and genetics, cancer modelling, neural nets, genetic networks, abstract relational biology, metabolic-replication systems, category theory^[13] applications in biology and medicine,^[14] automata theory, cellular automata, tessellation models^{[15] [16]} and complete self-reproduction^[17], chaotic systems in organisms, relational biology and organismic theories.^{[18] [19]} This published report also includes 390 references to peer-reviewed articles by a large number of authors.^{[6] [20] [21]}

Modeling cell and molecular biology

This area has received a boost due to the growing importance of molecular biology.^[9]

- Mechanics of biological tissues^[22]
- Theoretical enzymology and enzyme kinetics
- Cancer modelling and simulation^{[23] [24]}
- Modelling the movement of interacting cell populations^[25]
- Mathematical modelling of scar tissue formation^[26]
- Mathematical modelling of intracellular dynamics^[27]
- Mathematical modelling of the cell cycle^[28]

Modelling physiological systems

- Modelling of arterial disease^[29]
- Multi-scale modelling of the heart^[30]

Molecular set theory

Molecular set theory was introduced by Anthony Bartholomay, and its applications were developed in mathematical biology and especially in Mathematical Medicine.^[31] Molecular set theory (MST) is a mathematical formulation of the wide-sense chemical kinetics of biomolecular reactions in terms of sets of molecules and their chemical transformations represented by set-theoretical mappings between molecular sets. In a more general sense, MST is the theory of molecular categories defined as categories of molecular sets and their chemical transformations represented as set-theoretical mappings of molecular sets. The theory has also contributed to biostatistics and the formulation of clinical biochemistry problems in mathematical formulations of pathological, biochemical changes of interest to Physiology, Clinical Biochemistry and Medicine.^{[31] [32]}

Population dynamics

Population dynamics has traditionally been the dominant field of mathematical biology. Work in this area dates back to the 19th century. The Lotka–Volterra predator-prey equations are a famous example. In the past 30 years, population dynamics has been complemented by evolutionary game theory, developed first by John Maynard Smith. Under these dynamics, evolutionary biology concepts may take a deterministic mathematical form. Population dynamics overlap with another active area of research in mathematical biology: mathematical epidemiology, the study of infectious disease affecting populations. Various models of the spread of infections have been proposed and analyzed, and provide important results that may be applied to health policy decisions.

Mathematical methods

A model of a biological system is converted into a system of equations, although the word 'model' is often used synonymously with the system of corresponding equations. The solution of the equations, by either analytical or numerical means, describes how the biological system behaves either over time or at equilibrium. There are many different types of equations and the type of behavior that can occur is dependent on both the model and the equations used. The model often makes assumptions about the system. The equations may also make assumptions about the nature of what may occur.

Mathematical biophysics

The earlier stages of mathematical biology were dominated by mathematical biophysics, described as the application of mathematics in biophysics, often involving specific physical/mathematical models of biosystems and their components or compartments.

The following is a list of mathematical descriptions and their assumptions.

Deterministic processes (dynamical systems)

A fixed mapping between an initial state and a final state. Starting from an initial condition and moving forward in time, a deterministic process will always generate the same trajectory and no two trajectories cross in state space.

- Difference equations/Maps – discrete time, continuous state space.
- Ordinary differential equations – continuous time, continuous state space, no spatial derivatives. *See also:* Numerical ordinary differential equations.
- Partial differential equations – continuous time, continuous state space, spatial derivatives. *See also:* Numerical partial differential equations.

Stochastic processes (random dynamical systems)

A random mapping between an initial state and a final state, making the state of the system a random variable with a corresponding probability distribution.

- Non-Markovian processes – generalized master equation – continuous time with memory of past events, discrete state space, waiting times of events (or transitions between states) discretely occur and have a generalized probability distribution.
- Jump Markov process – master equation – continuous time with no memory of past events, discrete state space, waiting times between events discretely occur and are exponentially distributed. *See also:* Monte Carlo method for numerical simulation methods, specifically dynamic Monte Carlo method and Gillespie algorithm.
- Continuous Markov process – stochastic differential equations or a Fokker-Planck equation – continuous time, continuous state space, events occur continuously according to a random Wiener process.

Spatial modelling

One classic work in this area is Alan Turing's paper on morphogenesis entitled *The Chemical Basis of Morphogenesis*, published in 1952 in the Philosophical Transactions of the Royal Society.

- Travelling waves in a wound-healing assay^[33]
- Swarming behaviour^[34]
- A mechanochemical theory of morphogenesis^[35]
- Biological pattern formation^[36]
- Spatial distribution modeling using plot samples^[37]

Relational biology

Abstract Relational Biology (ARB)^[38] is concerned with the study of general, relational models of complex biological systems, usually abstracting out specific morphological, or anatomical, structures. Some of the simplest models in ARB are the Metabolic-Replication, or (**M,R**)--systems introduced by Robert Rosen in 1957-1958 as abstract, relational models of cellular and organismal organization.

Phylogenetics

Phylogenetics is an area that deals with the reconstruction and analysis of phylogenetic (evolutionary) trees and networks based on inherited characteristics^[39]

Model example: the cell cycle

The eukaryotic cell cycle is very complex and is one of the most studied topics, since its misregulation leads to cancers. It is possibly a good example of a mathematical model as it deals with simple calculus but gives valid results. Two research groups^{[40] [41]} have produced several models of the cell cycle simulating several organisms. They have recently produced a generic eukaryotic cell cycle model which can represent a particular eukaryote depending on the values of the parameters, demonstrating that the idiosyncrasies of the individual cell cycles are due to different protein concentrations and affinities, while the underlying mechanisms are conserved (Csikasz-Nagy et al., 2006).

By means of a system of ordinary differential equations these models show the change in time (dynamical system) of the protein inside a single typical cell; this type of model is called a deterministic process (whereas a model describing a statistical distribution of protein concentrations in a population of cells is called a stochastic process).

To obtain these equations an iterative series of steps must be done: first the several models and observations are combined to form a consensus diagram and the appropriate kinetic laws are chosen to write the differential equations, such as rate kinetics for stoichiometric reactions, Michaelis-Menten kinetics for enzyme substrate reactions and Goldbeter–Koshland kinetics for ultrasensitive transcription factors, afterwards the parameters of the equations (rate constants, enzyme efficiency coefficients and Michealis constants) must be fitted to match observations; when they cannot be fitted the kinetic equation is revised and when that is not possible the wiring diagram is modified. The parameters are fitted and validated using observations of both wild type and mutants, such as protein half-life and cell size.

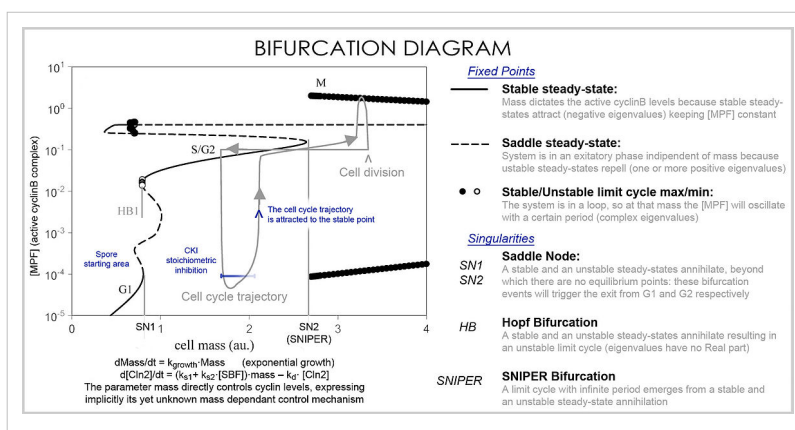
In order to fit the parameters the differential equations need to be studied. This can be done either by simulation or by analysis.

In a simulation, given a starting vector (list of the values of the variables), the progression of the system is calculated by solving the equations at each time-frame in small increments.

In analysis, the proprieties of the equations are used to investigate the behavior of the system depending of the values of the parameters and variables. A system of differential equations can be represented as a vector field, where each vector described the change (in concentration of two or more protein) determining where and how fast the trajectory (simulation) is heading. Vector fields can have several special points: a

stable point, called a sink, that attracts in all directions (forcing the concentrations to be at a certain value), an unstable point, either a source or a saddle point which repels (forcing the concentrations to change away from a certain value), and a limit cycle, a closed trajectory towards which several trajectories spiral towards (making the concentrations oscillate).

A better representation which can handle the large number of variables and parameters is called a bifurcation diagram (Bifurcation theory): the presence of these special steady-state points at certain values of a parameter (e.g. mass) is represented by a point and once the parameter passes a certain value, a qualitative change occurs, called a bifurcation, in which the nature of the space changes, with profound consequences for the protein concentrations: the cell cycle has phases (partially corresponding to G1 and G2) in which mass, via a stable point, controls cyclin levels, and phases (S and M phases) in which the concentrations change independently, but once the phase has changed at a bifurcation event (Cell cycle checkpoint), the system cannot go back to the previous levels since at the current mass the vector field is profoundly different and the mass cannot be reversed back through the bifurcation event, making a checkpoint irreversible. In particular the S and M checkpoints are regulated by means of special bifurcations called a Hopf bifurcation and an infinite period bifurcation.



Notes

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External links

- The Society for Mathematical Biology (<http://www.smb.org/>)
- Theoretical and mathematical biology website (<http://www.kli.ac.at/theorylab/index.html>)
- Complexity Discussion Group (<http://www.complex.vcu.edu/>)
- UCLA Biocybernetics Laboratory (<http://biocyb.cs.ucla.edu/research.html>)
- TUCS Computational Biomodelling Laboratory (<http://www.tucs.fi/research/labs/combio.php>)
- Nagoya University Division of Biomodeling (<http://www.agr.nagoya-u.ac.jp/english/e3senko-1.html>)
- Technische Universiteit Biomodeling and Informatics (<http://www.bmi2.bmt.tue.nl/Biomedinf/>)
- BioCybernetics Wiki, a vertical wiki on biomedical cybernetics and systems biology (<http://wiki.biological-cybernetics.de>)
- Bulletin of Mathematical Biology (<http://www.springerlink.com/content/119979/>)
- European Society for Mathematical and Theoretical Biology (<http://www.esmtb.org/>)
- Journal of Mathematical Biology (<http://www.springerlink.com/content/100436/>)
- Biomathematics Research Centre at University of Canterbury (<http://www.math.canterbury.ac.nz/bio/>)
- Centre for Mathematical Biology at Oxford University (<http://www.maths.ox.ac.uk/cmb/>)
- Mathematical Biology at the National Institute for Medical Research (<http://mathbio.nimr.mrc.ac.uk/>)
- Institute for Medical BioMathematics (<http://www.imbm.org/>)
- *Mathematical Biology Systems of Differential Equations* (<http://eqworld.ipmnet.ru/en/solutions/syspde/spde-toc2.pdf>) from EqWorld: The World of Mathematical Equations
- Systems Biology Workbench - a set of tools for modelling biochemical networks (<http://sbw.kgi.edu>)
- The Collection of Biostatistics Research Archive (<http://www.biostatsresearch.com/repository/>)
- Statistical Applications in Genetics and Molecular Biology (<http://www.bepress.com/sagmb/>)
- The International Journal of Biostatistics (<http://www.bepress.com/ijb/>)

- Theoretical Modeling of Cellular Physiology at Ecole Normale Supérieure, Paris (<http://www.biologie.ens.fr/bcsmcbs/>)

Lists of references

- A general list of Theoretical biology/Mathematical biology references, including an updated list of actively contributing authors (<http://www.kli.ac.at/theorylab/index.html>).
- A list of references for applications of category theory in relational biology (<http://planetmath.org/?method=l2h&from=objects&id=10746&op=getobj>).
- An updated list of publications of theoretical biologist Robert Rosen (<http://www.people.vcu.edu/~mikuleck/rosen.htm>)
- Theory of Biological Anthropology (Documents No. 9 and 10 in English) (<http://homepage.uibk.ac.at/~c720126/humanethologie/ws/medicus/block1/inhalt.html>)
- Drawing the Line Between Theoretical and Basic Biology (a forum article by Isidro T. Savillo) (<http://www.scientistsolutions.com/t5844-Drawing+the+line+between+Theoretical+and+Basic+Biology.html>)

Related journals

- Acta Biotheoretica (<http://www.springerlink.com/link.asp?id=102835>)
- Bioinformatics (<http://bioinformatics.oupjournals.org/>)
- Biological Theory (<http://www.mitpressjournals.org/loi/biot/>)
- BioSystems (<http://www.elsevier.com/locate/biosystems>)
- Bulletin of Mathematical Biology (<http://www.springerlink.com/content/119979/>)
- Ecological Modelling (<http://www.elsevier.com/locate/issn/03043800>)
- Journal of Mathematical Biology (<http://www.springerlink.com/content/100436/>)
- Journal of Theoretical Biology (<http://www.elsevier.com/locate/issn/0022-5193>)
- Journal of the Royal Society Interface (<http://publishing.royalsociety.org/index.cfm?page=1058#>)
- Mathematical Biosciences (<http://www.elsevier.com/locate/mbs>)
- Medical Hypotheses (<http://www.harcourt-international.com/journals/mehy/>)
- Rivista di Biologia-Biology Forum (<http://www.tilgher.it/biologiae.html>)
- Theoretical and Applied Genetics (<http://www.springerlink.com/content/100386/>)
- Theoretical Biology and Medical Modelling (<http://www.tbiomed.com/>)
- Theoretical Population Biology (<http://www.elsevier.com/locate/issn/00405809>)
- Theory in Biosciences (http://www.elsevier.com/wps/product/cws_home/701802) (formerly: Biologisches Zentralblatt)

Related societies

- ESMTB: European Society for Mathematical and Theoretical Biology (<http://www.esmtb.org/>)
- The Israeli Society for Theoretical and Mathematical Biology (<http://bioinformatics.weizmann.ac.il/istmb/>)
- Société Francophone de Biologie Théorique (<http://www.necker.fr/sfbt/>)
- International Society for Biosemiotic Studies (<http://www.biosemiotics.org/>)

Lotka–Volterra Differential Equations in Population Biology

The **Lotka–Volterra equations**, also known as the *predator–prey equations*, are a pair of first-order, non-linear, differential equations frequently used to describe the dynamics of biological systems in which two species interact, one a predator and one its prey. They evolve in time according to the pair of equations:

$$\begin{aligned}\frac{dx}{dt} &= x(\alpha - \beta y) \\ \frac{dy}{dt} &= -y(\gamma - \delta x)\end{aligned}$$

where,

- y is the number of some predator (for example, wolves);
- x is the number of its prey (for example, rabbits);
- $\frac{dy}{dt}$ and $\frac{dx}{dt}$ represent the growth of the two populations against time;
- t represents the time; and
- α, β, γ and δ are parameters representing the interaction of the two species.

The Lotka–Volterra system of equations is an example of a Kolmogorov model,^{[1] [2] [3]} which is a more general framework that can model the dynamics of ecological systems with predator-prey interactions, competition, disease, and mutualism.

History

The Lotka–Volterra predator–prey model was initially proposed by Alfred J. Lotka “in the theory of autocatalytic chemical reactions” in 1910.^{[4] [5]} This was effectively the logistic equation,^[6] which was originally derived by Pierre François Verhulst.^[7] In 1920 Lotka extended, via Kolmogorov (see above), the model to “organic systems” using a plant species and a herbivorous animal species as an example^[8] and in 1925 he utilised the equations to analyse predator-prey interactions in his book on biomathematics^[9] arriving at the equations that we know today. Vito Volterra, who made a statistical analysis of fish catches in the Adriatic^[5] independently investigated the equations in 1926.^{[10] [11]}

C.S. Holling extended this model yet again, in two 1959 papers, in which he proposed the idea of functional response.^{[12] [13]} Both the Lotka–Volterra model and Holling’s extensions have been used to model the moose and wolf populations in Isle Royale National Park,^[14] which with over 50 published papers is one of the best studied predator-prey relationships.

In economics

The Lotka–Volterra equations have a long history of use in economic theory; their initial application is commonly credited to Richard Goodwin in 1965^[15] or 1967.^{[16] [17]} In economics, links are between many if not all industries; a proposed way to model the dynamics of various industries has been by introducing trophic functions between various sectors,^[18] and ignoring smaller sectors by considering the interactions of only two industrial sectors.^[19]

Physical meanings of the equations

The Lotka–Volterra model makes a number of assumptions about the environment and evolution of the predator and prey populations:

1. The prey population finds ample food at all times.
2. The food supply of the predator population depends entirely on the prey populations.
3. The rate of change of population is proportional to its size.
4. During the process, the environment does not change in favour of one species and the genetic adaptation is sufficiently slow.

As differential equations are used, the solution is deterministic and continuous. This, in turn, implies that the generations of both the predator and prey are continually overlapping.^[20]

Prey

When multiplied out, the prey equation becomes:

$$\frac{dx}{dt} = \alpha x - \beta xy.$$

The prey are assumed to have an unlimited food supply, and to reproduce exponentially unless subject to predation; this exponential growth is represented in the equation above by the term αx . The rate of predation upon the prey is assumed to be proportional to the rate at which the predators and the prey meet; this is represented above by βxy . If either x or y is zero then there can be no predation.

With these two terms the equation above can be interpreted as: the change in the prey's numbers is given by its own growth minus the rate at which it is preyed upon.

Predators

The predator equation becomes:

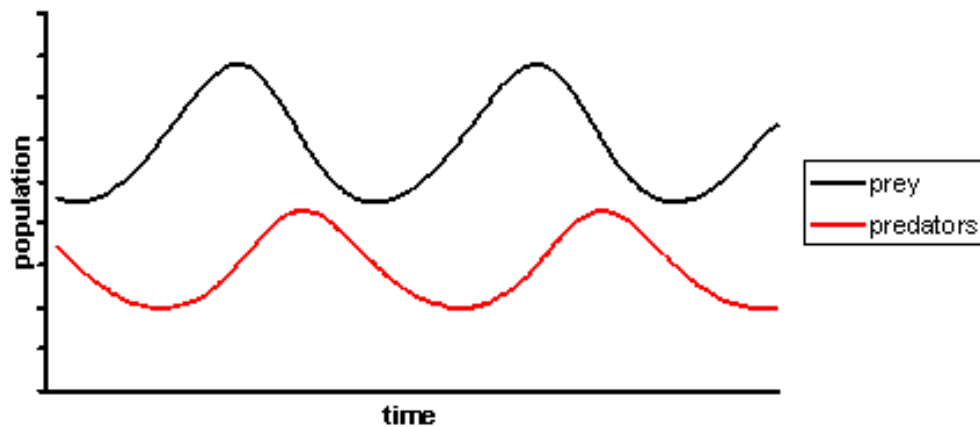
$$\frac{dy}{dt} = \delta xy - \gamma y.$$

In this equation, δxy represents the growth of the predator population. (Note the similarity to the predation rate; however, a different constant is used as the rate at which the predator population grows is not necessarily equal to the rate at which it consumes the prey). γy represents the loss rate of the predators due to either natural death or emigration; it leads to an exponential decay in the absence of prey.

Hence the equation expresses the change in the predator population as growth fueled by the food supply, minus natural death.

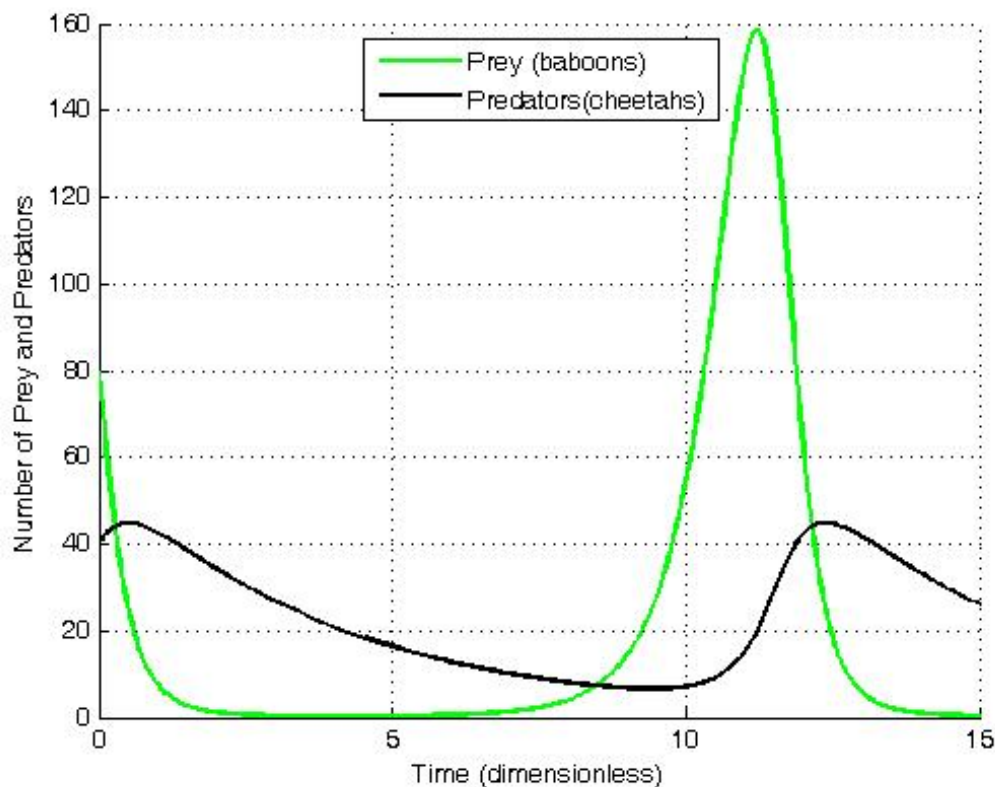
Solutions to the equations

The equations have periodic solutions which do not have a simple expression in terms of the usual trigonometric functions. However, a linearization of the equations yields a solution similar to simple harmonic motion^[21] with the population of predators following that of prey by 90° .



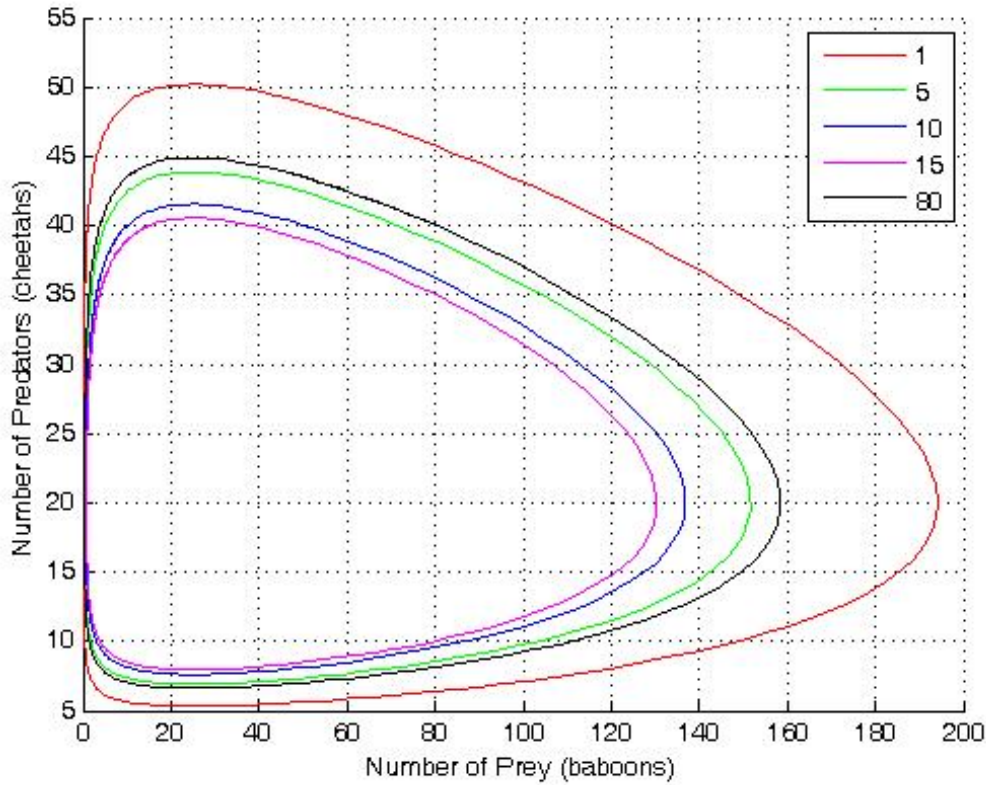
An example problem

Suppose there are two species of animals, a baboon (prey) and a cheetah (predator). If the initial conditions are 80 baboons and 40 cheetahs, one can plot the progression of the two species over time. The choice of time interval is arbitrary.



One can also plot a solution which corresponds to the oscillatory nature of the population of the two species. This solution is in a state of dynamic equilibrium. At any given time in this phase plane, the system is in a limit cycle and

lies somewhere on the inside of these elliptical solutions. There is no particular requirement on the system to begin within a limit cycle and thus in a stable solution, however, it will always reach one eventually.



These graphs clearly illustrate a serious problem with this as a biological model: in each cycle, the baboon population is reduced to extremely low numbers yet recovers (while the cheetah population remains sizeable at the lowest baboon density). Given chance fluctuations, discrete numbers of individuals, and the family structure and lifecycle of baboons, the baboons actually go extinct and by consequence the cheetahs as well. This modelling problem has been called the "atto-fox problem",^[22] an atto-fox being an imaginary 10^{-18} of a fox, in relation to rabies modelling in the UK.

Dynamics of the system

In the model system, the predators thrive when there are plentiful prey but, ultimately, outstrip their food supply and decline. As the predator population is low the prey population will increase again. These dynamics continue in a cycle of growth and decline.

Population equilibrium

Population equilibrium occurs in the model when neither of the population levels is changing, i.e. when both of the derivatives are equal to 0.

$$x(\alpha - \beta y) = 0$$

$$-y(\gamma - \delta x) = 0$$

When solved for x and y the above system of equations yields

$$\{y = 0, x = 0\}$$

and

$$\left\{ y = \frac{\alpha}{\beta}, x = \frac{\gamma}{\delta} \right\},$$

Hence, there are two equilibria.

The first solution effectively represents the extinction of both species. If both populations are at 0, then they will continue to be so indefinitely. The second solution represents a fixed point at which both populations sustain their current, non-zero numbers, and, in the simplified model, do so indefinitely. The levels of population at which this equilibrium is achieved depend on the chosen values of the parameters, α , β , γ , and δ .

Stability of the fixed points

The stability of the fixed point at the origin can be determined by performing a linearization using partial derivatives, while the other fixed point requires a slightly more sophisticated method.

The Jacobian matrix of the predator-prey model is

$$J(x, y) = \begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{bmatrix}.$$

First fixed point

When evaluated at the steady state of (0, 0) the Jacobian matrix J becomes

$$J(0, 0) = \begin{bmatrix} \alpha & 0 \\ 0 & -\gamma \end{bmatrix}.$$

The eigenvalues of this matrix are

$$\lambda_1 = \alpha, \quad \lambda_2 = -\gamma.$$

In the model α and γ are always greater than zero, and as such the sign of the eigenvalues above will always differ. Hence the fixed point at the origin is a saddle point.

The stability of this fixed point is of importance. If it were stable, non-zero populations might be attracted towards it, and as such the dynamics of the system might lead towards the extinction of both species for many cases of initial population levels. However, as the fixed point at the origin is a saddle point, and hence unstable, we find that the extinction of both species is difficult in the model. (In fact, this can only occur if the prey are artificially completely eradicated, causing the predators to die of starvation. If the predators are eradicated, the prey population grows without bound in this simple model).

Second fixed point

Evaluating J at the second fixed point we get

$$J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \begin{bmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\beta} & 0 \end{bmatrix}.$$

The eigenvalues of this matrix are

$$\lambda_1 = i\sqrt{\alpha\gamma}, \quad \lambda_2 = -i\sqrt{\alpha\gamma}.$$

As the eigenvalues are both purely imaginary, this fixed point is not hyperbolic, so no conclusions can be drawn from the linear analysis. However, the system admits a constant of motion

$$K = y^\alpha e^{-\beta y} x^\gamma e^{-\delta x},$$

and the level curves, where $K = \text{const}$, are closed trajectories surrounding the fixed point. Consequently, the levels of the predator and prey populations cycle, and oscillate around this fixed point.

The largest value of the constant K can be obtained by solving the optimization problem

$$y^{\alpha} e^{-\beta y} x^{\gamma} e^{-\delta x} = \frac{y^{\alpha} x^{\gamma}}{e^{\delta x + \beta y}} \longrightarrow \max_{x, y > 0}.$$

The maximal value of K is attained at the stationary point $\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)$ and it is given by

$$K^* = \left(\frac{\alpha}{\beta e}\right)^{\alpha} \left(\frac{\gamma}{\delta e}\right)^{\gamma},$$

where e is Euler's Number.

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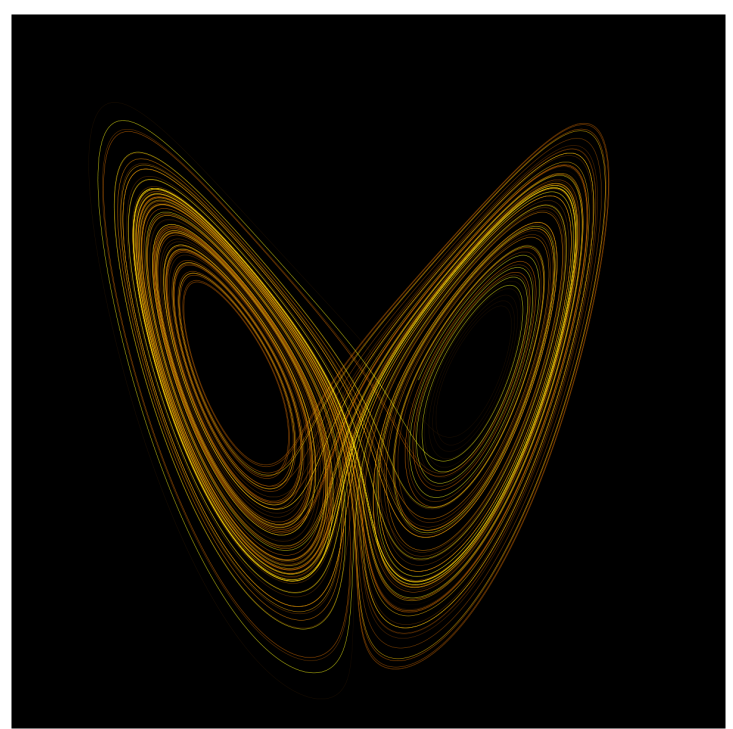
External links

- Lotka–Volterra Predator-Prey Model (<http://www.egwald.ca/nonlineardynamics/twodimensionaldynamics.php#predatorpreymodel>) by Elmer G. Wiens
- Lotka-Volterra Model (<http://math.fullerton.edu/mathews/n2003/Lotka-VolterraMod.html>)
- NANIA Lotka-Volterra applet (<http://www.ph.ed.ac.uk/nania/lv/lv.html>)
- Lotka Algorithmic Simulation (<http://jseed.sourceforge.net/lotka/index.html>) Similar program, in Javascript (requires an HTML5 browser).

Chaotic Dynamics

Chaos theory

Chaos theory is a field of study in applied mathematics, with applications in several disciplines including physics, economics, biology, and philosophy. Chaos theory studies the behavior of dynamical systems that are highly sensitive to initial conditions; an effect which is popularly referred to as the butterfly effect. Small differences in initial conditions (such as those due to rounding errors in numerical computation) yield widely diverging outcomes for chaotic systems, rendering long-term prediction impossible in general.^[1] This happens even though these systems are deterministic, meaning that their future behavior is fully determined by their initial conditions, with no random elements involved.^[2] In other words, the deterministic nature of these systems does not make them predictable.^[3] ^[4] This behavior is known as deterministic chaos, or simply *chaos*.



A plot of the Lorenz attractor for values $r = 28$, $\sigma = 10$, $b = 8/3$.

Chaotic behavior can be observed in many natural systems, such as the weather.^[5] Explanation of such behavior may be sought through analysis of a chaotic mathematical model, or through analytical techniques such as recurrence plots and Poincaré maps.

Applications



A conus textile shell, similar in appearance to Rule 30, a cellular automaton with chaotic behaviour.

Chaos theory is applied in many scientific disciplines: mathematics, programming, microbiology, biology, computer science, economics,^[6] ^[7] ^[8] engineering,^[9] finance,^[10] ^[11] philosophy, physics, politics, population dynamics, psychology, robotics,^[12] and meteorology.

Chaotic behavior has been observed in the laboratory in a variety of systems including electrical circuits, lasers, oscillating chemical reactions, fluid dynamics, and mechanical and magneto-mechanical devices, as well as computer models of chaotic processes. Observations of chaotic behavior in nature include changes in weather,^[5] the dynamics of satellites in the solar system, the time evolution of the

magnetic field of celestial bodies, population growth in ecology, the dynamics of the action potentials in neurons, and molecular vibrations. There is some controversy over the existence of chaotic dynamics in plate tectonics and in economics.^{[13] [14] [15]}

A successful application of chaos theory is in ecology where dynamical systems such as the Ricker model have been used to show how population growth under density dependence can lead to chaotic dynamics.

Chaos theory is also currently being applied to medical studies of epilepsy, specifically to the prediction of seemingly random seizures by observing initial conditions.^[16]

Quantum chaos theory studies how the correspondence between quantum mechanics and classical mechanics works in the context of chaotic systems.^[17] Recently, another field, called relativistic chaos,^[18] has emerged to describe systems that follow the laws of general relativity.

The motion of N stars in response to their self-gravity (the gravitational N -body problem) is generically chaotic.^[19]

In electrical engineering, chaotic systems are used in communications, random number generators, and encryption systems.

In numerical analysis, the Newton-Raphson method of approximating the roots of a function can lead to chaotic iterations if the function has no real roots.^[20]

Chaotic dynamics

In common usage, "chaos" means "a state of disorder".^[21] However, in chaos theory, the term is defined more precisely. Although there is no universally accepted mathematical definition of chaos, a commonly used definition says that, for a dynamical system to be classified as chaotic, it must have the following properties:^[22]

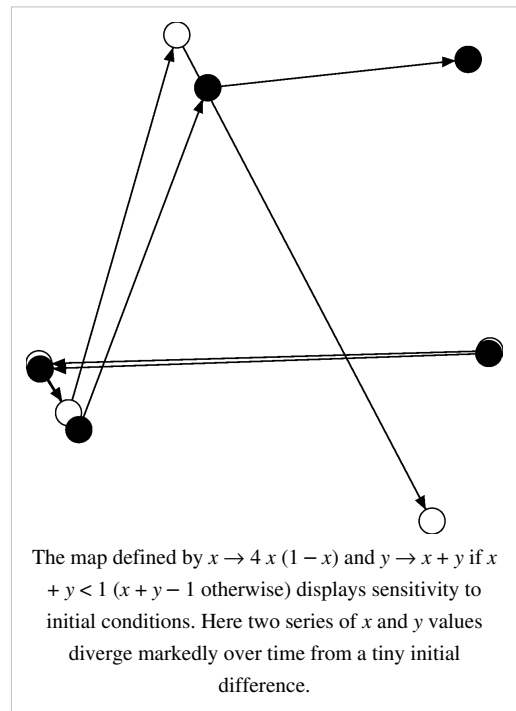
1. it must be sensitive to initial conditions;
2. it must be topologically mixing; and
3. its periodic orbits must be dense.

The requirement for sensitive dependence on initial conditions implies that there is a set of initial conditions of positive measure which do not converge to a cycle of any length.

Sensitivity to initial conditions

Sensitivity to initial conditions means that each point in such a system is arbitrarily closely approximated by other points with significantly different future trajectories. Thus, an arbitrarily small perturbation of the current trajectory may lead to significantly different future behaviour. However, it has been shown that the last two properties in the list above actually imply sensitivity to initial conditions^{[23] [24]} and if attention is restricted to intervals, the second property implies the other two^[25] (an alternative, and in general weaker, definition of chaos uses only the first two properties in the above list^[26]). It is interesting that the most practically significant condition, that of sensitivity to initial conditions, is actually redundant in the definition, being implied by two (or for intervals, one) purely topological conditions, which are therefore of greater interest to mathematicians.

Sensitivity to initial conditions is popularly known as the "butterfly effect", so called because of the title of a paper given by Edward Lorenz in 1972 to the American Association for the Advancement of Science in Washington, D.C. entitled *Predictability: Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?* The flapping wing



represents a small change in the initial condition of the system, which causes a chain of events leading to large-scale phenomena. Had the butterfly not flapped its wings, the trajectory of the system might have been vastly different.

A consequence of sensitivity to initial conditions is that if we start with only a finite amount of information about the system (as is usually the case in practice), then beyond a certain time the system will no longer be predictable. This is most familiar in the case of weather, which is generally predictable only about a week ahead.^[27]

The Lyapunov exponent characterises the extent of the sensitivity to initial conditions. Quantitatively, two trajectories in phase space with initial separation $\delta\mathbf{Z}_0$ diverge

$$|\delta\mathbf{Z}(t)| \approx e^{\lambda t} |\delta\mathbf{Z}_0|$$

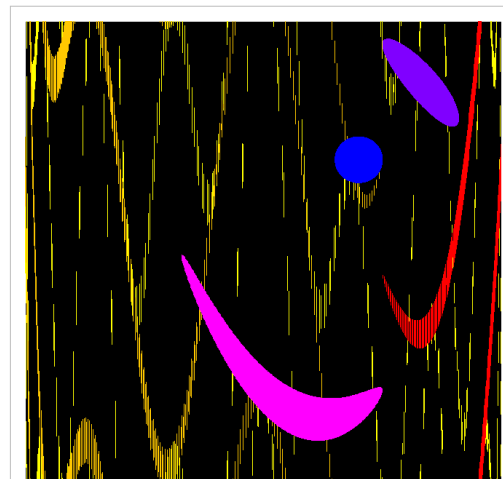
where λ is the Lyapunov exponent. The rate of separation can be different for different orientations of the initial separation vector. Thus, there is a whole spectrum of Lyapunov exponents — the number of them is equal to the number of dimensions of the phase space. It is common to just refer to the largest one, i.e. to the Maximal Lyapunov exponent (MLE), because it determines the overall predictability of the system. A positive MLE is usually taken as an indication that the system is chaotic.

There are also measure-theoretic mathematical conditions (discussed in ergodic theory) such as mixing or being a K-system which relate to sensitivity of initial conditions and chaos^[4]

Topological mixing

Topological mixing (or *topological transitivity*) means that the system will evolve over time so that any given region or open set of its phase space will eventually overlap with any other given region. This mathematical concept of "mixing" corresponds to the standard intuition, and the mixing of colored dyes or fluids is an example of a chaotic system.

Topological mixing is often omitted from popular accounts of chaos, which equate chaos with sensitivity to initial conditions. However, sensitive dependence on initial conditions alone does not give chaos. For example, consider the simple dynamical system produced by repeatedly doubling an initial value. This system has sensitive dependence on initial conditions everywhere, since any pair of nearby points will eventually become widely separated. However, this example has no topological mixing, and therefore has no chaos. Indeed, it has extremely simple behaviour: all points except 0 tend to infinity.



The map defined by $x \rightarrow 4x(1-x)$ and $y \rightarrow x+y$ if $x+y < 1$ ($x+y-1$ otherwise) also displays topological mixing. Here the blue region is transformed by the dynamics first to the purple region, then to the pink and red regions, and eventually to a cloud of points scattered across the space.

Density of periodic orbits

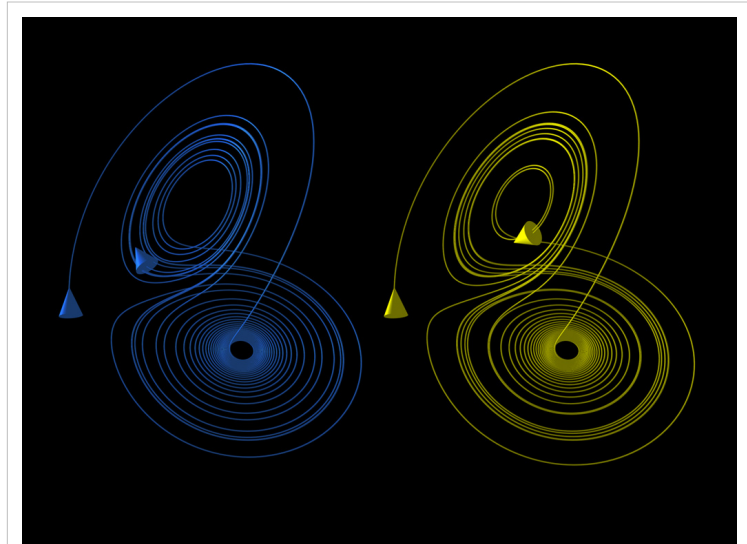
Density of periodic orbits means that every point in the space is approached arbitrarily closely by periodic orbits. Topologically mixing systems failing this condition may not display sensitivity to initial conditions, and hence may not be chaotic. For example, an irrational rotation of the circle is topologically transitive, but does not have dense periodic orbits, and hence does not have sensitive dependence on initial conditions.^[28] The one-dimensional logistic map defined by $x \rightarrow 4x(1-x)$ is one of the simplest systems with density of periodic orbits. For example, $\frac{5+\sqrt{5}}{8} \rightarrow \frac{5+\sqrt{5}}{8} \rightarrow \frac{5-\sqrt{5}}{8}$ (or approximately $0.3454915 \rightarrow 0.9045085 \rightarrow 0.3454915$) is an (unstable) orbit of period 2, and similar orbits exist for periods 4, 8, 16, etc. (indeed, for all the periods specified by Sharkovskii's theorem).^[29]

Sharkovskii's theorem is the basis of the Li and Yorke^[30] (1975) proof that any one-dimensional system which exhibits a regular cycle of period three will also display regular cycles of every other length as well as completely chaotic orbits.

Strange attractors

Some dynamical systems, like the one-dimensional logistic map defined by $x \rightarrow 4x(1-x)$, are chaotic everywhere, but in many cases chaotic behaviour is found only in a subset of phase space. The cases of most interest arise when the chaotic behaviour takes place on an attractor, since then a large set of initial conditions will lead to orbits that converge to this chaotic region.

An easy way to visualize a chaotic attractor is to start with a point in the basin of attraction of the attractor, and then simply plot its subsequent orbit. Because of the topological transitivity condition, this is likely to produce a picture of the entire final attractor, and indeed both orbits shown in the figure on the right give a picture of the



The Lorenz attractor displays chaotic behavior. These two plots demonstrate sensitive dependence on initial conditions within the region of phase space occupied by the attractor.

general shape of the Lorenz attractor. This attractor results from a simple three-dimensional model of the Lorenz weather system. The Lorenz attractor is perhaps one of the best-known chaotic system diagrams, probably because it was not only one of the first, but it is also one of the most complex and as such gives rise to a very interesting pattern which looks like the wings of a butterfly.

Unlike fixed-point attractors and *limit cycles*, the attractors which arise from chaotic systems, known as *strange attractors*, have great detail and complexity. Strange attractors occur in both continuous dynamical systems (such as the Lorenz system) and in some discrete systems (such as the Hénon map). Other discrete dynamical systems have a repelling structure called a Julia set which forms at the boundary between basins of attraction of fixed points – Julia sets can be thought of as *strange repellers*. Both strange attractors and Julia sets typically have a fractal structure, and a fractal dimension can be calculated for them.

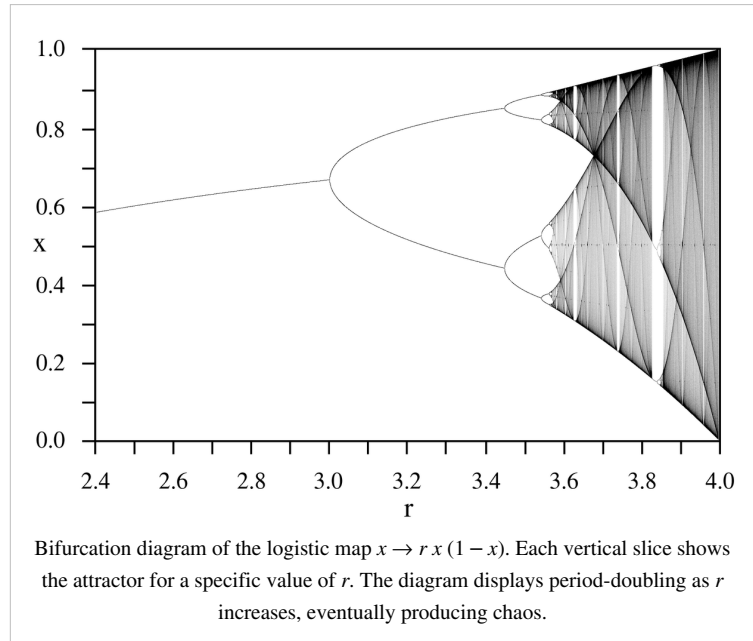
Minimum complexity of a chaotic system

Discrete chaotic systems, such as the logistic map, can exhibit strange attractors whatever their dimensionality. However, the Poincaré-Bendixson theorem shows that a strange attractor can only arise in a continuous dynamical system (specified by differential equations) if it has three or more dimensions. Finite dimensional linear systems are never chaotic; for a dynamical system to display chaotic behaviour it has to be either nonlinear, or infinite-dimensional.

The Poincaré–Bendixson theorem states that a two dimensional differential equation has very regular behavior. The Lorenz attractor discussed above is generated by a system of three differential equations with a total of seven terms on the right hand side, five of

which are linear terms and two of which are quadratic (and therefore nonlinear). Another well-known chaotic attractor is generated by the Rossler equations with seven terms on the right hand side, only one of which is (quadratic) nonlinear. Sprott^[31] found a three dimensional system with just five terms on the right hand side, and with just one quadratic nonlinearity, which exhibits chaos for certain parameter values. Zhang and Heidel^{[32] [33]} showed that, at least for dissipative and conservative quadratic systems, three dimensional quadratic systems with only three or four terms on the right hand side cannot exhibit chaotic behavior. The reason is, simply put, that solutions to such systems are asymptotic to a two dimensional surface and therefore solutions are well behaved.

While the Poincaré–Bendixson theorem means that a continuous dynamical system on the Euclidean plane cannot be chaotic, two-dimensional continuous systems with non-Euclidean geometry can exhibit chaotic behaviour. Perhaps surprisingly, chaos may occur also in linear systems, provided they are infinite-dimensional.^[34] A theory of linear chaos is being developed in the functional analysis, a branch of mathematical analysis.



History

An early proponent of chaos theory was Henri Poincaré. In the 1880s, while studying the three-body problem, he found that there can be orbits which are nonperiodic, and yet not forever increasing nor approaching a fixed point.^[35]^[36] In 1898 Jacques Hadamard published an influential study of the chaotic motion of a free particle gliding frictionlessly on a surface of constant negative curvature.^[37] In the system studied, "Hadamard's billiards", Hadamard was able to show that all trajectories are unstable in that all particle trajectories diverge exponentially from one another, with a positive Lyapunov exponent.

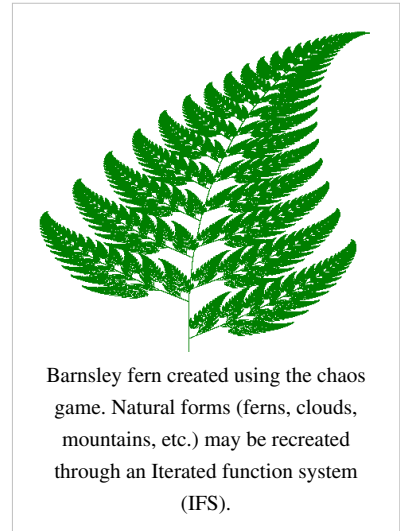
Much of the earlier theory was developed almost entirely by mathematicians, under the name of ergodic theory. Later studies, also on the topic of nonlinear differential equations, were carried out by G.D. Birkhoff,^[38] A. N. Kolmogorov,^[39] [40] [41] M.L. Cartwright and J.E. Littlewood,^[42] and Stephen Smale.^[43] Except for Smale, these studies were all directly inspired by physics: the three-body problem in the case of Birkhoff, turbulence and astronomical problems in the case of Kolmogorov, and radio engineering in the case of Cartwright and Littlewood. Although chaotic planetary motion had not been observed, experimentalists had encountered turbulence in fluid motion and nonperiodic oscillation in radio circuits without the benefit of a theory to explain what they were seeing.

Despite initial insights in the first half of the twentieth century, chaos theory became formalized as such only after mid-century, when it first became evident for some scientists that linear theory, the prevailing system theory at that time, simply could not explain the observed behaviour of certain experiments like that of the logistic map. What had been beforehand excluded as measure imprecision and simple "noise" was considered by chaos theories as a full component of the studied systems.

The main catalyst for the development of chaos theory was the electronic computer. Much of the mathematics of chaos theory involves the repeated iteration of simple mathematical formulas, which would be impractical to do by hand. Electronic computers made these repeated calculations practical, while figures and images made it possible to visualize these systems.



Turbulence in the tip vortex from an airplane wing. Studies of the critical point beyond which a system creates turbulence was important for Chaos theory, analyzed for example by the Soviet physicist Lev Landau who developed the Landau-Hopf theory of turbulence. David Ruelle and Floris Takens later predicted, against Landau, that fluid turbulence could develop through a strange attractor, a main concept of chaos theory.



Barnsley fern created using the chaos game. Natural forms (ferns, clouds, mountains, etc.) may be recreated through an Iterated function system (IFS).

An early pioneer of the theory was Edward Lorenz whose interest in chaos came about accidentally through his work on weather prediction in 1961.^[44] Lorenz was using a simple digital computer, a Royal McBee LGP-30, to run his weather simulation. He wanted to see a sequence of data again and to save time he started the simulation in the middle of its course. He was able to do this by entering a printout of the data corresponding to conditions in the middle of his simulation which he had calculated last time.

To his surprise the weather that the machine began to predict was completely different from the weather calculated before. Lorenz tracked this down to the computer printout. The computer worked with 6-digit precision, but the printout rounded variables off to a 3-digit number, so a value like 0.506127 was printed as 0.506. This difference is tiny and the consensus at the time would have been that it should have had practically no effect. However Lorenz had discovered that small changes in initial conditions produced large changes in the

long-term outcome.^[45] Lorenz's discovery, which gave its name to Lorenz attractors, showed that even detailed atmospheric modelling cannot in general make long-term weather predictions. Weather is usually predictable only about a week ahead.^[27]

The year before, Benoît Mandelbrot found recurring patterns at every scale in data on cotton prices.^[46] Beforehand, he had studied information theory and concluded noise was patterned like a Cantor set: on any scale the proportion of noise-containing periods to error-free periods was a constant – thus errors were inevitable and must be planned for by incorporating redundancy.^[47] Mandelbrot described both the "Noah effect" (in which sudden discontinuous changes can occur) and the "Joseph effect" (in which persistence of a value can occur for a while, yet suddenly change afterwards).^[48] ^[49] This challenged the idea that changes in price were normally distributed. In 1967, he published "How long is the coast of Britain? Statistical self-similarity and fractional dimension", showing that a coastline's length varies with the scale of the measuring instrument, resembles itself at all scales, and is infinite in length for an infinitesimally small measuring device.^[50] Arguing that a ball of twine appears to be a point when viewed from far away (0-dimensional), a ball when viewed from fairly near (3-dimensional), or a curved strand (1-dimensional), he argued that the dimensions of an object are relative to the observer and may be fractional. An object whose irregularity is constant over different scales ("self-similarity") is a fractal (for example, the Koch curve or "snowflake", which is infinitely long yet encloses a finite space and has fractal dimension equal to circa 1.2619, the Menger sponge and the Sierpiński gasket). In 1975 Mandelbrot published *The Fractal Geometry of Nature*, which became a classic of chaos theory. Biological systems such as the branching of the circulatory and bronchial systems proved to fit a fractal model.

Chaos was observed by a number of experimenters before it was recognized; e.g., in 1927 by van der Pol^[51] and in 1958 by R.L. Ives.^[52] ^[53] However, as a graduate student in Chihiro Hayashi's laboratory at Kyoto University, Yoshisuke Ueda was experimenting with analog computers (that is, vacuum tubes) and noticed, on Nov. 27, 1961, what he called "randomly transitional phenomena". Yet his advisor did not agree with his conclusions at the time, and did not allow him to report his findings until 1970.^[54] ^[55]

In December 1977 the New York Academy of Sciences organized the first symposium on Chaos, attended by David Ruelle, Robert May, James A. Yorke (coiner of the term "chaos" as used in mathematics), Robert Shaw (a physicist, part of the Eudaemons group with J. Doyne Farmer and Norman Packard who tried to find a mathematical method to beat roulette, and then created with them the Dynamical Systems Collective in Santa Cruz, California), and the meteorologist Edward Lorenz.

The following year, Mitchell Feigenbaum published the noted article "Quantitative Universality for a Class of Nonlinear Transformations", where he described logistic maps.^[56] Feigenbaum had applied fractal geometry to the study of natural forms such as coastlines. Feigenbaum notably discovered the universality in chaos, permitting an application of chaos theory to many different phenomena.

In 1979, Albert J. Libchaber, during a symposium organized in Aspen by Pierre Hohenberg, presented his experimental observation of the bifurcation cascade that leads to chaos and turbulence in convective Rayleigh–Benard systems. He was awarded the Wolf Prize in Physics in 1986 along with Mitchell J. Feigenbaum "for his brilliant experimental demonstration of the transition to turbulence and chaos in dynamical systems".^[57]

Then in 1986 the New York Academy of Sciences co-organized with the National Institute of Mental Health and the Office of Naval Research the first important conference on Chaos in biology and medicine. There, Bernardo Huberman presented a mathematical model of the eye tracking disorder among schizophrenics.^[58] This led to a renewal of physiology in the 1980s through the application of chaos theory, for example in the study of pathological cardiac cycles.

In 1987, Per Bak, Chao Tang and Kurt Wiesenfeld published a paper in *Physical Review Letters*^[59] describing for the first time self-organized criticality (SOC), considered to be one of the mechanisms by which complexity arises in nature. Alongside largely lab-based approaches such as the Bak–Tang–Wiesenfeld sandpile, many other investigations have focused on large-scale natural or social systems that are known (or suspected) to display

scale-invariant behaviour. Although these approaches were not always welcomed (at least initially) by specialists in the subjects examined, SOC has nevertheless become established as a strong candidate for explaining a number of natural phenomena, including: earthquakes (which, long before SOC was discovered, were known as a source of scale-invariant behaviour such as the Gutenberg–Richter law describing the statistical distribution of earthquake sizes, and the Omori law^[60] describing the frequency of aftershocks); solar flares; fluctuations in economic systems such as financial markets (references to SOC are common in econophysics); landscape formation; forest fires; landslides; epidemics; and biological evolution (where SOC has been invoked, for example, as the dynamical mechanism behind the theory of "punctuated equilibria" put forward by Niles Eldredge and Stephen Jay Gould). Worryingly, given the implications of a scale-free distribution of event sizes, some researchers have suggested that another phenomenon that should be considered an example of SOC is the occurrence of wars. These "applied" investigations of SOC have included both attempts at modelling (either developing new models or adapting existing ones to the specifics of a given natural system), and extensive data analysis to determine the existence and/or characteristics of natural scaling laws.

The same year, James Gleick published *Chaos: Making a New Science*, which became a best-seller and introduced the general principles of chaos theory as well as its history to the broad public. At first the domain of work of a few, isolated individuals, chaos theory progressively emerged as a transdisciplinary and institutional discipline, mainly under the name of nonlinear systems analysis. Alluding to Thomas Kuhn's concept of a paradigm shift exposed in *The Structure of Scientific Revolutions* (1962), many "chaologists" (as some described themselves) claimed that this new theory was an example of such a shift, a thesis upheld by J. Gleick.

The availability of cheaper, more powerful computers broadens the applicability of chaos theory. Currently, chaos theory continues to be a very active area of research, involving many different disciplines (mathematics, topology, physics, population biology, biology, meteorology, astrophysics, information theory, etc.).

Distinguishing random from chaotic data

It can be difficult to tell from data whether a physical or other observed process is random or chaotic, because in practice no time series consists of pure 'signal.' There will always be some form of corrupting noise, even if it is present as round-off or truncation error. Thus any real time series, even if mostly deterministic, will contain some randomness.^{[61] [62]}

All methods for distinguishing deterministic and stochastic processes rely on the fact that a deterministic system always evolves in the same way from a given starting point.^{[61] [63]} Thus, given a time series to test for determinism, one can:

1. pick a test state;
2. search the time series for a similar or 'nearby' state; and
3. compare their respective time evolutions.

Define the error as the difference between the time evolution of the 'test' state and the time evolution of the nearby state. A deterministic system will have an error that either remains small (stable, regular solution) or increases exponentially with time (chaos). A stochastic system will have a randomly distributed error.^[64]

Essentially all measures of determinism taken from time series rely upon finding the closest states to a given 'test' state (e.g., correlation dimension, Lyapunov exponents, etc.). To define the state of a system one typically relies on phase space embedding methods.^[65] Typically one chooses an embedding dimension, and investigates the propagation of the error between two nearby states. If the error looks random, one increases the dimension. If you can increase the dimension to obtain a deterministic looking error, then you are done. Though it may sound simple it is not really. One complication is that as the dimension increases the search for a nearby state requires a lot more computation time and a lot of data (the amount of data required increases exponentially with embedding dimension) to find a suitably close candidate. If the embedding dimension (number of measures per state) is chosen too small (less than the 'true' value) deterministic data can appear to be random but in theory there is no problem choosing the

dimension too large – the method will work.

When a non-linear deterministic system is attended by external fluctuations, its trajectories present serious and permanent distortions. Furthermore, the noise is amplified due to the inherent non-linearity and reveals totally new dynamical properties. Statistical tests attempting to separate noise from the deterministic skeleton or inversely isolate the deterministic part risk failure. Things become worse when the deterministic component is a non-linear feedback system.^[66] In presence of interactions between nonlinear deterministic components and noise, the resulting nonlinear series can display dynamics that traditional tests for nonlinearity are sometimes not able to capture.^[67]

The question of how to distinguish deterministic chaotic systems from stochastic systems has also been discussed in philosophy.^[68]

Cultural references

Chaos theory has been mentioned in numerous movies and works of literature. Examples include the book *Jurassic Park*, the film *The Butterfly Effect*, the sitcom *The Big Bang Theory*, Tom Stoppard's play *Arcadia* and the video game Tom Clancy's *Splinter Cell: Chaos Theory*. The influence of chaos theory in shaping the popular understanding of the world we live in was the subject of the BBC documentary *High Anxieties - The Mathematics of Chaos* directed by David Malone. Chaos Theory is also the subject of discussion in the BBC documentary "The Secret Life of Chaos" presented by the physicist Jim Al-Khalili.

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External links

- Nonlinear Dynamics Research Group (<http://lagrange.physics.drexel.edu>) with Animations in Flash
- The Chaos group at the University of Maryland (<http://www.chaos.umd.edu>)
- The Chaos Hypertextbook (<http://hypertextbook.com/chaos/>). An introductory primer on chaos and fractals.
- Society for Chaos Theory in Psychology & Life Sciences (<http://www.societyforchaostheory.org/>)
- Nonlinear Dynamics Research Group at CSDC (<http://www.csdc.unifi.it/mdswitch.html?newlang=eng>), Florence Italy
- Interactive live chaotic pendulum experiment (<http://physics.mercer.edu/pendulum/>), allows users to interact and sample data from a real working damped driven chaotic pendulum
- Nonlinear dynamics: how science comprehends chaos (<http://www.creatingtechnology.org/papers/chaos.htm>), talk presented by Sunny Auyang, 1998.
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- Systems Analysis, Modelling and Prediction Group (<http://www.eng.ox.ac.uk/samp>) at the University of Oxford.
- A page about the Mackey-Glass equation (<http://www.mgix.com/snippets/?MackeyGlass>).
- High Anxieties - The Mathematics of Chaos (<http://www.youtube.com/user/thedebtgeneration?feature=mhum>) (2008) BBC documentary directed by David Malone.

Chaos theory in organizational development

Chaos theory in organizational development refers to a subset of chaos theory which incorporates principles of quantum mechanics and presents them in a complex systems environment.

Background

Viewing a chimp as a complex system in itself, and magnifying the interactional effects of primates and waves to reflect the interactions of different elements making up a complex system, such as an organization, assists us in seeing parallels between chaos theory and organizational relationships. What must be pointed out, however, is that these "parallels" between organizations and the sub-atomic particles exist largely in terms of analogy (metaphorically) between two very different domains of activity; the interactional effects of sub-atomic particles, in quantum mechanics, are expressed in terms of math; bringing these theories into the domain of human activity can be seen as problematical. Although these parallels are easily witnessed in regard to complex organizational systems, it is difficult to see evidence of irrational quantum-effects in everyday life. If you roll a ball forward, it rolls forward in the general direction intended. As a whole, Newtonian principles of interaction stand solidly within the bounds of macrophysics. But at the sub-atomic level, things do not act as expected. "At the subatomic level, the objectivity found in classical physics is replaced by quantum subjectivity." (Shelton, 2003) The introduction of chaos theory brings the principles of quantum physics to the pragmatic world. These complex systems have a rather random appearance and, until recently, have been labeled and discarded as chaotic and unintelligible. With the advent of computer systems and powerful processors, it has become easier to map chaotic behavior and find interesting underpinnings of order. The newly discovered underlying order to chaos sparked new interest and inspired more research in the field of chaos theory. The recent focus of most of the research on chaos theory is primarily rooted in these underlying patterns found in an otherwise chaotic environment, more specifically, concepts such as self-organization, bifurcation, and self-similarity.

Elements of organization

Self-organization

Self-organization, as opposed to natural or social selection, is a dynamic change within the organization where system changes are made by recalculating, re-inventing and modifying its structure in order to adapt, survive, grow, and develop. Self-organization is the result of re-invention and creative adaptation due to the introduction of, or being in a constant state of, perturbed equilibrium. One example of an organization which exists in a constant state of perturbation is that of the learning organization, which is "one that allows self-organization, rather than attempting to control the bifurcation through planned change." (Dooley, 1995) Being "off-balance" lends itself to regrouping and re-evaluating the system's present state in order to make needed adjustments and regain control and equilibrium. By understanding and introducing the element of punctuated equilibrium (chaos) while facilitating networks for growth, an organization can change gears from "cruise" to "turbo" in regard to speed and intensity of organizational change. While maintaining an equilibrial state seems to be an intuitively rational method for enabling an organization to gain a sense of consistency and solidarity, existing on the edge of a chaotic state remains the most beneficial environment for systems to flourish develop and grow.

For instance, two competing organizations that differ in regard to their levels of homeostasis will not be in competition for long. Generally speaking, the organization with the less-stable structure will come out ahead while the constant stability of the latter will eventually lead to its own demise. Although quite similar, small differences in homeostasis levels are enough to make a tremendous difference in future outcomes for each organization. The notion of similarity in origin vs. dissimilar results comes to fruition with the emergence of bifurcation.

Bifurcation

The concept of bifurcation cannot be explained without discussion of the term frequently labeled "sensitivity to initial conditions." Sensitivity to initial conditions refers to the high level of importance of primary conditions from which the future path and direction of a system stems. This sensitivity to initial conditions is commonly referred to as the "Butterfly Effect," in which a butterfly flaps its tiny wings in one end of the world which results in a typhoon or hurricane somewhere else on the globe. While this is an entertaining notion, sensitivity to initial conditions remains in reality a very abstract concept without the presence of bifurcation, which is mathematically labeled as the actual splitting point of two near-identical entities which, due to the sensitivity of initial conditions, tend to take two very distinct paths and result in two totally different geographically or even evolutionary places.

Imagine dropping two identical coins from your fingertips off a 25-story balcony at the same time. Unless they are glued together, they will each take a different path towards the ground. Even though the force of gravity determines their general direction and speed, a host of uncontrollable variables such as wind and dust particles affect each coin independently. The infinitesimal and perhaps unidentifiable difference in starting conditions exponentially amplifies the effects of all other variables encountered which then feed back and add even more variation to the system resulting in very different paths taken to the ground. The moment the two coins split paths is known as the bifurcation point. The importance of this point lies in its implication of change and new direction.

Applications and pitfalls

The primary goal of an organizational development (OD) consultant is to initiate, facilitate, and support successful change in an organization. Using chaos theory as the sole model for change may be far too risky for any stakeholder buy-in. The concept of uncertainty on which chaos theory relies is not an appealing motive for change compared to many alternative "safer" models of organizational change which entail less risk. By careful planning and management of disorder a successful intervention is possible, but only with a truly dedicated arsenal of talented and creative resources. By permitting or actively forcing an organization to enter a chaotic state, change becomes inevitable and bifurcation imminent; but the question remains, "Will the new direction be the one intended?" In order to account for the direction of the new thrust, most planning attention should be focused on attractors instead of the initiation of disorder.

Although chaos eventually gives way to self-organization, how can we control the duration, intensity, and shape of its outcome? It seems that punctuating equilibrium and instilling disorder in an organization is risky business. Throwing an organization off balance could possibly send it in a downward spiral towards dissemination by ultimately compromising the structural integrity (i.e. identity) of the system to the point of no return. The only way to reap the benefits of chaos theory in OD while maintaining a sense of security is to adjust the organization towards a state of existence which lies "on the edge of chaos".

By existing on the edge of chaos, organizations are forced to find new, creative ways to compete and stay ahead. Good examples of such learning organizations are found throughout the field of technology as well as the airline industry, namely organizations such as Southwest Airlines, which used re-invention not just for survival, but also to prosper in an otherwise dismal market. In contrast, there are organizations which, due to extended periods of equilibrium, find themselves struggling for survival. Telephone companies, for instance, were once solid and static entities that dominated the communication market. While the rest of the world was developing new communication technology, telephone companies did not creatively grow at the same rate. The result is an organization that is battling to stay alive unless they embrace the element of chaos due to crisis, and allow creative adaptability to function freely so that self-organization and re-invention can occur.

While organizations existing on the edge of chaos are known to be the most creative and adaptive of organizations, how do their members feel about constant evolution and re-invention? Is it possible to identify with, and stay loyal to, an organization that constantly changes shape? The short answer is yes. As long as the organization does not change its core essence, its identifiable, shared purpose, its members will still experience the organization as a

developing system that changes shape but retains the same familiar face.

Perhaps the safest way to use chaos theory in OD is not in the instigation of organizational change, but in the use of its principles in dealing with issues that arise within the organization. By embracing organizational phenomena previously seen as dysfunctional, such as interpersonal conflict, and using it as a source for transformational change by applying principles found in chaos theory (Shelton, 2003), an organization can make "lemonade out of lemons" and become more responsive to change agents while continuously moving ahead and growing from the inside out without the fear of complete chaos.

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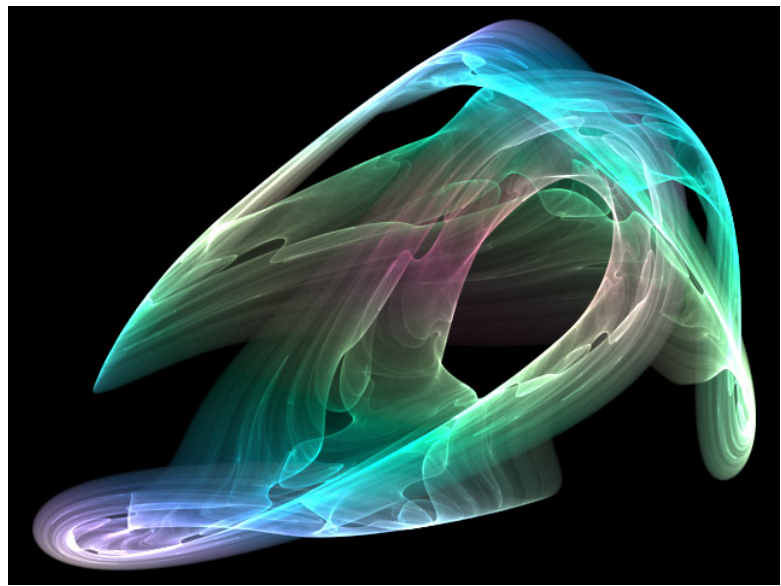
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Attractor

An **attractor** is a set towards which a dynamical system evolves over time. That is, points that get close enough to the attractor remain close even if slightly disturbed. Geometrically, an attractor can be a point, a curve, a manifold, or even a complicated set with a fractal structure known as a *strange attractor*. Describing the attractors of chaotic dynamical systems has been one of the achievements of chaos theory.

A trajectory of the dynamical system in the attractor does not have to satisfy any special constraints except for remaining on the attractor. The trajectory may be periodic or chaotic or of any other type. If this condition is met, but the flow in the neighbourhood is away from the set, the set is called a **repeller** (or *repellor*).



Visual representation of a strange attractor

Motivation

A dynamical system is generally described by one or more differential or difference equations. The equations of a given dynamic system specify its behavior over any given short period of time. To determine the system's behavior for a longer period, it is necessary to integrate the equations, either through analytical means or through iteration, often with the aid of computers.

Dynamical systems in the physical world tend to be dissipative: if it were not for some driving force, the motion would cease. (Dissipation may come from internal friction, thermodynamic losses, or loss of material, among many causes.) The dissipation and the driving force tend to combine to kill out initial transients and settle the system into its typical behavior. This one part of the phase space of the dynamical system corresponding to the typical behavior is the **attracting section** or **attractee**.

Invariant sets and limit sets are similar to the attractor concept. An *invariant set* is a set that evolves to itself under the dynamics. Attractors may contain invariant sets. A *limit set* is a set of points such that there exists some initial state that ends up arbitrarily close to the limit set (i.e. to each point of the set) as time goes to infinity. Attractors are limit sets, but not all limit sets are attractors: It is possible to have some points of a system converge to a limit set, but different points when perturbed slightly off the limit set may get knocked off and never return to the vicinity of the limit set.

For example, the damped pendulum has two invariant points: the point x_0 of minimum height and the point x_1 of maximum height. The point x_0 is also a limit set, as trajectories converge to it; the point x_1 is not a limit set. Because of the dissipation, the point x_0 is also an attractor. If there were no dissipation, x_0 would not be an attractor.

Mathematical definition

Let $f(t, \bullet)$ be a function which specifies the dynamics of the system. That is, if a is a point in the phase space, so that the state of the system at a certain time, then $f(0, a) = a$ and for a positive value of t , $f(t, a)$ is the result of the evolution of this state after t units of time. For example, if the system is a free particle in one dimension then the phase space is the plane \mathbf{R}^2 with coordinates (x, v) , where x is the position of the particle and v is its velocity, and the evolution is given by

$$f(t, (x, v)) = (x + tv, v).$$

An **attractor** is a subset A of the phase space characterized by the following three conditions:

- A is *forward invariant* under f : if a is an element of A then so is $f(t, a)$, for all $t > 0$.
- There exists a neighborhood of A , called the **basin of attraction** for A and denoted $B(A)$, which consists of all points b that "enter A in the limit $t \rightarrow \infty$ ". More formally, $B(A)$ is the set of all points b in the phase space with the following property:

For any open neighborhood N of A , there is a positive constant T such that $f(t, b) \in N$ for all real $t > T$.

- There is no proper subset of A having the first two properties.

Since the basin of attraction contains an open set containing A , every point that is sufficiently close to A is attracted to A . The definition of an attractor uses a metric on the phase space, but the resulting notion usually depends only on the topology of the phase space. In the case of \mathbf{R}^n , the Euclidean norm is typically used.

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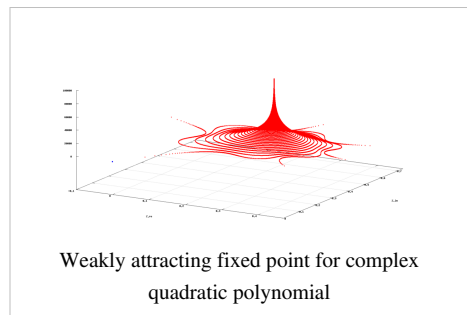
Types of attractors

Attractors are parts of the phase space of the dynamical system. Until the 1960s, as evidenced by textbooks of that era, attractors were thought of as being geometrical subsets of the phase space: points, lines, surfaces, volumes. The (topologically) wild sets that had been observed were thought to be fragile anomalies. Stephen Smale was able to show that his horseshoe map was robust and that its attractor had the structure of a Cantor set.

Two simple attractors are the fixed point and the limit cycle. There can be many other geometrical sets that are attractors. When these sets (or the motions on them), are hard to describe, then the attractor is a *strange attractor*, as described in the section below.

Fixed point

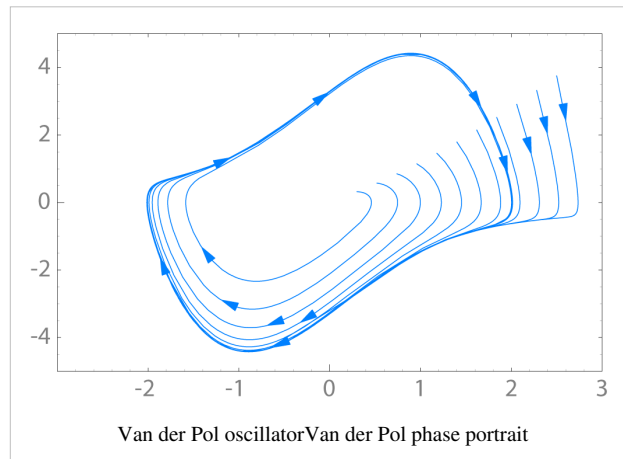
A fixed point is a point of a function that does not change under some transformation. If we regard the evolution of a dynamical system as a series of transformations, then there may or may not be a point which remains fixed under the whole series of transformation. In general there would not be such a point, but there may be one. The final state that a dynamical system evolves towards, such as the final states of a falling pebble, a damped pendulum, or the water in a glass corresponds to a fixed point of the evolution function, and will occur at the attractor, but the two concepts are not equivalent. A marble rolling around in a basin may have a fixed point in phase space even if it doesn't in physical space. Once it has lost momentum and settled into the bottom of the bowl it then has a fixed point in physical space, phase space, and is located at the attractor for that system.



Limit cycle

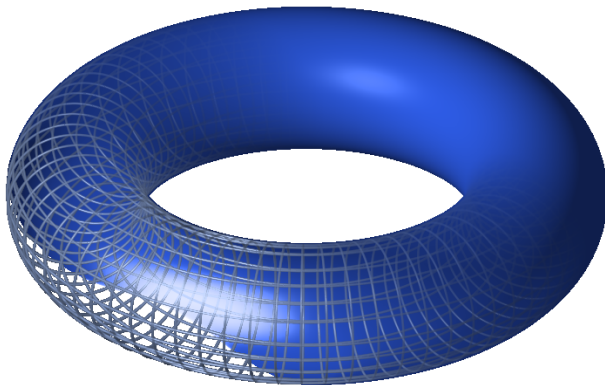
See main article limit cycle

A limit cycle is a periodic orbit of the system that is isolated. Examples include the swings of a pendulum clock, the tuning circuit of a radio, and the heartbeat while resting. The ideal pendulum is not an example because its orbits are not isolated. In phase space of the ideal pendulum, near any point of a periodic orbit there is another point that belongs to a different periodic orbit.



Limit tori

There may be more than one frequency in the periodic trajectory of the system through the state of a limit cycle. If two of these frequencies form an irrational fraction (i.e. they are incommensurate), the trajectory is no longer closed, and the limit cycle becomes a limit torus. We call this kind of attractor N_t -torus if there are N_t incommensurate frequencies. For example here is a 2-torus:



A time series corresponding to this attractor is a quasiperiodic series: A discretely sampled sum of N_t periodic functions (not necessarily sine waves) with incommensurate frequencies. Such a time series does not have a strict periodicity, but its power spectrum still consists only of sharp lines.

Strange attractor

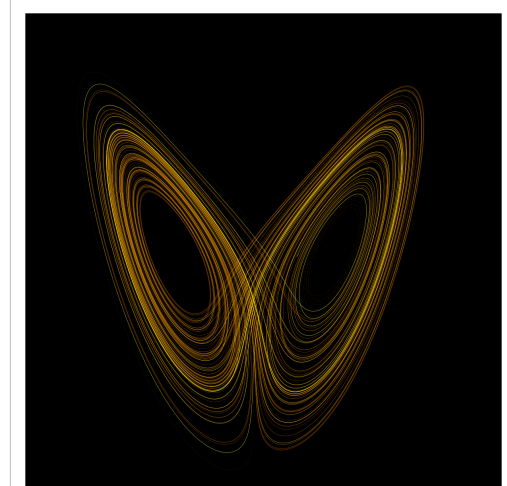
An attractor is informally described as **strange** if it has non-integer dimension. This is often the case when the dynamics on it are chaotic, but there exist also strange attractors that are not chaotic. The term was coined by David Ruelle and Floris Takens to describe the attractor that resulted from a series of bifurcations of a system describing fluid flow. Strange attractors are often differentiable in a few directions, but some are like a Cantor dust, and therefore not differentiable.

Examples of strange attractors include the Hénon attractor, Rössler attractor, Lorenz attractor, Tamari attractor.

Partial differential equations

Parabolic partial differential equations may have finite-dimensional attractors. The diffusive part of the equation damps higher frequencies and in some cases leads to a global attractor. The *Ginzburg–Landau*, the *Kuramoto–Sivashinsky*, and the two-dimensional, forced Navier–Stokes equations are all known to have global attractors of finite dimension.

For the three-dimensional, incompressible Navier–Stokes equation with periodic boundary conditions, if it has a global attractor, then this attractor will be of finite dimensions.



A plot of Lorenz's strange attractor for values $\rho=28$, $\sigma=10$, $\beta=8/3$

References

- *Attractor*^[1] at Scholarpedia, curated by John Milnor.
- David Ruelle and Floris Takens (1971). "On the nature of turbulence". *Communications of Mathematical Physics* **20**: 167–192. doi:10.1007/BF01646553.
- D. Ruelle (1981). "Small random perturbations of dynamical systems and the definition of attractors". *Communications of Mathematical Physics* **82**: 137–151. doi:10.1007/BF01206949.
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- David Ruelle (1989). *Elements of Differentiable Dynamics and Bifurcation Theory*. Academic Press. ISBN 0-12-601710-7.
- Ruelle, David (August 2006). "What is...a Strange Attractor?"^[2] (PDF). *Notices of the American Mathematical Society* **53** (7): 764–765. Retrieved 2008-01-16.
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Further reading

- Edward N. Lorenz (1996) *The Essence of Chaos* ISBN 0-295-97514-8
- James Gleick (1988) *Chaos: Making a New Science* ISBN 0-140-09250-1

External links

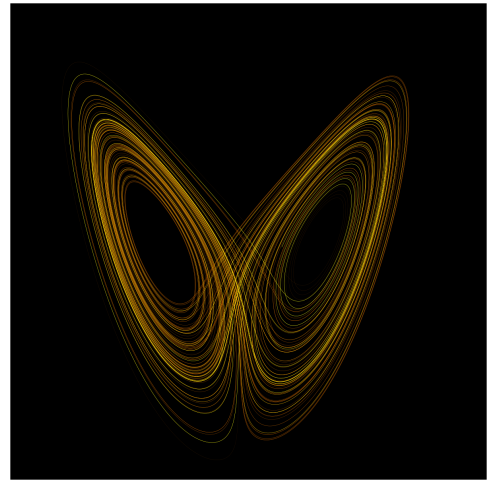
- Basin of attraction on Scholarpedia ^[3]
- A gallery of trigonometric strange attractors ^[4]
- A gallery of polynomial strange attractors ^[5]
- Animated Pickover Strange Attractors ^[6]
- Chaoscope, a 3D Strange Attractor rendering freeware ^[7]
- 1D, 2D and 3D of strange attractors, include Tamari Attractor ^[8]
- Research abstract ^[9] and software laboratory ^[10]
- A java generator for strange attractors ^[11]
- Online strange attractors generator ^[12]
- Tamari attractor ^[13]

References

- [1] <http://www.scholarpedia.org/article/Attractor>
- [2] <http://www.ams.org/notices/200607/what-is-ruelle.pdf>
- [3] http://www.scholarpedia.org/article/Basin_of_attraction
- [4] http://slide.nethium.pl/album_en.net?gNwADMfFmY
- [5] http://ccrma-www.stanford.edu/~stilti/images/chaotic_attractors/poly.html
- [6] <http://www.aidansamuel.com/strange.php>
- [7] <http://www.chaoscope.org>
- [8] <http://www.ecometry.biz/attractors>
- [9] <http://ronrecord.com/PhD/intro.html>
- [10] <ftp://ftp2.sco.com/pub/skunkware/src/x11/misc/mathrec-1.1c.tar.gz>
- [11] <http://www.dse.nl/~rolandb/attractort/attractor.html>
- [12] http://wokos.nethium.pl/attractors_en.net
- [13] <http://www.bentamari.com/attractors.html>

Lorenz attractor

The **Lorenz attractor**, named for Edward N. Lorenz, is an example of a non-linear dynamic system corresponding to the long-term behavior of the **Lorenz oscillator**. The Lorenz oscillator is a 3-dimensional dynamical system that exhibits chaotic flow, noted for its lemniscate shape. The map shows how the state of a dynamical system (the three variables of a three-dimensional system) evolves over time in a complex, non-repeating pattern.



A plot of the trajectory Lorenz system for values $\rho=28$, $\sigma = 10$, $\beta = 8/3$

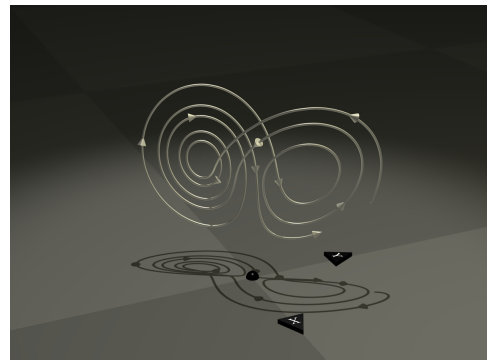
Overview

The attractor itself, and the equations from which it is derived, were introduced in 1963 by Edward Lorenz, who derived it from the simplified equations of convection rolls arising in the equations of the atmosphere.

In addition to its interest to the field of non-linear mathematics, the Lorenz model has important implications for climate and weather prediction. The model is an explicit statement that planetary and stellar atmospheres may exhibit a variety of quasi-periodic regimes that are, although fully deterministic, subject to abrupt and seemingly random change.

From a technical standpoint, the Lorenz oscillator is nonlinear, three-dimensional and deterministic. For a certain set of parameters, the system exhibits chaotic behavior and displays what is today called a strange attractor. The strange attractor in this case is a fractal of Hausdorff dimension between 2 and 3. Grassberger (1983) has estimated the Hausdorff dimension to be 2.06 ± 0.01 and the correlation dimension to be 2.05 ± 0.01 .

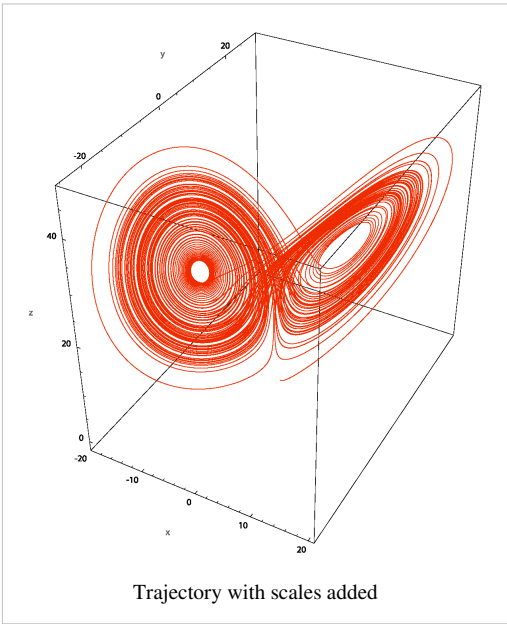
The system also arises in simplified models for lasers (Haken 1975) and dynamos (Knobloch 1981).



A trajectory of Lorenz's equations, rendered as a metal wire to show direction and 3D structure

Equations

The equations that govern the Lorenz oscillator are:

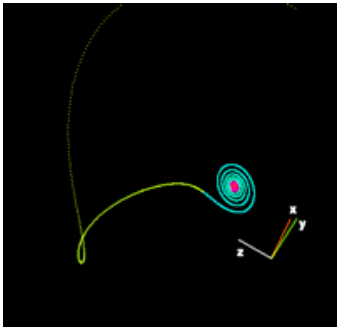
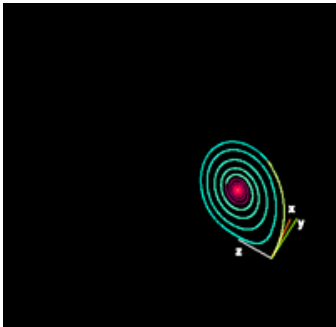
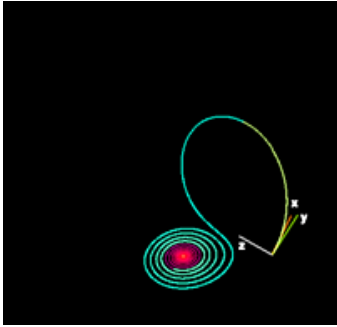
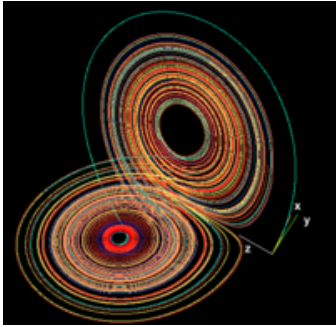


$$\begin{aligned}\frac{dx}{dt} &= \sigma(y - x) \\ \frac{dy}{dt} &= x(\rho - z) - y \\ \frac{dz}{dt} &= xy - \beta z\end{aligned}$$

where σ is called the **Prandtl number** and ρ is called the **Rayleigh number**. All $\sigma, \rho, \beta > 0$, but usually $\sigma = 10$, $\beta = 8/3$ and ρ is varied. The system exhibits chaotic behavior for $\rho = 28$ but displays knotted periodic orbits for other values of ρ . For example, with $\rho = 99.96$ it becomes a $T(3,2)$ torus knot. When $\sigma \neq 0$ and $\beta(\rho - 1) \geq 0$, the equations generate three critical points. The critical points at $(0,0,0)$ correspond to no convection, and the critical points at $(\pm\sqrt{\beta(\rho - 1)}, \pm\sqrt{\beta(\rho - 1)}, \rho - 1)$ correspond to steady convection. This pair is stable only if $\rho = \sigma \frac{\sigma + \beta + 3}{\sigma - \beta - 1}$, which can hold only for positive ρ if $\sigma > \beta + 1$.

Sensitive dependence on the initial condition		
Time t=1 (Enlarge)	Time t=2 (Enlarge)	Time t=3 (Enlarge)
These figures — made using $\rho=28$, $\sigma = 10$ and $\beta = 8/3$ — show three time segments of the 3-D evolution of 2 trajectories (one in blue, the other in yellow) in the Lorenz attractor starting at two initial points that differ only by 10^{-5} in the x-coordinate. Initially, the two trajectories seem coincident (only the yellow one can be seen, as it is drawn over the blue one) but, after some time, the divergence is obvious.		
Java animation of the Lorenz attractor shows the continuous evolution. ^[1]		

Rayleigh number

The Lorenz attractor for different values of ρ	
	
$\rho=14, \sigma=10, \beta=8/3$ (Enlarge)	$\rho=13, \sigma=10, \beta=8/3$ (Enlarge)
	
$\rho=15, \sigma=10, \beta=8/3$ (Enlarge)	$\rho=28, \sigma=10, \beta=8/3$ (Enlarge)
For small values of ρ , the system is stable and evolves to one of two fixed point attractors. When ρ is larger than 24.28, the fixed points become repulsors and the trajectory is repelled by them in a very complex way, evolving without ever crossing itself.	
Java animation showing evolution for different values of ρ ^[1]	

Source code

The source code to simulate the Lorenz attractor in GNU Octave follows.

```
% Lorenz Attractor equations solved by ODE Solve
%% x' = sigma*(y-x)
%% y' = x*(rho - z) - y
%% z' = x*y - beta*z
function dx = lorenzatt(X)
    rho = 28; sigma = 10; beta = 8/3;
    dx = zeros(3,1);
    dx(1) = sigma*(X(2) - X(1));
    dx(2) = X(1)*(rho - X(3)) - X(2);
    dx(3) = X(1)*X(2) - beta*X(3);
    return
end

% Using LSODE to solve the ODE system.
clear all
close all
lsode_options("absolute tolerance",1e-3)
```

```
lsode_options("relative tolerance",1e-4)
t = linspace(0,25,1e3); X0 = [0,1,1.05];
[X,T,MSG]=lsode(@lorenzatt,X0,t);
T
MSG
plot3(X(:,1),X(:,2),X(:,3))
view(45,45)
```

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- Tucker, W. (2002). "A Rigorous ODE Solver and Smale's 14th Problem" ^[3]. *Found. Comp. Math.* **2**: 53–117.

External links

- Weisstein, Eric W., "Lorenz attractor" ^[4] from MathWorld.
- Lorenz attractor ^[5] by Rob Morris, Wolfram Demonstrations Project.
- Lorenz equation ^[6] on planetmath.org
- For drawing the Lorenz attractor, or coping with a similar situation ^[7] using ANSI C and gnuplot.
- Synchronized Chaos and Private Communications, with Kevin Cuomo ^[8]. The implementation of Lorenz attractor in an electronic circuit.
- Lorenz attractor interactive animation ^[9] (you need the Adobe Shockwave plugin)
- Levitated.net: computational art and design ^[10]
- 3D Attractors: Mac program to visualize and explore the Lorenz attractor in 3 dimensions ^[11]
- 3D VRML Lorenz attractor ^[12] (you need a VRML viewer plugin)
- Essay on Lorenz attractors in J ^[13] - see J programming language
- Applet for non-linear simulations ^[14] (select "Lorenz attractor" preset), written by Viktor Bachraty in Jython
- Lorenz Attractor implemented in analog electronic ^[15]
- Visualizing the Lorenz attractor in 3D with Python and VTK ^[16]
- Lorenz Attractor implemented in Flash ^[17]

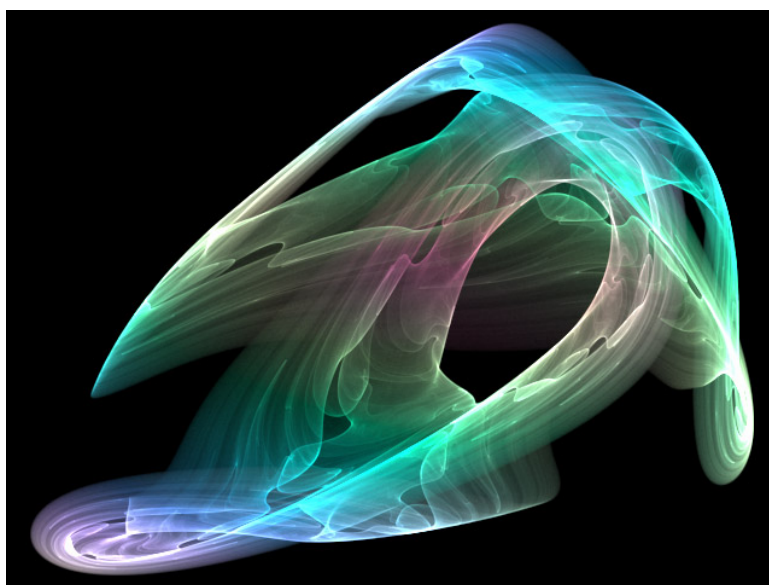
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- [2] <http://www.teorfys.uu.se/en/node/467>
- [3] <http://www.math.uu.se/~warwick/main/rodes.html>
- [4] <http://mathworld.wolfram.com/LorenzAttractor.html>
- [5] <http://demonstrations.wolfram.com/LorenzAttractor/>
- [6] <http://planetmath.org/encyclopedia/LorenzEquation.html>
- [7] <http://www.mizuno.org/c/la/index.shtml>
- [8] http://video.google.com/videoplay?docid=2875296564158834562&q=strogatz&ei=xr9OSJ_SOpeG2wKB3Iy2DA&hl=en
- [9] <http://toxi.co.uk/lorenz/>
- [10] <http://www.levitated.net/daily/levLorenzAttractor.html>
- [11] <http://amath.colorado.edu/faculty/juanga/3DAttractors.html>
- [12] <http://ibiblio.org/e-notes/VRML/Lorenz/Lorenz.htm>
- [13] http://www.jsoftware.com/jwiki/Essays/Lorenz_Attractor
- [14] <http://student.fiit.stuba.sk/~bachratv02/mes/applet.html>
- [15] <http://frank.harvard.edu/~paulh/misc/lorenz.htm>
- [16] <http://www.martinlaprise.info/2010/02/28/visualizing-the-lorentz-attractor-with-vtk/>
- [17] <http://911web.org/lorenz-attractor.php>

Strange attractor

An **attractor** is a set towards which a dynamical system evolves over time. That is, points that get close enough to the attractor remain close even if slightly disturbed. Geometrically, an attractor can be a point, a curve, a manifold, or even a complicated set with a fractal structure known as a *strange attractor*. Describing the attractors of chaotic dynamical systems has been one of the achievements of chaos theory.

A trajectory of the dynamical system in the attractor does not have to satisfy any special constraints except for remaining on the attractor. The trajectory may be periodic or chaotic or of any other type. If this condition is met, but the flow in the neighbourhood is away from the set, the set is called a **repeller** (or *repellor*).



Visual representation of a strange attractor

Motivation

A dynamical system is generally described by one or more differential or difference equations. The equations of a given dynamic system specify its behavior over any given short period of time. To determine the system's behavior for a longer period, it is necessary to integrate the equations, either through analytical means or through iteration, often with the aid of computers.

Dynamical systems in the physical world tend to be dissipative: if it were not for some driving force, the motion would cease. (Dissipation may come from internal friction, thermodynamic losses, or loss of material, among many causes.) The dissipation and the driving force tend to combine to kill out initial transients and settle the system into its typical behavior. This one part of the phase space of the dynamical system corresponding to the typical behavior is the **attracting section** or **attractee**.

Invariant sets and limit sets are similar to the attractor concept. An *invariant set* is a set that evolves to itself under the dynamics. Attractors may contain invariant sets. A *limit set* is a set of points such that there exists some initial state that ends up arbitrarily close to the limit set (i.e. to each point of the set) as time goes to infinity. Attractors are limit sets, but not all limit sets are attractors: It is possible to have some points of a system converge to a limit set, but different points when perturbed slightly off the limit set may get knocked off and never return to the vicinity of the limit set.

For example, the damped pendulum has two invariant points: the point x_0 of minimum height and the point x_1 of maximum height. The point x_0 is also a limit set, as trajectories converge to it; the point x_1 is not a limit set. Because of the dissipation, the point x_0 is also an attractor. If there were no dissipation, x_0 would not be an attractor.

Mathematical definition

Let $f(t, \bullet)$ be a function which specifies the dynamics of the system. That is, if a is a point in the phase space, so that the state of the system at a certain time, then $f(0, a) = a$ and for a positive value of t , $f(t, a)$ is the result of the evolution of this state after t units of time. For example, if the system is a free particle in one dimension then the phase space is the plane \mathbf{R}^2 with coordinates (x, v) , where x is the position of the particle and v is its velocity, and the evolution is given by

$$f(t, (x, v)) = (x + tv, v).$$

An **attractor** is a subset A of the phase space characterized by the following three conditions:

- A is *forward invariant* under f : if a is an element of A then so is $f(t, a)$, for all $t > 0$.
- There exists a neighborhood of A , called the **basin of attraction** for A and denoted $B(A)$, which consists of all points b that "enter A in the limit $t \rightarrow \infty$ ". More formally, $B(A)$ is the set of all points b in the phase space with the following property:

For any open neighborhood N of A , there is a positive constant T such that $f(t, b) \in N$ for all real $t > T$.

- There is no proper subset of A having the first two properties.

Since the basin of attraction contains an open set containing A , every point that is sufficiently close to A is attracted to A . The definition of an attractor uses a metric on the phase space, but the resulting notion usually depends only on the topology of the phase space. In the case of \mathbf{R}^n , the Euclidean norm is typically used.

Many other definitions of attractor occur in the literature. For example, some authors require that an attractor have positive measure (preventing a point from being an attractor), others relax the requirement that $B(A)$ be a neighborhood.

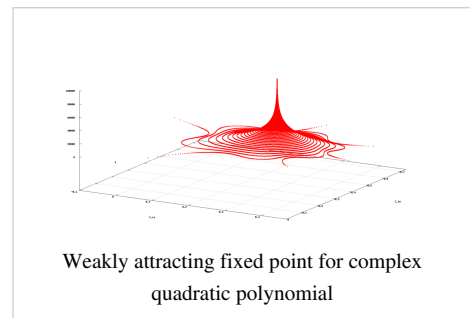
Types of attractors

Attractors are parts of the phase space of the dynamical system. Until the 1960s, as evidenced by textbooks of that era, attractors were thought of as being geometrical subsets of the phase space: points, lines, surfaces, volumes. The (topologically) wild sets that had been observed were thought to be fragile anomalies. Stephen Smale was able to show that his horseshoe map was robust and that its attractor had the structure of a Cantor set.

Two simple attractors are the fixed point and the limit cycle. There can be many other geometrical sets that are attractors. When these sets (or the motions on them), are hard to describe, then the attractor is a *strange attractor*, as described in the section below.

Fixed point

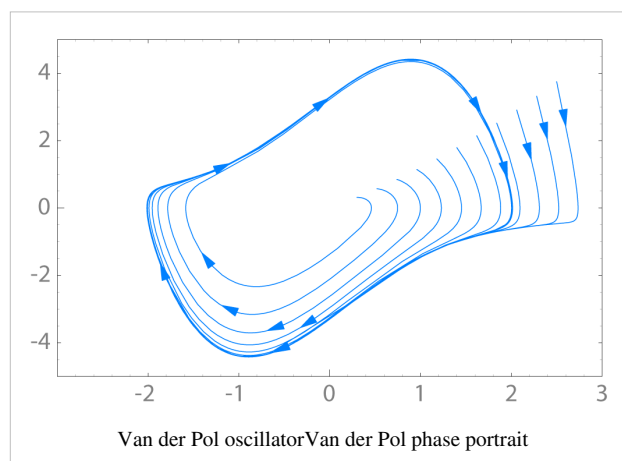
A fixed point is a point of a function that does not change under some transformation. If we regard the evolution of a dynamical system as a series of transformations, then there may or may not be a point which remains fixed under the whole series of transformation. In general there would not be such a point, but there may be one. The final state that a dynamical system evolves towards, such as the final states of a falling pebble, a damped pendulum, or the water in a glass corresponds to a fixed point of the evolution function, and will occur at the attractor, but the two concepts are not equivalent. A marble rolling around in a basin may have a fixed point in phase space even if it doesn't in physical space. Once it has lost momentum and settled into the bottom of the bowl it then has a fixed point in physical space, phase space, and is located at the attractor for that system.



Limit cycle

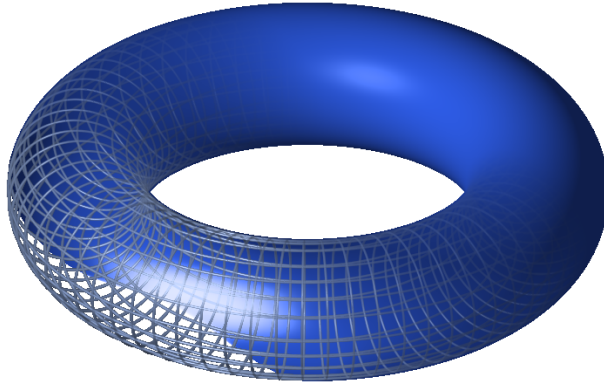
See main article limit cycle

A limit cycle is a periodic orbit of the system that is isolated. Examples include the swings of a pendulum clock, the tuning circuit of a radio, and the heartbeat while resting. The ideal pendulum is not an example because its orbits are not isolated. In phase space of the ideal pendulum, near any point of a periodic orbit there is another point that belongs to a different periodic orbit.



Limit tori

There may be more than one frequency in the periodic trajectory of the system through the state of a limit cycle. If two of these frequencies form an irrational fraction (i.e. they are incommensurate), the trajectory is no longer closed, and the limit cycle becomes a limit torus. We call this kind of attractor N_t -torus if there are N_t incommensurate frequencies. For example here is a 2-torus:



A time series corresponding to this attractor is a quasiperiodic series: A discretely sampled sum of N_t periodic functions (not necessarily sine waves) with incommensurate frequencies. Such a time series does not have a strict periodicity, but its power spectrum still consists only of sharp lines.

Strange attractor

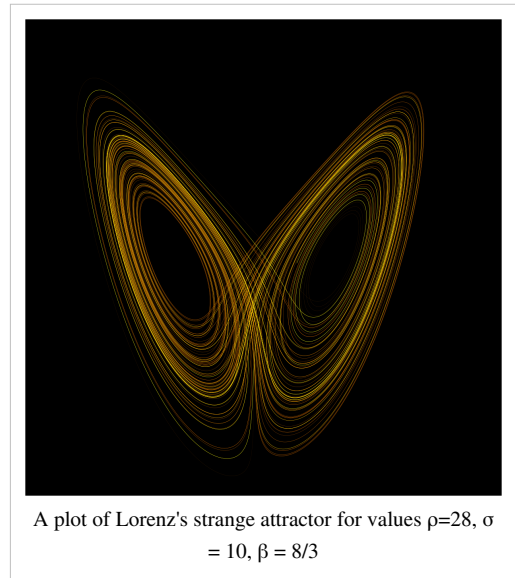
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Examples of strange attractors include the Hénon attractor, Rössler attractor, Lorenz attractor, Tamari attractor.

Partial differential equations

Parabolic partial differential equations may have finite-dimensional attractors. The diffusive part of the equation damps higher frequencies and in some cases leads to a global attractor. The *Ginzburg–Landau*, the *Kuramoto–Sivashinsky*, and the two-dimensional, forced Navier–Stokes equations are all known to have global attractors of finite dimension.

For the three-dimensional, incompressible Navier–Stokes equation with periodic boundary conditions, if it has a global attractor, then this attractor will be of finite dimensions.



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Further reading

- Edward N. Lorenz (1996) *The Essence of Chaos* ISBN 0-295-97514-8
- James Gleick (1988) *Chaos: Making a New Science* ISBN 0-140-09250-1

External links

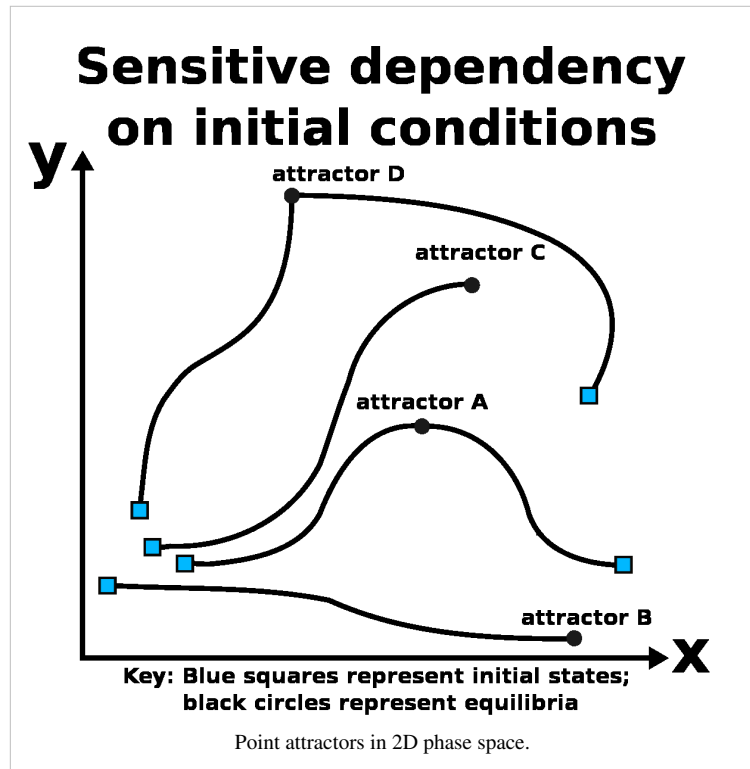
- Basin of attraction on Scholarpedia ^[3]
- A gallery of trigonometric strange attractors ^[4]
- A gallery of polynomial strange attractors ^[5]
- Animated Pickover Strange Attractors ^[6]
- Chaoscope, a 3D Strange Attractor rendering freeware ^[7]
- 1D, 2D and 3D of strange attractors, include Tamari Attractor ^[8]
- Research abstract ^[9] and software laboratory ^[10]
- A java generator for strange attractors ^[11]
- Online strange attractors generator ^[12]
- Tamari attractor ^[13]

Butterfly effect

In chaos theory, the **butterfly effect** is the *sensitive dependence on initial conditions*; where a small change at one place in a nonlinear system can result in large differences to a later state. For example, the presence or absence of a butterfly flapping its wings could lead to creation or absence of a hurricane.

Although the butterfly effect may appear to be an esoteric and unusual behavior, it is exhibited by very simple systems: for example, a ball placed at the crest of a hill might roll into any of several valleys depending on slight differences in initial position.

The butterfly effect is a common trope in fiction when presenting scenarios involving time travel and with "what if" cases where one storyline diverges at the moment of a seemingly minor event resulting in two significantly different outcomes.



Theory

Recurrence, the approximate return of a system towards its initial conditions, together with sensitive dependence on initial conditions, are the two main ingredients for chaotic motion. They have the practical consequence of making complex systems, such as the weather, difficult to predict past a certain time range (approximately a week in the case of weather), since it is impossible to measure the starting atmospheric conditions completely accurately.

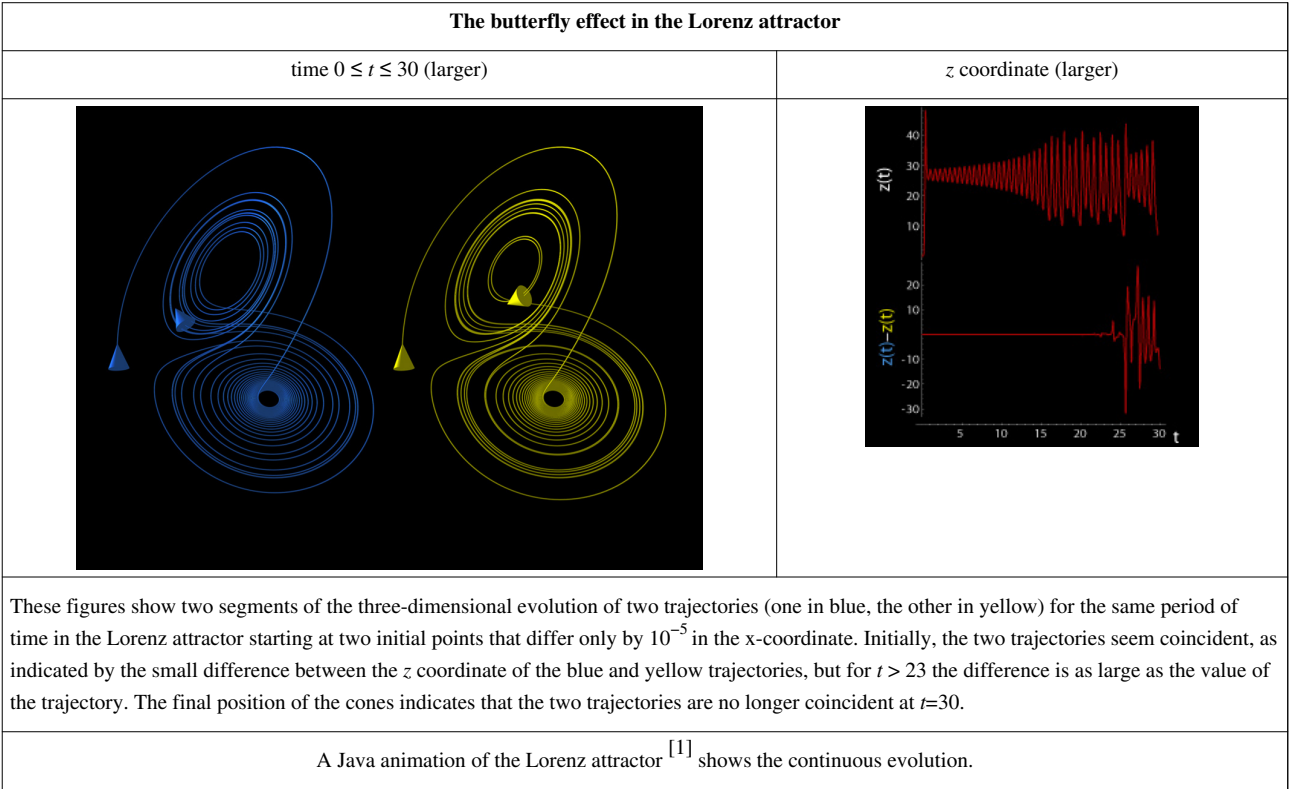
Origin of the concept and the term

The term "butterfly effect" itself is related to the work of Edward Lorenz, and it is based in chaos theory and sensitive dependence on initial conditions, already described in the literature in a particular case of the three-body problem by Henri Poincaré in 1890.^[1] He later proposed that such phenomena could be common, say in meteorology. In 1898,^[1] Jacques Hadamard noted general divergence of trajectories in spaces of negative curvature, and Pierre Duhem discussed the possible general significance of this in 1908.^[1] The idea that one butterfly could eventually have a far-reaching ripple effect on subsequent historic events seems first to have appeared in "A Sound of Thunder", a 1952 short story by Ray Bradbury about time travel (see Literature and print here) although Lorenz made the term popular. In 1961, Lorenz was using a numerical computer model to rerun a weather prediction, when, as a shortcut on a number in the sequence, he entered the decimal .506 instead of entering the full .506127. The result was a completely different weather scenario.^[2] Lorenz published his findings in a 1963 paper^[3] for the New York Academy of Sciences noting that "One meteorologist remarked that if the theory were correct, one flap of a seagull's wings could change the course of weather forever." Later speeches and papers by Lorenz used the more poetic butterfly. According to Lorenz, when Lorenz failed to provide a title for a talk he was to present at the 139th meeting of the American Association for the Advancement of Science in 1972, Philip Merilees concocted *Does the*

flap of a butterfly's wings in Brazil set off a tornado in Texas? as a title. Although a butterfly flapping its wings has remained constant in the expression of this concept, the location of the butterfly, the consequences, and the location of the consequences have varied widely.^[4]

The phrase refers to the idea that a butterfly's wings might create tiny changes in the atmosphere that may ultimately alter the path of a tornado or delay, accelerate or even prevent the occurrence of a tornado in another location. The flapping wing represents a small change in the initial condition of the system, which causes a chain of events leading to large-scale alterations of events (compare: domino effect). Had the butterfly not flapped its wings, the trajectory of the system might have been vastly different. While the butterfly does not "cause" the tornado in the sense of providing the energy for the tornado, it does "cause" it in the sense that the flap of its wings is an essential part of the initial conditions resulting in a tornado, and without that flap that particular tornado would not have existed.

Illustration



Mathematical definition

A dynamical system with evolution map f^t displays sensitive dependence on initial conditions if points arbitrarily close together become separate with increasing t at an exponential rate. The definition is not topological, but essentially metrical.

If M is the state space for the map f^t , then f^t displays sensitive dependence to initial conditions if for any x in M and any $\delta > 0$, there are y in M , with $0 < d(x, y) < \delta$ such that

$$d(f^{\tau}(x), f^{\tau}(y)) > exp(a\tau)d(x, y).$$

The definition does not require that all points from a neighborhood separate from the base point x , but it requires one positive Lyapunov exponent.

Examples

The butterfly effect is most familiar in terms of weather; it can easily be demonstrated in standard weather prediction models, for example.^[5]

The potential for sensitive dependence on initial conditions (the butterfly effect) has been studied in a number of cases in semiclassical and quantum physics including atoms in strong fields and the anisotropic Kepler problem.^{[6] [7]} Some authors have argued that extreme (exponential) dependence on initial conditions is not expected in pure quantum treatments;^{[8] [9]} however, the sensitive dependence on initial conditions demonstrated in classical motion is included in the semiclassical treatments developed by Martin Gutzwiller^[10] and Delos and co-workers.^[11]

Other authors suggest that the butterfly effect can be observed in quantum systems. Karkuszewski et al. consider the time evolution of quantum systems which have slightly different Hamiltonians. They investigate the level of sensitivity of quantum systems to small changes in their given Hamiltonians.^[12] Poulin et al. present a quantum algorithm to measure fidelity decay, which “measures the rate at which identical initial states diverge when subjected to slightly different dynamics.” They consider fidelity decay to be “the closest quantum analog to the (purely classical) butterfly effect.”^[13] Whereas the classical butterfly effect considers the effect of a small change in the position and/or velocity of an object in a given Hamiltonian system, the quantum butterfly effect considers the effect of a small change in the Hamiltonian system with a given initial position and velocity.^{[14] [15]} This quantum butterfly effect has been demonstrated experimentally.^[16] Quantum and semiclassical treatments of system sensitivity to initial conditions are known as quantum chaos.^{[8] [14]}

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Further reading

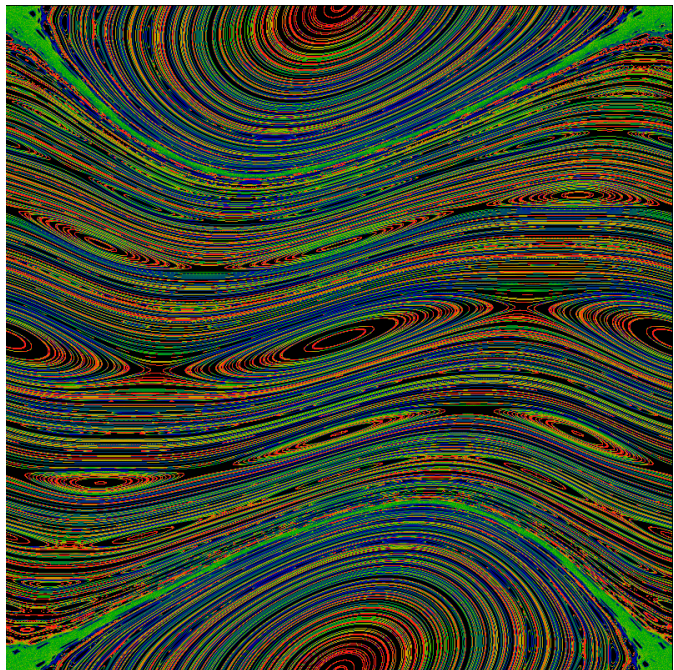
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External links

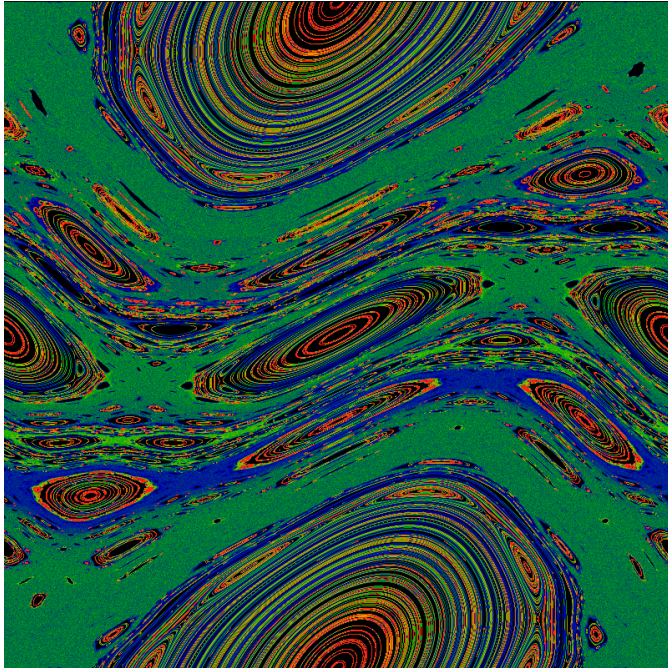
- The meaning of the butterfly: Why pop culture loves the 'butterfly effect,' and gets it totally wrong (http://www.boston.com/bostonglobe/ideas/articles/2008/06/08/the_meaning_of_the_butterfly/?page=full), Peter Dizikes, *Boston Globe*, June 8, 2008
- From butterfly wings to single e-mail (<http://www.news.cornell.edu/releases/Feb04/AAAS.Kleinberg.ws.html>) (Cornell University)
- New England Complex Systems Institute - Concepts: Butterfly Effect (<http://necsi.edu/guide/concepts/butterflyeffect.html>)
- The Chaos Hypertextbook (<http://hypertextbook.com/chaos/>). An introductory primer on chaos and fractals.
- Weisstein, Eric W., "Butterfly Effect (<http://mathworld.wolfram.com/ButterflyEffect.html>)" from MathWorld.

Standard map

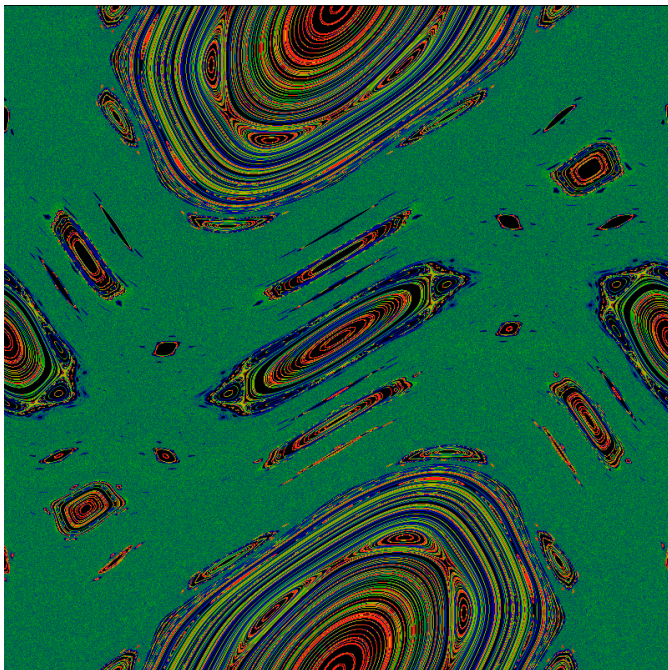
The **standard map** is an area-preserving chaotic map from a square with side 2π onto itself. It is defined by:



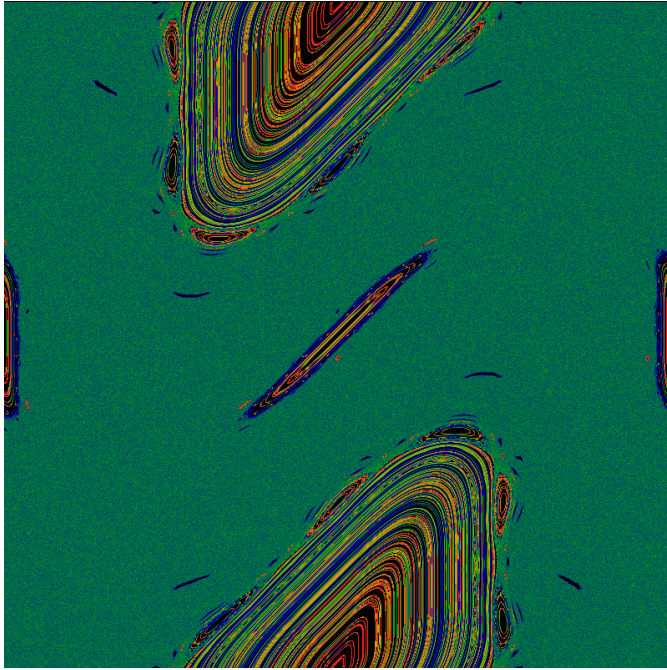
Orbits of the standard map for $K = 0.6$.



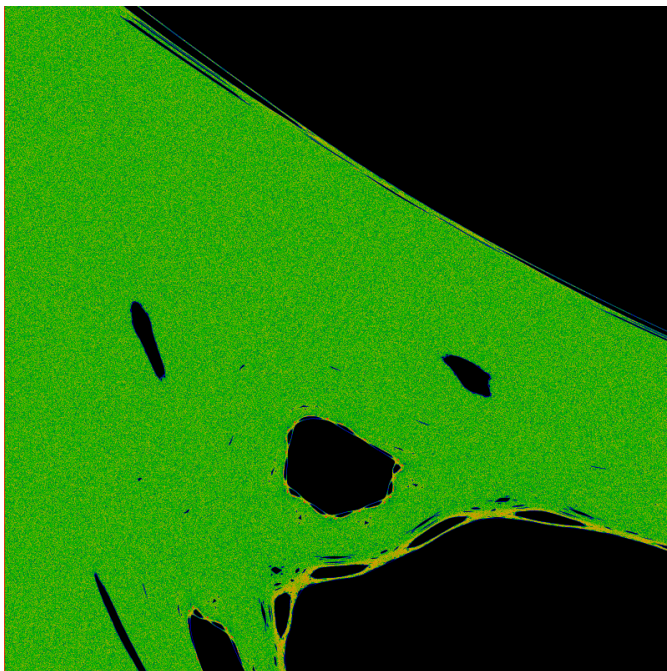
Orbits of the standard map for $K = 0.971635$.



Orbits of the standard map for $K = 1.2$.



Orbits of the standard map for $K = 2.0$. The large green region is the main chaotic region of the map.



A single orbit of the standard map for $K=2.0$. Magnified close-up centered at $\theta = 0.282$, $p = 0.666$, of total width/height 0.02. Note the extremely uniform distribution of the orbit.

$$\begin{aligned} p_{n+1} &= p_n + K \sin(\theta_n) \\ \theta_{n+1} &= \theta_n + p_{n+1} \end{aligned}$$

where p_n and θ_n are taken modulo 2π .

This map is also known as the **Chirikov–Taylor map** or as the **Chirikov standard map**. The properties of chaos of the standard map were established by Boris Chirikov in 1969. See more details at Scholarpedia entry ^[1]. The

quantized map is also known as the kicked rotator in the quantum chaos community.

Physical model

This map describes the motion of a simple mechanical system known as the kicked rotator. It consists of a stick that is free of the gravitational force, which can rotate frictionlessly in a plane around an axis located in one of its tips, and which is periodically kicked on the other tip. The variables θ_n and p_n respectively determine the angular position of the stick and its angular momentum after the n -th kick. The constant K measures the intensity of the kicks.

The kicked rotator approximates systems studied in the fields of mechanics of particles, accelerator physics, plasma physics, and solid state physics. For example, circular particle accelerators accelerate particles by applying periodic kicks, as they circulate in the beam tube. Thus, the structure of the beam can be approximated by the kicked rotor. However, this map is interesting from a fundamental point of view in physics and mathematics because it is a very simple model of a conservative system that displays Hamiltonian chaos. It is therefore useful to study the development of chaos in this kind of system.

Main properties

For $K = 0$ the map is linear and only periodic and quasiperiodic orbits are possible. When plotted in phase space (the θ - p plane), periodic orbits appear as closed curves, and quasiperiodic orbits as necklaces of closed curves whose centers lie in another larger closed curve. Which type of orbit is observed depends on the map's initial conditions.

Nonlinearity of the map increases with K , and with it the possibility to observe chaotic dynamics for appropriate initial conditions. This is illustrated in the figure, which displays a collection of different orbits allowed to the standard map for various values of $K > 0$. All the orbits shown are periodic or quasiperiodic, with the exception of the green one that is chaotic and develops in a large region of phase space as an apparently random set of points. Particularly remarkable is the extreme uniformity of the distribution in the chaotic region, although this can be deceptive: even within the chaotic regions, there are an infinite number of diminishingly small islands that are never visited during iteration, as shown in the close-up.

Circle map

The standard map is related to the circle map, which has a single, similar iterated equation:

$$\theta_{n+1} = \theta_n + \Omega - K \sin(\theta_n)$$

as compared to

$$\theta_{n+1} = \theta_n + p_n + K \sin(\theta_n)$$

$$p_{n+1} = \theta_{n+1} - \theta_n$$

for the standard map, the equations reordered to emphasize similarity. In essence, the circle map forces the momentum to a constant.

References

[1] http://www.scholarpedia.org/article/Chirikov_standard_map

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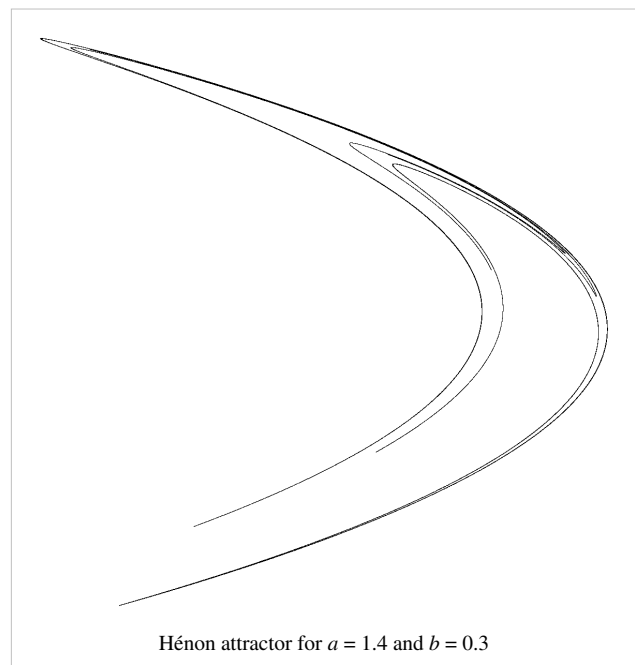
External links

- Standard map (<http://mathworld.wolfram.com/StandardMap.html>) at MathWorld
- Chirikov standard map (http://www.scholarpedia.org/article/Chirikov_standard_map) at Scholarpedia (<http://www.scholarpedia.org>)
- Website dedicated to Boris Chirikov (<http://www.quantware.ups-tlse.fr/chirikov/>)
- Interactive Java Applet visualizing orbits of the Standard Map (<http://complexity.xozzox.de/nonlinmappings.html>), by Achim Luhn

Hénon map

The **Hénon map** is a discrete-time dynamical system. It is one of the most studied examples of dynamical systems that exhibit chaotic behavior. The Hénon map takes a point (x_n, y_n) in the plane and maps it to a new point

$$\begin{aligned}x_{n+1} &= y_n + 1 - ax_n^2, \\ y_{n+1} &= bx_n.\end{aligned}$$



The map depends on two parameters, a and b , which for the **canonical Hénon map** have values of $a = 1.4$ and $b = 0.3$. For the canonical values the Hénon map is chaotic. For other values of a and b the map may be chaotic, intermittent, or converge to a periodic orbit. An overview of the type of behavior of the map at different parameter values may be obtained from its orbit diagram.

The map was introduced by Michel Hénon as a simplified model of the Poincaré section of the Lorenz model. For the canonical map, an initial point of the plane will either approach a set of points known as the Hénon strange attractor, or diverge to infinity. The Hénon attractor is a fractal, smooth in one direction and a Cantor set in another. Numerical estimates yield a correlation dimension of 1.25 ± 0.02 ^[1] and a Hausdorff dimension of 1.261 ± 0.003 ^[2] for the attractor of the canonical map.

As a dynamical system, the canonical Hénon map is interesting because, unlike the logistic map, its orbits defy a simple description.

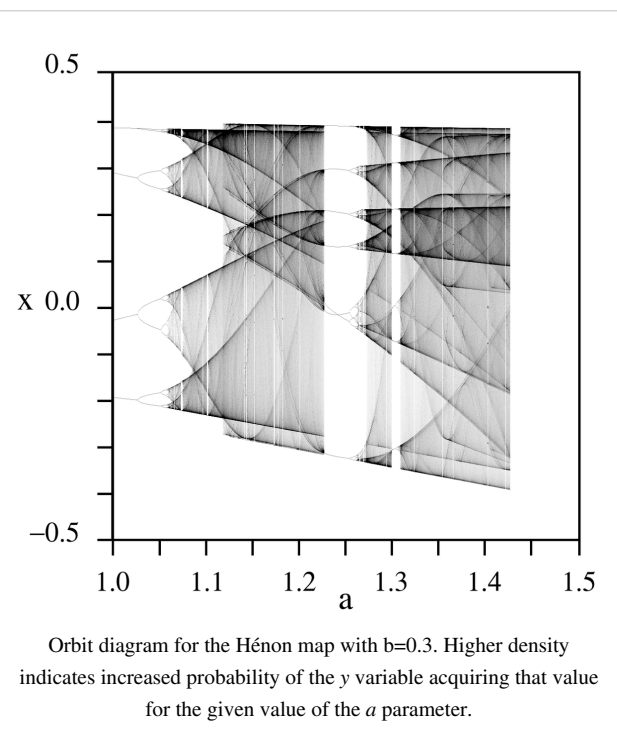
Attractor

The Hénon map maps two points into themselves: these are the invariant points. For the canonical values of a and b of the Hénon map, one of these points is on the attractor:

$$x = 0.631354477... \text{ and } y = 0.189406343...$$

This point is unstable. Points close to this fixed point and along the slope 1.924 will approach the fixed point and points along the slope -0.156 will move away from the fixed point. These slopes arise from the linearizations of the stable manifold and unstable manifold of the fixed point. The unstable manifold of the fixed point in the attractor is contained in the strange attractor of the Hénon map.

The Hénon map does not have a strange attractor for all values of the parameters a and b . For example, by keeping b fixed at 0.3 the bifurcation diagram shows that for $a = 1.25$ the Hénon map has a stable periodic orbit as an attractor.



Cvitanović et al. have shown how the structure of the Hénon strange attractor can be understood in terms of unstable periodic orbits within the attractor.

Decomposition

The Hénon map may be decomposed into an area-preserving bend:

$$(x_1, y_1) = (x, 1 - ax^2 + y),$$

a contraction in the x direction:

$$(x_2, y_2) = (bx_1, y_1),$$

and a reflection in the line $y = x$:

$$(x_3, y_3) = (y_2, x_2).$$

References

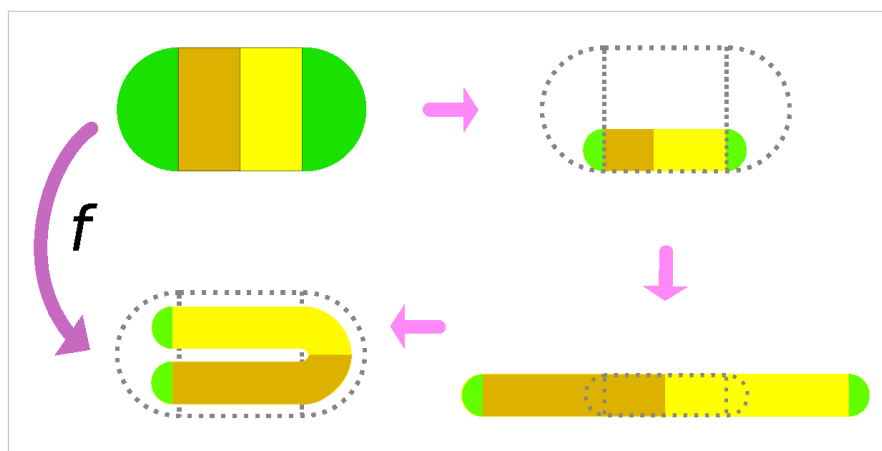
- [1] P. Grassberger and I. Procaccia (1983). "Measuring the strangeness of strange attractors". *Physica* **9D** (1-2): 189–208. Bibcode 1983PhyD....9..189G. doi:10.1016/0167-2789(83)90298-1.
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External links

- Interactive Henon map (<http://ibiblio.org/e-notes/Chaos/henon.htm>) and Henon attractor (<http://ibiblio.org/e-notes/Chaos/strange.htm>) in Chaotic Maps (<http://ibiblio.org/e-notes/Chaos/contents.htm>)
- Another interactive iteration of the Henon Map (<http://complexity.xozzox.de/nonlinmappings.html>) by A. Luhn
- Orbit Diagram of the Hénon Map (<http://demonstrations.wolfram.com/OrbitDiagramOfTheHenonMap/>) by C. Pellicer-Lostao and R. Lopez-Ruiz after work by Ed Pegg Jr, The Wolfram Demonstrations Project.

Horseshoe map

In the mathematics of chaos theory, a **horseshoe map** is any member of a class of chaotic maps of the square into itself. It is a core example in the study of dynamical systems. The map was introduced by Stephen Smale while studying the behavior of the orbits of the van der Pol oscillator. The action of the map is defined geometrically by squishing the square, then stretching the result into a long strip, and finally folding the strip into the shape of a horseshoe.



Most points eventually leave the square under the action of the map. They go to the side caps where they will, under iteration, converge to a fixed point in one of the caps. The points that remain in the square under repeated iteration form a fractal set and are part of the invariant set of the map.

The squishing, stretching and folding of the horseshoe map are the essential elements that must be present in a chaotic system. In the horseshoe map the squeezing and stretching are uniform. They compensate each other so that the area of the square does not change. The folding is done neatly, so that the orbits that remain forever in the square can be simply described.

For a horseshoe map:

- there are an infinite number of periodic orbits;
- periodic orbits of arbitrarily long period exist;
- the number of periodic orbits grows exponentially with the period; and
- close to any point of the fractal invariant set there is a point of a periodic orbit.

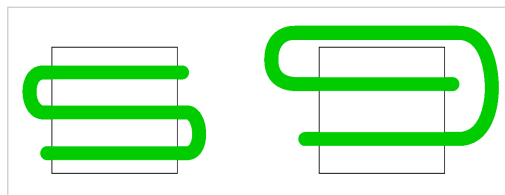
The horseshoe map

The horseshoe map f is a diffeomorphism defined from a region S of the plane into itself. The region S is a square capped by two semi-disks. The action of f is defined through the composition of three geometrically defined transformations. First the square is contracted along the vertical direction by a factor $a < 1/2$. The caps are contracted so as to remain semi-disks attached to the resulting rectangle. Contracting by a factor smaller than one half assures that there will be a gap between the branches of the horseshoe. Next the rectangle is stretched horizontally by a factor of $1/a$; the caps remain unchanged. Finally the resulting strip is folded into a horseshoe-shape and placed back into S .

The interesting part of the dynamics is the image of the square into itself. Once that part is defined, the map can be extended to a diffeomorphism by defining its action on the caps. The caps are made to contract and eventually map inside one of the caps (the left one in the figure). The extension of f to the caps adds a fixed point to the non-wandering set of the map. To keep the class of horseshoe maps simple, the curved region of the horseshoe should not map back into the square.

The horseshoe map is one-to-one: any point in the domain has a unique image, even though not all points of the domain are the image of a point. The inverse of the horseshoe map, denoted by f^{-1} cannot have as its domain the entire region S , instead it must be restricted to the image of S under f , that is, the domain of f^{-1} is $f(S)$.

By folding the contracted and stretched square in different ways, other types of horseshoe maps are possible.



The contracted square cannot overlap itself to assure that it remains one-to-one. When the action on the square is extended to a diffeomorphism, the extension cannot always be done on the plane. For example, the map on the right needs to be extended to a diffeomorphism of the sphere by using a “cap” that wraps around the equator.

The horseshoe map is an Axiom A diffeomorphism that serves as a model for the general behavior at a transverse homoclinic point, where the stable and unstable manifolds of a periodic point intersect.

Dynamics of the map

The horseshoe map was designed to reproduce the chaotic dynamics of a flow in the neighborhood of a given periodic orbit. The neighborhood is chosen to be a small disk perpendicular to the orbit. As the system evolves, points in this disk remain close to the given periodic orbit, tracing out orbits that eventually intersect the disk once again. Other orbits diverge.

The behavior of all the orbits in the disk can be determined by considering what happens to the disk. The intersection of the disk with the given periodic orbit comes back to itself every period of the orbit and so do points in its neighborhood. When this neighborhood returns, its shape is transformed. Among the points back inside the disk are some points that will leave the disk neighborhood and others that will continue to return. The set of points that never leaves the neighborhood of the given periodic orbit form a fractal.

A symbolic name can be given to all the orbits that remain in the neighborhood. The initial neighborhood disk can be divided into a small number of regions. Knowing the sequence in which the orbit visits these regions allows the orbit to be pinpointed exactly. The visitation sequence of the orbits provide a symbolic representation of the dynamics, known as symbolic dynamics.

Orbits

It is possible to describe the behavior of all initial conditions of the horseshoe map. An initial point $u_0 = \{x, y\}$ gets mapped into the point $u_1 = f(u_0)$. Its iterate is the point $u_2 = f(u_1) = f^2(u_0)$, and repeated iteration generates the orbit u_0, u_1, u_2, \dots

Under repeated iteration of the horseshoe map, most orbits end up at the fixed point in the left cap. This is because the horseshoe maps the left cap into itself by an affine transformation, which has exactly one fixed point. Any orbit that lands on the left cap never leaves it and converges to the fixed point in the left cap under iteration. Points in the right cap get mapped into the left cap on the next iteration, and most points in the square get mapped into the caps. Under iteration, most points will be part of orbits that converge to the fixed point in the left cap, but some points of the square never leave.

Iterating the square

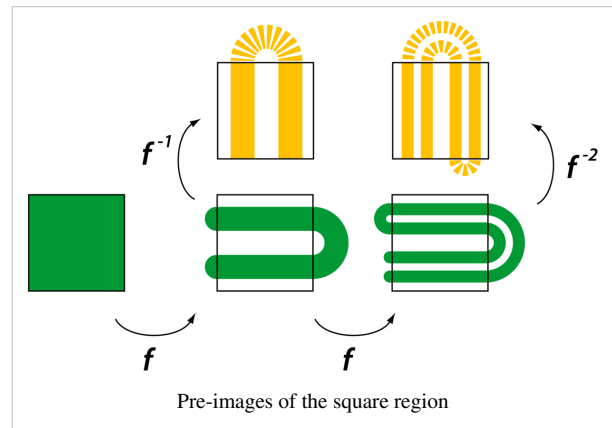
Under forward iterations of the horseshoe map, the original square gets mapped into a series of horizontal strips. The points in these horizontal strips come from vertical strips in the original square. Let S_0 be the original square, map it forward n times, and consider only the points that fall back into the square S_0 , which is a set of horizontal stripes

$$H_n = f^n(S_0) \cap S_0.$$

The points in the horizontal stripes came from the vertical stripes

$$V_n = f^{-n}(H_n),$$

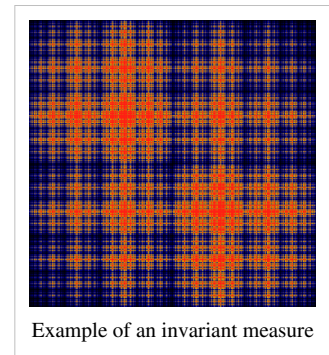
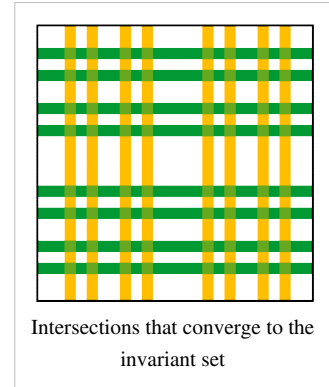
which are the horizontal stripes H_n mapped backwards n times. That is, a point in V_n will, under n iterations of the horseshoe, end up in the set H_n of vertical strips.



Invariant set

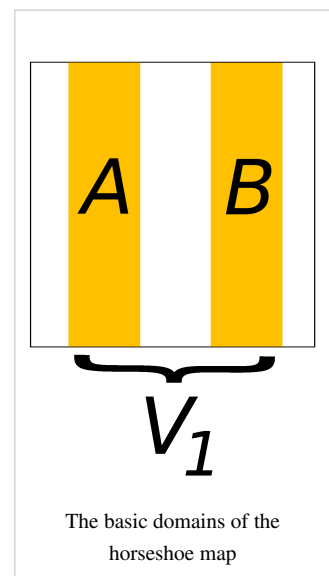
If a point is to remain indefinitely in the square, then it must belong to a set Λ that maps to itself. Whether this set is empty or not has to be determined. The vertical strips V_1 map into the horizontal strips H_1 , but not all points of V_1 map back into V_1 . Only the points in the intersection of V_1 and H_1 may belong to Λ , as can be checked by following points outside the intersection for one more iteration.

The intersection of the horizontal and vertical stripes, $H_n \cap V_n$, are squares that converge in the limit $n \rightarrow \infty$ to the invariant set Λ . The structure of this set can be better understood by introducing a system of labels for all the intersections—a symbolic dynamics.



Symbolic dynamics

The intersection $H_n \cap V_n$ is contained in V_1 . So any point that is in Λ under iteration must land in the left vertical strip A of V_1 , or on the right vertical strip B . The lower horizontal strip of H_1 is the image of A and the upper horizontal strip is the image of B , so $H_1 = f(A) \cup f(B)$. The strips A and B can be used to label the four squares in the intersection of V_1 and H_1 :



$$\Lambda_{A \cdot A} = f(A) \cap A \quad \Lambda_{A \cdot B} = f(A) \cap B$$

$$\Lambda_{B \cdot A} = f(B) \cap A \quad \Lambda_{B \cdot B} = f(B) \cap B$$

The set $\Lambda_{B \cdot A}$ consist of points from strip A that were in strip B in the previous iteration. A dot is used to separate the region the point of an orbit is in from the region the point came from.

The notation can be extended to higher iterates of the horseshoe map. The vertical strips can be named according to the sequence of visits to strip A or strip B . For example, the set $ABB \subset V_3$ consists of the points from A that will all land in B in one iteration and remain in B in the iteration after that:

$$ABB = \{ x \in A \mid f(x) \in B \text{ and } f^2(x) \in B \}$$

Working backwards from that trajectory determines a small region, the set ABB , within V_3 .

The horizontal strips are named from their vertical strip pre-images. In this notation, the intersection of V_2 and H_2 consists of 16 squares, one of which is

$$\Lambda_{AB \bullet BB} = f^2(AB) \cap BB.$$

All the points in $\Lambda_{AB \bullet BB}$ are in B and will continue to be in B for at least one more iteration. Their previous trajectory before landing in BB was A followed by B .

Periodic orbits

Any one of the intersections $\Lambda_{P \bullet F}$ of a horizontal strip with a vertical strip, where P and F are sequences of A s and B s, is an affine transformation of a small region in V_I . If P has k symbols in it, and if $f^{-k}(\Lambda_{P \bullet F})$ and $\Lambda_{P \bullet F}$ intersect, the region $\Lambda_{P \bullet F}$ will have a fixed point. This happens when the sequence P is the same as F . For example, $\Lambda_{ABAB \bullet ABAB} \subset V_4 \cap H_4$ has at least one fixed point. This point is also the same as the fixed point in $\Lambda_{AB \bullet AB}$. By including more and more AB s in the P and F part of the label of intersection, the area of the intersection can be made as small as needed. It converges to a point that is part of a periodic orbit of the horseshoe map. The periodic orbit can be labeled by the simplest sequence of A s and B s that labels one of the regions the periodic orbit visits.

For every sequence of A s and B s there is a periodic orbit.

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External links

- *Smale Horseshoe*^[1] at Scholarpedia.
- Interactive Smale horseshoe^[2] with Java applet and comments

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- [1] http://www.scholarpedia.org/article/Smale_Horseshoe
 [2] <http://www.ibiblio.org/e-notes/Chaos/homoclinic.htm>

Coupled map lattice

A **coupled map lattice (CML)** is a dynamical system that models the behavior of non-linear systems (especially partial differential equations). They are predominantly used to qualitatively study the chaotic dynamics of spatially extended systems. This includes the dynamics of spatiotemporal ^[1] chaos where the number of effective degrees of freedom diverge as the size of the system increases ^[2]. Features of the CML are discrete time dynamics, discrete underlying spaces (lattices or networks), and real (number or vector), local, continuous state variables ^[3]. Studied systems include populations, chemical reactions, convection, fluid flow and biological networks. Even recently, CMLs have been applied to computational networks ^[4] identifying detrimental attack methods and cascading failures.

CML's are comparable to cellular automata models in terms of their discrete features ^[5]. However, the value of each site in a cellular automata network is strictly dependent on its neighbor (s) from the previous time step. Each site of the CML is only dependent upon its neighbors relative to the coupling term in the recurrence equation. However, the similarities can be compounded when considering multicomponent dynamical systems.

Introduction

The modeling of a CML generally incorporates a system of equations (coupled or uncoupled), a finite number of variables, a global or local coupling scheme and the corresponding coupling terms. The dimension of the underlying lattice can exist in infinite dimensions, but for this observation we restrict the lattice to two. Mappings of interest in CMLs generally demonstrate a chaotic behavior. Such maps can be found here: List of chaotic maps.

A logistic mapping demonstrates chaotic behavior, easily identifiable in one dimension for parameter $r > 3.57$ (see Logistic map). It is graphed across a small lattice and decoupled with respect to neighboring sites. The recurrence equation is homogeneous, albeit randomly seeded. The parameter r is updated every time step (see Figure 1, Enlarge, Summary):

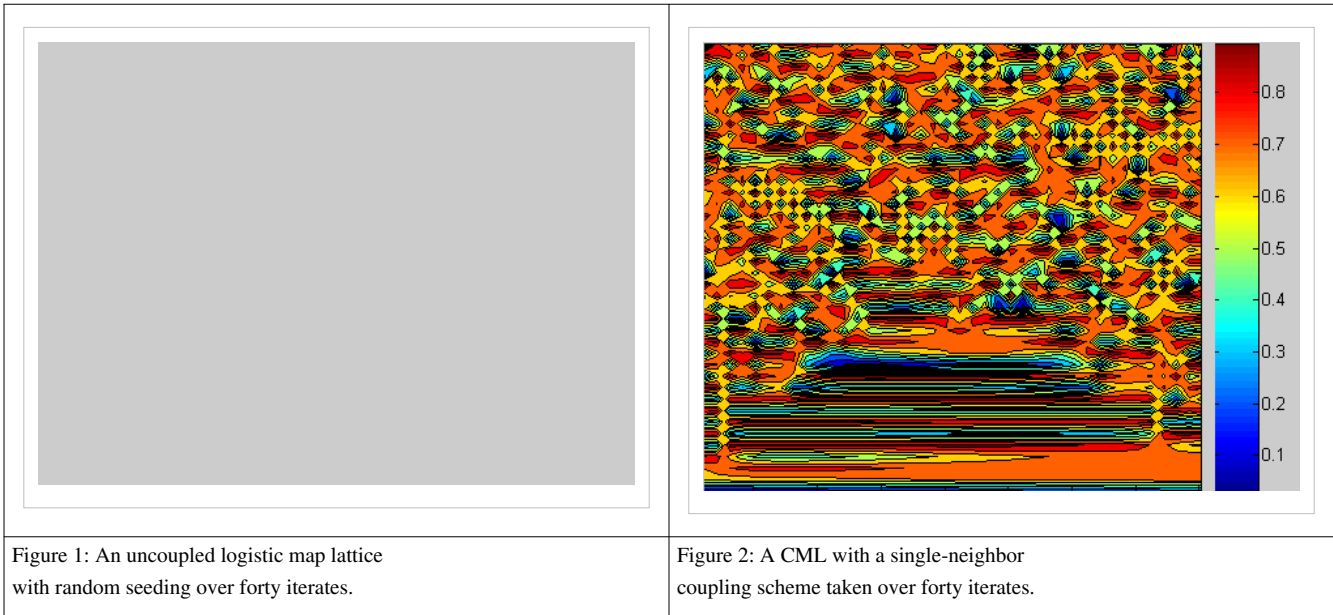
$$x_{n+1} = rx_n(1 - x_n)$$

The result is a raw form of chaotic behavior in a map lattice. The range of the function is bounded so similar contours through the lattice is expected. However, there are no significant spatial correlations or pertinent fronts to the chaotic behavior. No obvious order is apparent.

For a basic coupling, we consider a 'single neighbor' coupling where the value at any given site s is mapped recursively with respect to itself and the neighboring site $s - 1$. The coupling parameter $\epsilon = 0.5$ is equally weighted.

$$x_{n+1} = (\epsilon)[rx_n(1 - x_n)]_s + (1 - \epsilon)[rx_n(1 - x_n)]_{s-1}$$

Even though each native recursion is chaotic, a more solid form develops in the evolution. Elongated convective spaces persist throughout the lattice (see Figure 2).



History

CMLs were first introduced in the mid 1980's through a series of closely released publications [6] [7] [8] [9]. Kapral used CMLs for modeling chemical spatial phenomena. Kuznetsov sought to apply CMLs to electrical circuitry by developing a renormalization group approach (similar to Feigenbaum's universality to spatially extended systems). Kaneko's focus was more broad and he is still known as the most active researcher in this area [10]. The most examined CML model was introduced by Kaneko in 1983 where the recurrence equation is as follows:

$$u_s^{t+1} = (1 - \varepsilon)f(u_s^t) + \frac{\varepsilon}{2} (f(u_{s+1}^t) + f(u_{s-1}^t)) \quad t \in \mathbb{N}, \varepsilon \in [0, 1]$$

where $u_s^t \in \mathbb{R}$, and f is a real mapping.

The applied CML strategy was as follows:

- Choose a set of field variables on the lattice at a macroscopic level. The dimension (not limited by the CML system) should be chosen to correspond to the physical space being researched.
- Decompose the process (underlying the phenomena) into independent components.
- Replace each component by a nonlinear transformation of field variables on each lattice point and the coupling term on suitable, chosen neighbors.
- Carry out each unit dynamics ("procedure") successively.

Classification

The CML system evolves through discrete time by a mapping on vector sequences. These mappings are a recursive function of two competing terms: an individual nonlinear reaction, and a spatial interaction (coupling) of variable intensity. CMLs can be classified by the strength of this coupling parameter(s).

Much of the current published work in CMLs is based in weak coupled systems [11] where diffeomorphism of the state space close to identity are studied. Weak coupling with monotonic (bistable) dynamical regimes demonstrate spatial chaos phenomena and are popular in neural models [12]. Weak coupling unimodal maps are characterized by their stable periodic points and are used by genetic regulatory network models. Space-time chaotic phenomena can be demonstrated from chaotic mappings subject to weak coupling coefficients and are popular in phase transition phenomena models.

Intermediate and strong coupling interactions are less prolific areas of study. Intermediate interactions are studied with respect to fronts and traveling waves, riddled basins, riddled bifurcations, clusters and non-unique phases.

Strong coupling interactions are most well known to model synchronization effects of dynamic spatial systems such as the Kuramoto model.

These classifications do not reflect the local or global (GMLs^[13]) coupling nature of the interaction. Nor do they consider the frequency of the coupling which can exist as a degree of freedom in the system^[14]. Finally, they do not distinguish between sizes of the underlying space or boundary conditions.

Surprisingly the dynamics of CMLs have little to do with the local maps that constitute their elementary components. With each model a rigorous mathematical investigation is needed to identify a chaotic state (beyond visual interpretation). Rigorous proofs have been performed to this effect. By example: the existence of space-time chaos in weak space interactions of one-dimensional maps with strong statistical properties was proven by Bunimovich and Sinai in 1988^[15]. Similar proofs exist for weakly hyperbolic maps under the same conditions.

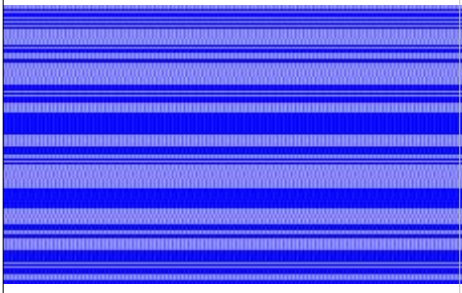
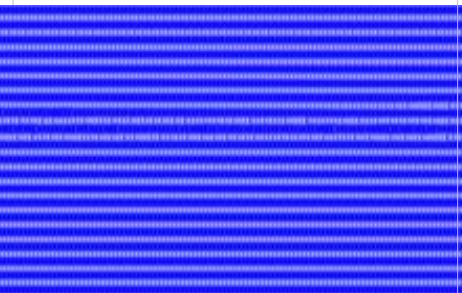
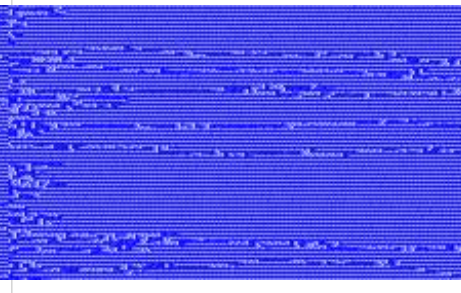
Unique CML qualitative classes

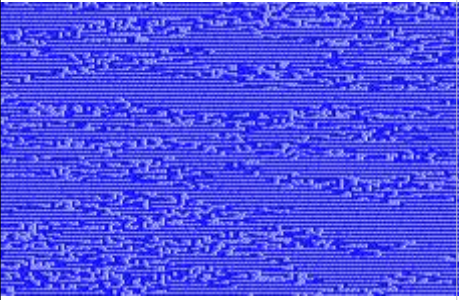
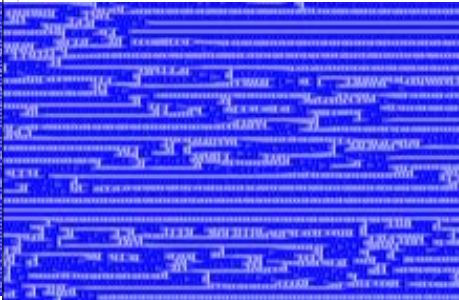
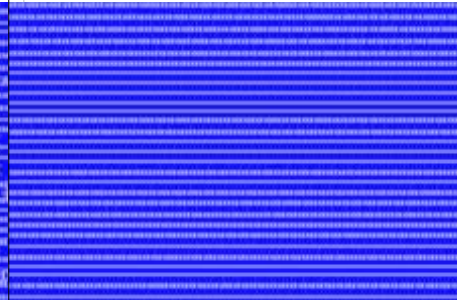
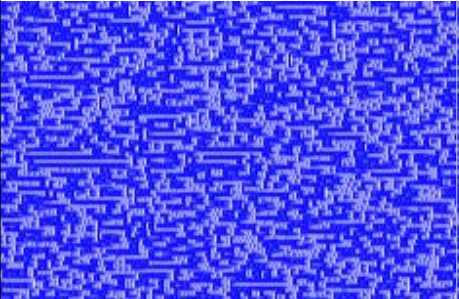

CMLs have revealed novel qualitative universality classes in (CML) phenomenology. Such classes include:

- Spatial bifurcation and frozen chaos
- Pattern Selection
- Selection of zig-zag patterns and chaotic diffusion of defects
- Spatio-temporal intermittency
- Soliton turbulence
- Global traveling waves generated by local phase slips
- Spatial bifurcation to down-flow in open flow systems.

Visual phenomena

The unique qualitative classes listed above can be visualized. By applying the Kaneko 1983 model to the logistic $f(x_n) = 1 - \alpha x^2$ map, several of the CML qualitative classes may be observed. These are demonstrated below, note the unique parameters:

Frozen Chaos	Pattern Selection	Chaotic Brownian Motion of Defect
		
Figure 1: Sites are divided into non-uniform clusters, where the divided patterns are regarded as attractors. Sensitivity to initial conditions exist relative to $\alpha < 1.5$.	Figure 2: Near uniform sized clusters ($\alpha = 1.71, \varepsilon = 0.4$).	Figure 3: Deflects exist in the system and fluctuate chaotically akin to Brownian motion ($\alpha = 1.85, \varepsilon = 0.1$).
Defect Turbulence	Spatiotemporal Intermittency I	Spatiotemporal Intermittency II

		
Figure 4: Many defects are generated and turbulently collide ($a = 1.895$, $\varepsilon = 0.1$).	Figure 5: Each site transits between a coherent state and chaotic state intermittently ($a = 1.75$, $\varepsilon = 0.6$), Phase I.	Figure 6: The coherent state, Phase II.
Fully Developed Spatiotemporal Chaos	Traveling Wave	
		
Figure 7: Most sites independently oscillate chaotically ($a = 2.00$, $\varepsilon = 0.3$).	Figure 8: The wave of clusters travels at 'low' speeds ($a = 1.47$, $\varepsilon = 0.5$).	

Quantitative analysis quantifiers

Coupled map lattices being a prototype of spatially extended systems easy to simulate have represented a benchmark for the definition and introduction of many indicators of spatio-temporal chaos, the most relevant ones are

- The power spectrum in space and time
- Lyapunov spectra^[16]
- Dimension density
- Kolmogorov–Sinai entropy density
- Distributions of patterns
- Pattern entropy
- Propagation speed of finite and infinitesimal disturbance
- Mutual information and correlation in space-time
- Lyapunov exponents, localization of Lyapunov vectors
- Comoving and sub-space time Lyapunov exponents.
- Spatial and temporal Lyapunov exponents^[17]

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- H.G. Schuster and W. Just, *Deterministic Chaos* (<http://www.whsmith.co.uk/CatalogAndSearch/ProductDetails.aspx?productID=9783527404155>), John Wiley and Sons Ltd, 2005, ISBN 3527404155
- Introduction to Chaos and Nonlinear Dynamics (<http://brain.cc.kogakuin.ac.jp/~kanamaru/Chaos/e/>)

External links

- Kaneko Laboratory (<http://chaos.c.u-tokyo.ac.jp/>)
- Institut Henri Poincaré, Paris, June 21 – July 2, 2004 (<http://www.cpht.polytechnique.fr/cpth/cml2004/>)
- Istituto dei Sistemi Complessi (<http://www.fi.isc.cnr.it/>), Florence, Italy

Software

- Java CML/GML web-app (<http://brain.cc.kogakuin.ac.jp/~kanamaru/Chaos/e/CMLGCM/>)
- AnT 4.669 – A simulation and Analysis Tool for Dynamical Systems (<http://ant4669.de/>)

List of chaotic maps

In mathematics, a chaotic map is a map (= evolution function) that exhibits some sort of chaotic behavior. Maps may be parameterized by a discrete-time or a continuous-time parameter. Discrete maps usually take the form of iterated functions. Chaotic maps often occur in the study of dynamical systems.

Chaotic maps often generate fractals. Although a fractal may be constructed by an iterative procedure, some fractals are studied in and of themselves, as sets rather than in terms of the map that generates them. This is often because there are several different iterative procedures to generate the same fractal.

List of chaotic maps

Map	Time domain	Space domain	Number of space dimensions	Also known as
Arnold's cat map	discrete	real	2	
Baker's map	discrete	real	2	
Bogdanov map				
Chossat-Golubitsky symmetry map				
Circle map	discrete	real	1	
Cobweb map				
Complex quadratic map	discrete	complex	1	
Complex squaring map	discrete	complex	1	
Complex Cubic map				
Degenerate Double Rotor map				
Double Rotor map				
Duffing map	discrete	real	2	
Duffing equation	continuous	real	1	
Dyadic transformation	discrete	real	1	2x mod 1 map, Bernoulli map, doubling map, sawtooth map
Exponential map	discrete	complex	2	
Gauss map	discrete	real	1	mouse map, Gaussian map
Generalized Baker map				
Gingerbreadman map	discrete	real	2	
Gumowski/Mira map				
Hénon map	discrete	real	2	
Hénon with 5th order polynomial				
Hitzl-Zele map				
Horseshoe map	discrete	real	2	
Ikeda map	discrete	real	2	
Interval exchange map	discrete	real	1	
Kaplan-Yorke map	discrete	real	2	
Linear map on unit square				

Logistic map	discrete	real	1	
Lorenz attractor	continuous	real	3	
Lorenz system's Poincare Return map				
Lozi map				
Nordmark truncated map				
Pomeau-Manneville maps for intermittent chaos	discrete	real	1 and 2	Normal-form maps for intermittency (Types I, II and III)
Pulsed rotor				
Quasiperiodicity map				
Rabinovich-Fabrikant equations	continuous	real	3	
Random Rotate map				
Rössler map	continuous	real	3	
Shobu-Ose-Mori piecewise-linear map	discrete	real	1	piecewise-linear approximation for Pomeau-Manneville Type I map
Sinai map - See [1]				
Symplectic map				
Standard map, Kicked rotor	discrete	real	2	Chirikov standard map, Chirikov-Taylor map
Tangent map				
Tent map	discrete	real	1	
Tinkerbell map	discrete	real	2	
Triangle map				
Van der Pol oscillator	continuous	real	1	
Zaslavskii map	discrete	real	2	
Zaslavskii rotation map				

List of fractals

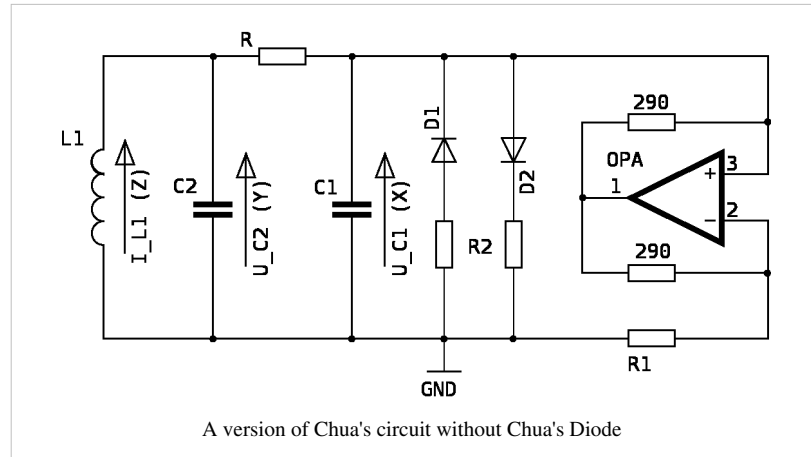
- Cantor set
- de Rham curve
- Gravity set, or Mitchell-Green gravity set
- Julia set - derived from complex quadratic map
- Newton fractal
- Nova fractal - derived from Newton fractal
- Koch snowflake - special case of de Rham curve
- Lyapunov fractal
- Mandelbrot set - derived from complex quadratic map
- Menger sponge
- Sierpinski carpet
- Sierpinski triangle

References

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Chua's circuit

Chua's circuit is a simple electronic circuit that exhibits classic chaos theory behavior. It was introduced in 1983 by Leon O. Chua, who was a visitor at Waseda University in Japan at that time.^[1] The ease of construction of the circuit has made it a ubiquitous real-world example of a chaotic system, leading some to declare it "a paradigm for chaos."^[2]



Chaotic criteria

An autonomous circuit made from standard components (resistors, capacitors, inductors) must satisfy three criteria before it can display chaotic behaviour. It must contain:

1. one or more nonlinear elements
2. one or more locally active resistors
3. **three** or more energy-storage elements.

Chua's circuit is the simplest electronic circuit meeting these criteria. As shown in the figure, the energy storage elements are two capacitors (labeled C1 and C2) and an inductor (labeled L1). There is an active resistor (labeled R). There is a nonlinear resistor made of two linear resistors and two diodes. At the far right is a negative impedance converter made from three linear resistors and an operational amplifier.

Model

By means of the application of the laws of electromagnetism, the dynamics of Chua's circuit can be accurately modeled by means of a system of three nonlinear ordinary differential equations in the variables $x(t)$, $y(t)$ and $z(t)$, which give the voltages across the capacitors C1 and C2, and the intensity of the electrical current in the inductor L1, respectively. These equations read:

$$\begin{aligned}\frac{dx}{dt} &= \alpha[y - x - f(x)] \\ \frac{dy}{dt} &= x - y + z \\ \frac{dz}{dt} &= -\beta y\end{aligned}$$

The function $f(x)$ describes the electrical response of the nonlinear resistor, and its shape depends on the particular configuration of its components. The parameters α and β are determined by the particular values of the circuit components.

A chaotic attractor, known as "The Double Scroll" because of its shape in the (x,y,z) space, was first observed in a circuit containing a nonlinear element such that $f(x)$ was a 3-segment piecewise-linear function.^[3]

The easy experimental implementation of the circuit, combined with the existence of a simple and accurate theoretical model, makes Chua's circuit a useful system to study many fundamental and applied issues of chaos theory. Because of this, it has been object of much study, and appears widely referenced in the literature.

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Books

- *Chaos synchronization in Chua's circuit*, Leon O Chua, Berkeley : Electronics Research Laboratory, College of Engineering, University of California, [1992], OCLC: 44107698

External links

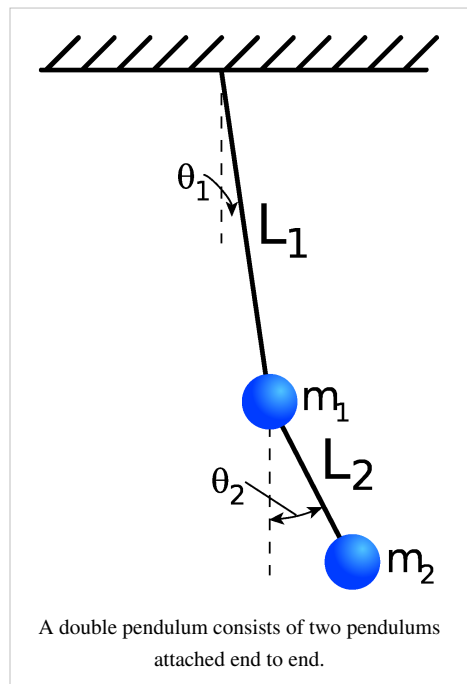
- Chua's Circuit: Diagram and discussion (http://www.cmp.caltech.edu/~mcc/chaos_new/Chua.html)
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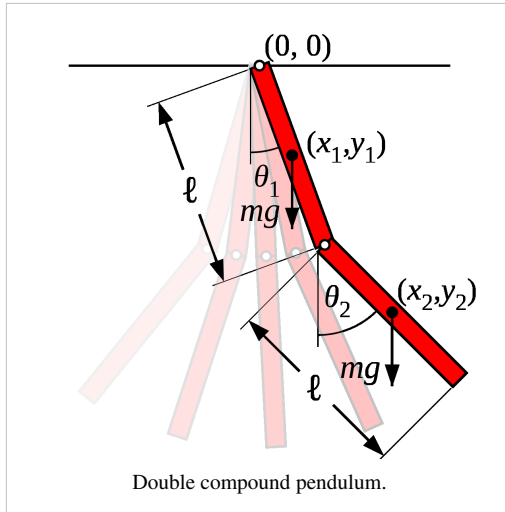
Double pendulum

In mathematics, in the area of dynamical systems, a **double pendulum** is a pendulum with another pendulum attached to its end, and is a simple physical system that exhibits rich dynamic behavior with a strong sensitivity to initial conditions. The motion of a double pendulum is governed by a set of coupled ordinary differential equations. For certain energies its motion is chaotic.

Analysis

Several variants of the double pendulum may be considered; the two limbs may be of equal or unequal lengths and masses, they may be simple pendulums or compound pendulums (also called complex pendulums) and the motion may be in three dimensions or restricted to the vertical plane. In the following analysis, the limbs are taken to be identical compound pendulums of length ℓ and mass m , and the motion is restricted to two dimensions.





In a compound pendulum, the mass is distributed along its length. If the mass is evenly distributed, then the centre of mass of each limb is at its midpoint, and the limb has a moment of inertia of $I = \frac{1}{12}m\ell^2$ about that point.

It is convenient to use the angle between each limb and the vertical as the generalized coordinates defining the configuration of the system. These angles are denoted θ_1 and θ_2 . The position of the centre of mass of each rod may be written in terms of these two coordinates. If the origin of the Cartesian coordinate system is taken to be at the point of suspension of the first pendulum, then the centre of mass of this pendulum is at:

$$x_1 = \frac{\ell}{2} \sin \theta_1,$$

$$y_1 = -\frac{\ell}{2} \cos \theta_1$$

and the centre of mass of the second pendulum is at

$$x_2 = \ell \left(\sin \theta_1 + \frac{1}{2} \sin \theta_2 \right),$$

$$y_2 = -\ell \left(\cos \theta_1 + \frac{1}{2} \cos \theta_2 \right).$$

This is enough information to write out the Lagrangian.

Lagrangian

The Lagrangian is

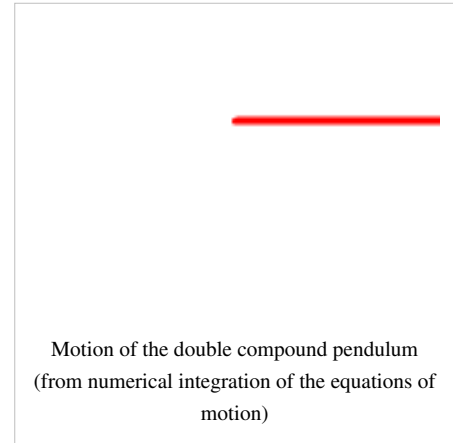
$$\begin{aligned} L &= \text{Kinetic Energy} - \text{Potential Energy} \\ &= \frac{1}{2}m(v_1^2 + v_2^2) + \frac{1}{2}I(\dot{\theta}_1^2 + \dot{\theta}_2^2) - mg(y_1 + y_2) \\ &= \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}I(\dot{\theta}_1^2 + \dot{\theta}_2^2) - mg(y_1 + y_2) \end{aligned}$$

The first term is the *linear* kinetic energy of the center of mass of the bodies and the second term is the *rotational* kinetic energy around the center of mass of each rod. The last term is the potential energy of the bodies in a uniform gravitational field. The dot-notation indicates the time derivative of the variable in question.

Substituting the coordinates above and rearranging the equation gives

$$L = \frac{1}{6}m\ell^2 \left[\dot{\theta}_2^2 + 4\dot{\theta}_1^2 + 3\dot{\theta}_1\dot{\theta}_2 \cos(\theta_1 - \theta_2) \right] + \frac{1}{2}mg\ell (3 \cos \theta_1 + \cos \theta_2).$$

There is only one conserved quantity (the energy), and no conserved momenta. The two momenta may be written as



Long exposure of double pendulum exhibiting chaotic motion (tracked with an LED)

$$p_{\theta_1} = \frac{\partial L}{\partial \dot{\theta}_1} = \frac{1}{6} m \ell^2 [8\dot{\theta}_1 + 3\dot{\theta}_2 \cos(\theta_1 - \theta_2)]$$

and

$$p_{\theta_2} = \frac{\partial L}{\partial \dot{\theta}_2} = \frac{1}{6} m \ell^2 [2\dot{\theta}_2 + 3\dot{\theta}_1 \cos(\theta_1 - \theta_2)] .$$

These expressions may be inverted to get

$$\dot{\theta}_1 = \frac{6}{m \ell^2} \frac{2p_{\theta_1} - 3 \cos(\theta_1 - \theta_2) p_{\theta_2}}{16 - 9 \cos^2(\theta_1 - \theta_2)}$$

and

$$\dot{\theta}_2 = \frac{6}{m \ell^2} \frac{8p_{\theta_2} - 3 \cos(\theta_1 - \theta_2) p_{\theta_1}}{16 - 9 \cos^2(\theta_1 - \theta_2)} .$$

The remaining equations of motion are written as

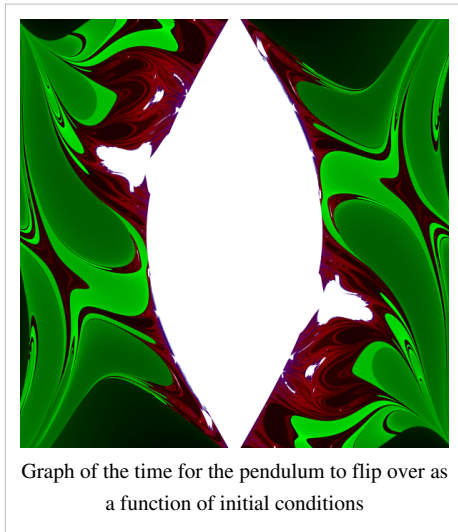
$$\dot{p}_{\theta_1} = \frac{\partial L}{\partial \theta_1} = -\frac{1}{2} m \ell^2 \left[\dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + 3 \frac{g}{\ell} \sin \theta_1 \right]$$

and

$$\dot{p}_{\theta_2} = \frac{\partial L}{\partial \theta_2} = -\frac{1}{2} m \ell^2 \left[-\dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + \frac{g}{\ell} \sin \theta_2 \right] .$$

These last four equations are explicit formulae for the time evolution of the system given its current state. It is not possible to go further and integrate these equations analytically, to get formulae for θ_1 and θ_2 as functions of time. It is however possible to perform this integration numerically using the Runge Kutta method or similar techniques.

Chaotic motion



The double pendulum undergoes chaotic motion, and shows a sensitive dependence on initial conditions. The image to the right shows the amount of elapsed time before the pendulum "flips over," as a function of initial conditions. Here, the initial value of θ_1 ranges along the x -direction, from -3 to 3 . The initial value θ_2 ranges along the y -direction, from -3 to 3 . The colour of each pixel indicates whether either pendulum flips within $10\sqrt{\ell/g}$ (green), within $100\sqrt{\ell/g}$ (red), $1000\sqrt{\ell/g}$ (purple) or $10000\sqrt{\ell/g}$ (blue). Initial conditions that don't lead to a flip within $10000\sqrt{\ell/g}$ are plotted white.

The boundary of the central white region is defined in part by energy conservation with the following curve:

$$3 \cos \theta_1 + \cos \theta_2 = 2.$$

Within the region defined by this curve, that is if

$$3 \cos \theta_1 + \cos \theta_2 > 2,$$

then it is energetically impossible for either pendulum to flip. Outside this region, the pendulum can flip, but it is a complex question to determine when it will flip.

The lack of a natural excitation frequency has led to the use of double pendulum systems in seismic resistance designs in buildings, where the building itself is the primary inverted pendulum, and a secondary mass is connected to complete the double pendulum.

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External links

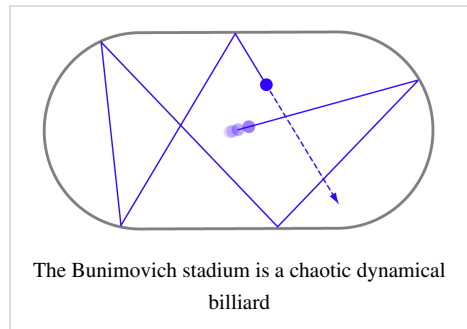
- Animations and explanations of a double pendulum ^[6] and a physical double pendulum (two square plates) ^[7] by Mike Wheatland (Univ. Sydney)
- Video ^[8] of a double square pendulum with three (almost) identical starting conditions.
- Double pendulum physics simulation from www.myphysicslab.com ^[9]
- Simulation, equations and explanation of Rott's pendulum ^[10]
- Comparison videos of a double pendulum with the same initial starting conditions on YouTube ^[11]
- Double Pendulum Simulator ^[12] - An open source simulator written in C++ using the Qt toolkit.
- Online Java simulator ^[13] of the Imaginary exhibition.

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- [6] http://www.physics.usyd.edu.au/~wheat/dpend_html/
- [7] <http://www.physics.usyd.edu.au/~wheat/sdepend/>
- [8] <http://www.youtube.com/watch?v=Uzlccwt5SKc&NR=1>
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- [10] <http://www.chris-j.co.uk/rott.php>
- [11] <http://www.youtube.com/watch?v=O2ySvbL3-yA>
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Dynamical billiards

A **billiard** is a dynamical system in which a particle alternates between motion in a straight line and specular reflections from a boundary. When the particle hits the boundary it reflects from it without loss of speed. Billiard dynamical systems are Hamiltonian idealizations of the game of billiards, but where the region contained by the boundary can have shapes other than rectangular and even be multidimensional. Dynamical billiards may also be studied on non-Euclidean geometries; indeed, the very first studies of billiards established their ergodic motion on surfaces of constant negative curvature. The study of billiards which are kept out of a region, rather than being kept in a region, is known as outer billiard theory.



The motion of the particle in the billiard is a straight line, with constant energy, between reflections with the boundary (a geodesic if the Riemannian metric of the billiard table is not flat). All reflections are specular: the angle of incidence just before the collision is equal to the angle of reflection just after the collision. The sequence of reflections is described by the **billiard map** that completely characterizes the motion of the particle.

Billiards capture all the complexity of Hamiltonian systems, from integrability to chaotic motion, without the difficulties of integrating the equations of motion to determine its Poincaré map. Birkhoff showed that a billiard system with an elliptic table is integrable.

Equations of motion

The Hamiltonian for a particle of mass m moving freely without friction on a surface is:

$$H(p, q) = \frac{p^2}{2m} + V(q),$$

where $V(q)$ is a potential designed to be zero inside the region Ω in which the particle can move, and infinity otherwise:

$$V(q) = \begin{cases} 0 & q \in \Omega, \\ \infty & q \notin \Omega. \end{cases}$$

This form of the potential guarantees a specular reflection on the boundary. The kinetic term guarantees that the particle moves in a straight line, without any change in energy. If the particle is to move on a non-Euclidean manifold, then the Hamiltonian is replaced by:

$$H(p, q) = \frac{p^i p^j g_{ij}(q)}{2m} + V(q),$$

where $g_{ij}(q)$ is the metric tensor at point $q \in \Omega$. Because of the very simple structure of this Hamiltonian, the equations of motion for the particle, the Hamilton–Jacobi equations, are nothing other than the geodesic equations on the manifold: the particle moves along geodesics.

Notable billiard tables

Hadamard's billiards

Hadamard's billiards concern the motion of a free point particle on a surface of constant negative curvature, in particular, the simplest compact Riemann surface with negative curvature, a surface of genus 2 (a two-holed donut). The model is exactly solvable, and is given by the geodesic flow on the surface. It is the earliest example of deterministic chaos ever studied, having been introduced by Jacques Hadamard in 1898.

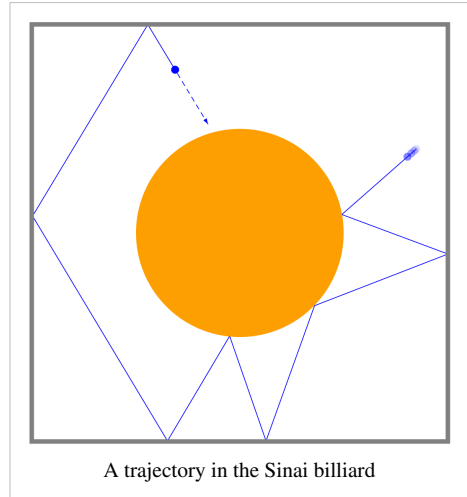
Artin's billiard

Artin's billiard considers the free motion of a point particle on a surface of constant negative curvature, in particular, the simplest non-compact Riemann surface, a surface with one cusp. It is notable for being exactly solvable, and yet not only ergodic but also strongly mixing. It is an example of an Anosov system. This system was first studied by Emil Artin in 1924.

Sinai billiard

The table of the **Sinai billiard** is a square with a disk removed from its center; the table is flat, having no curvature. The billiard arises from studying the behavior of two interacting disks bouncing inside a square, reflecting off the boundaries of the square and off each other. By eliminating the center of mass as a configuration variable, the dynamics of two interacting disks reduces to the dynamics in the Sinai billiard.

The billiard was introduced by Yakov G. Sinai as an example of an interacting Hamiltonian system that displays physical thermodynamic properties: all of its possible trajectories are ergodic and it has a positive Lyapunov exponent. As a model of a classical gas, the Sinai billiard is sometimes called the **Lorentz gas**.



A trajectory in the Sinai billiard

Sinai's great achievement with this model was to show that the classical Boltzmann–Gibbs ensemble for an ideal gas is essentially the maximally chaotic Hadamard billiards.

Bunimovich stadium

The table called the **Bunimovich stadium** is a rectangle capped by semicircles. Until it was introduced by Leonid Bunimovich, billiards with positive Lyapunov exponents were thought to need convex scatters, such as the disk in the Sinai billiard, to produce the exponential divergence of orbits. Bunimovich showed that by considering the orbits beyond the focusing point of a concave region it was possible to obtain exponential divergence.

Generalized billiards

Generalized billiards (GB) describe a motion of a mass point (a particle) inside a closed domain $\Pi \in \mathbb{R}^n$ with the piece-wise smooth boundary Γ . On the boundary Γ the velocity of point is transformed as the particle underwent the action of generalized billiard law. GB were introduced by Lev D. Pustyl'nikov in the general case in ^[1] (in Notes), and, in the case when Π is a parallelepiped in ^[2] (in Notes) in connection with the justification of the second law of thermodynamics (the law of entropy increase). From the physical point of view, GB describe a gas consisting of finitely many particles moving in a vessel, while the walls of the vessel heat up or cool down. The essence of the generalization is the following. As the particle hits the boundary Γ , its velocity transforms with the help of a given function $f(\gamma, t)$, defined on the direct product $\Gamma \times \mathbb{R}^1$ (where \mathbb{R}^1 is the real line, $\gamma \in \Gamma$ is a point of the boundary and $t \in \mathbb{R}^1$ is time), according to the following law. Suppose that the trajectory of the particle, which moves with the velocity v , intersects Γ at the point $\gamma \in \Gamma$ at time t^* . Then at time t^* the particle acquires the velocity v^* , as if it underwent an elastic push from the infinitely-heavy plane Γ^* , which is tangent to Γ at the point γ , and at time t^* moves along the normal to Γ at γ with the velocity $\frac{\partial f}{\partial t}(\gamma, t^*)$. We emphasize that the

position of the boundary itself is fixed, while its action upon the particle is defined through the function f . We take the positive direction of motion of the plane Γ^* to be towards the *interior* of Π . Thus if the derivative $\frac{\partial f}{\partial t}(\gamma, t) > 0$, then the particle accelerates after the impact.

If the velocity v^* , acquired by the particle as the result of the above reflection law, is directed to the interior of the domain Π , then the particle will leave the boundary and continue moving in Π until the next collision with Γ . If the velocity v^* is directed towards the outside of Π , then the particle remains on Γ at the point γ until at some time $\tilde{t} > t^*$ the interaction with the boundary will force the particle to leave it.

If the function $f(\gamma, t)$ does not depend on time t , i.e., $\partial f / \partial t = 0$, the generalized billiard coincides with the classical one.

This generalized reflection law is very natural. First, it reflects an obvious fact that the walls of the vessel with gas are motionless. Second the action of the wall on the particle is still the classical elastic push. In the essence, we consider infinitesimally moving boundaries with given velocities.

It is considered the reflection from the boundary Γ both in the framework of classical mechanics (Newtonian case) and the theory of relativity (relativistic case).

Main results: in the Newtonian case the energy of particle is bounded, the Gibbs entropy is a constant,^{[2] [3] [4]} (in Notes) and in relativistic case the energy of particle, the Gibbs entropy, the entropy with respect to the phase volume grow to infinity,^{[2] [4]} (in Notes), references to generalized billiards.

Quantum chaos

The quantum version of the billiards is readily studied in several ways. The classical Hamiltonian for the billiards, given above, is replaced by the stationary-state Schrödinger equation $H\psi = E\psi$ or, more precisely,

$$-\frac{\hbar^2}{2m}\Delta\psi_n(q) = E_n\psi_n(q),$$

where Δ is the Laplacian. The potential that is infinite outside the region Ω but zero inside it translates to the Dirichlet boundary conditions:

$$\psi_n(q) = 0 \quad \text{for } q \notin \Omega.$$

As usual, the wavefunctions are taken to be orthonormal:

$$\int_{\Omega} \overline{\psi_m(q)} \psi_n(q) dq = \delta_{mn}.$$

Curiously, the free-field Schrödinger equation is the same as the Helmholtz equation,

$$(\Delta + k^2)\psi = 0,$$

with

$$k^2 = \frac{2mE_n}{\hbar^2}.$$

This implies that two and three-dimensional quantum billiards can be modelled by the classical resonance modes of a radar cavity of a given shape, thus opening a door to experimental verification. (The study of radar cavity modes must be limited to the transverse magnetic (TM) modes, as these are the ones obeying the Dirichlet boundary conditions).

The semi-classical limit corresponds to $\hbar \rightarrow 0$ which can be seen to be equivalent to $m \rightarrow \infty$, the mass increasing so that it behaves classically.

As a general statement, one may say that whenever the classical equations of motion are integrable (e.g. rectangular or circular billiard tables), then the quantum-mechanical version of the billiards is completely solvable. When the classical system is chaotic, then the quantum system is generally not exactly solvable, and presents numerous difficulties in its quantization and evaluation. The general study of chaotic quantum systems is known as quantum chaos.

A particularly striking example of scarring on an elliptical table is given by the observation of the so-called quantum mirage.

Applications

The most practical application of theory of quantum billiards is related with double-clad fibers. In such a fiber laser, the small core with low Numerical Aperture confines the signal, and the wide cladding confines the multi-mode pump. In the paraxial approximation, the complex field of pump in the cladding behaves like a wave function in the quantum billiard. The modes of the cladding with scarring may avoid the core, and symmetrical configurations enhance this effect. The chaotic fibers^[5] provide good coupling; in the first approximation, such a fiber can be described with the same equations as an idealized billiard. The coupling is especially poor in fibers with circular symmetry while the spiral-shaped fiber—with the core close to the chunk of the spiral—shows good coupling properties. The small spiral deformation forces all the scars to be coupled with the core.^[6]

Notes

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-

Bifurcation theory

Bifurcation theory is the mathematical study of changes in the qualitative or topological structure of a given family, such as the integral curves of a family of vector fields, and the solutions of a family of differential equations. Most commonly applied to the mathematical study of dynamical systems, a **bifurcation** occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its behaviour.^[1] Bifurcations occur in both continuous systems (described by ODEs, DDEs or PDEs), and discrete systems (described by maps).

Bifurcation types

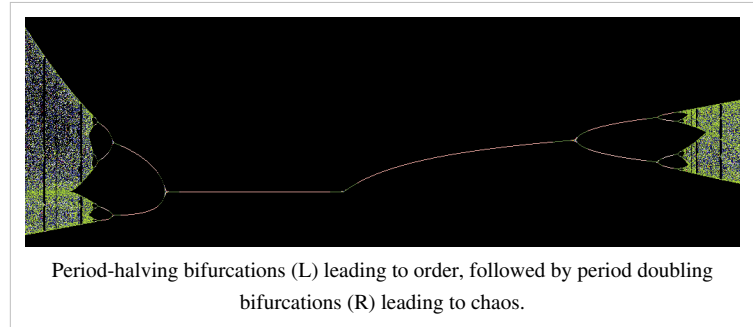
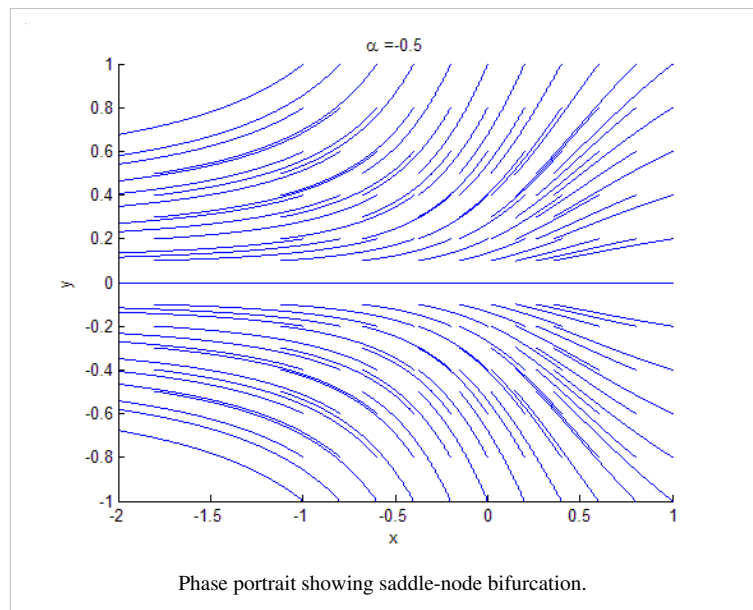
It is useful to divide bifurcations into two principal classes:

- Local bifurcations, which can be analysed entirely through changes in the local stability properties of equilibria, periodic orbits or other invariant sets as parameters cross through critical thresholds; and
- Global bifurcations, which often occur when larger invariant sets of the system 'collide' with each other, or with equilibria of the system. They cannot be detected purely by a stability analysis of the equilibria (fixed points).

Local bifurcations

A local bifurcation occurs when a parameter change causes the stability of an equilibrium (or fixed point) to change. In continuous systems, this corresponds to the real part of an eigenvalue of an equilibrium passing through zero. In discrete systems (those described by maps rather than ODEs), this corresponds to a fixed point having a Floquet multiplier with modulus equal to one. In both cases, the equilibrium is *non-hyperbolic* at the bifurcation point. The topological changes in the phase portrait of the system can be confined to arbitrarily small neighbourhoods of the bifurcating fixed points by moving the bifurcation parameter close to the bifurcation point (hence 'local').

More technically, consider the continuous dynamical system described by the ODE



$$\dot{x} = f(x, \lambda) \quad f : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n.$$

A local bifurcation occurs at (x_0, λ_0) if the Jacobian matrix df_{x_0, λ_0} has an eigenvalue with zero real part. If the eigenvalue is equal to zero, the bifurcation is a steady state bifurcation, but if the eigenvalue is non-zero but purely imaginary, this is a Hopf bifurcation.

For discrete dynamical systems, consider the system

$$x_{n+1} = f(x_n, \lambda).$$

Then a local bifurcation occurs at (x_0, λ_0) if the matrix df_{x_0, λ_0} has an eigenvalue with modulus equal to one. If the eigenvalue is equal to one, the bifurcation is either a saddle-node (often called fold bifurcation in maps), transcritical or pitchfork bifurcation. If the eigenvalue is equal to -1 , it is a period-doubling (or flip) bifurcation, and otherwise, it is a Hopf bifurcation.

Examples of local bifurcations include:

- Saddle-node (fold) bifurcation
- Transcritical bifurcation
- Pitchfork bifurcation
- Period-doubling (flip) bifurcation
- Hopf bifurcation
- Neimark (secondary Hopf) bifurcation

Global bifurcations

Global bifurcations occur when 'larger' invariant sets, such as periodic orbits, collide with equilibria. This causes changes in the topology of the trajectories in the phase space which cannot be confined to a small neighbourhood, as is the case with local bifurcations. In fact, the changes in topology extend out to an arbitrarily large distance (hence 'global').

Examples of global bifurcations include:

- Homoclinic bifurcation in which a limit cycle collides with a saddle point.
- Heteroclinic bifurcation in which a limit cycle collides with two or more saddle points.
- Infinite-period bifurcation in which a stable node and saddle point simultaneously occur on a limit cycle.
- Blue sky catastrophe in which a limit cycle collides with a nonhyperbolic cycle.

Global bifurcations can also involve more complicated sets such as chaotic attractors.

Codimension of a bifurcation

The codimension of a bifurcation is the number of parameters which must be varied for the bifurcation to occur. This corresponds to the codimension of the parameter set for which the bifurcation occurs within the full space of parameters. Saddle-node bifurcations and Hopf bifurcations are the only generic local bifurcations which are really codimension-one (the others all having higher codimension). However, often transcritical and pitchfork bifurcations are also often thought of as codimension-one, because the normal forms can be written with only one parameter.

An example of a well-studied codimension-two bifurcation is the Bogdanov–Takens bifurcation.

Applications in semiclassical and quantum physics

Bifurcation theory has been applied to connect quantum systems to the dynamics of their classical analogues in atomic systems,^{[2] [3] [4]} molecular systems,^[5] and resonant tunneling diodes.^[6] Bifurcation theory has also been applied to the study of laser dynamics^[7] and a number of theoretical examples which are difficult to access experimentally such as the kicked top^[8] and coupled quantum wells.^[9] The dominant reason for the link between quantum systems and bifurcations in the classical equations of motion is that at bifurcations, the signature of classical orbits becomes large, as Martin Gutzwiller points out in his classic^[10] work on quantum chaos.^[11] Many

kinds of bifurcations have been studied with regard to links between classical and quantum dynamics including saddle node bifurcations, Hopf bifurcations, umbilic bifurcations, period doubling bifurcations, reconnection bifurcations, tangent bifurcations, and cusp bifurcations.

Notes

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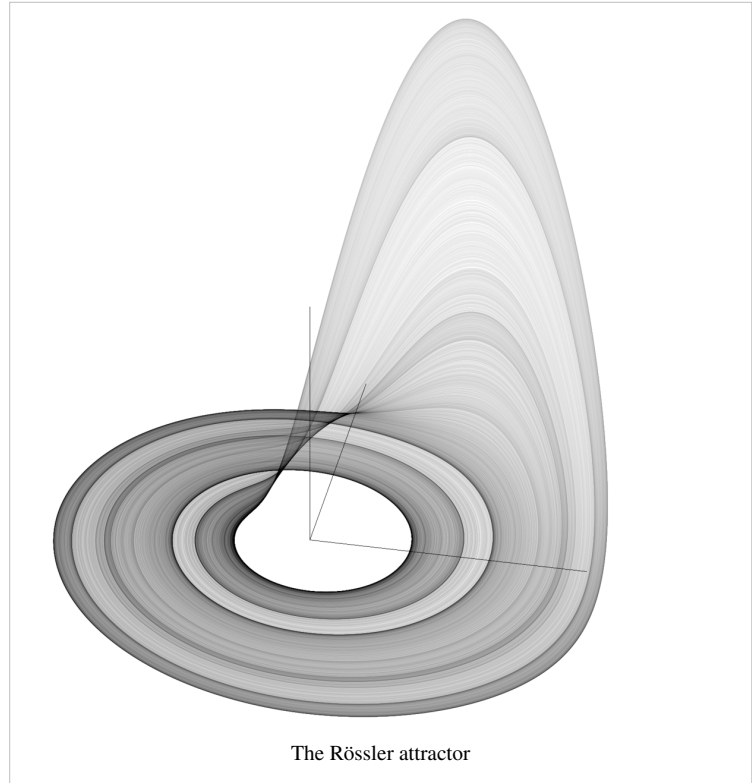
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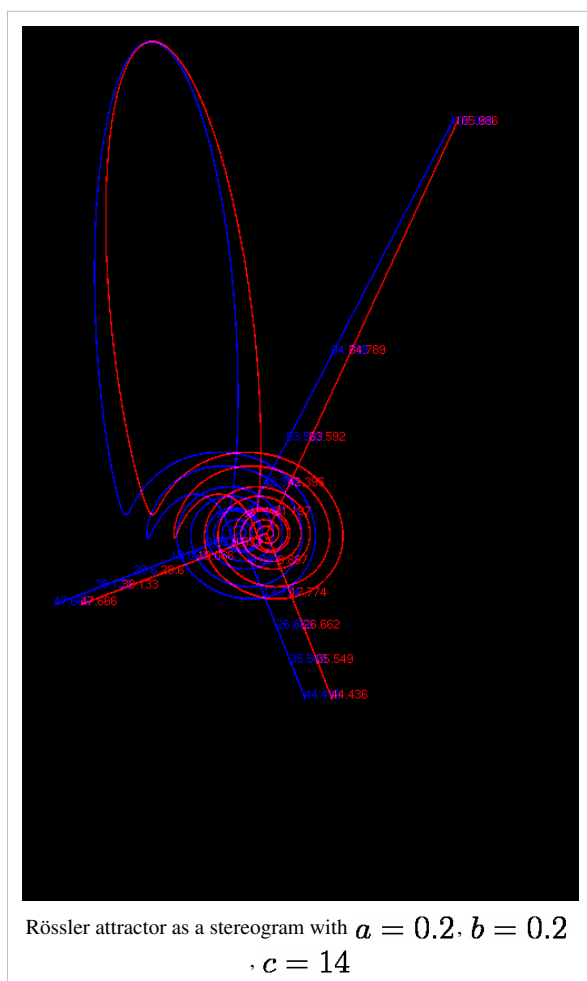
Rössler attractor

The **Rössler attractor** (pronounced /ˈrɒslər/) is the attractor for the **Rössler system**, a system of three non-linear ordinary differential equations. These differential equations define a continuous-time dynamical system that exhibits chaotic dynamics associated with the fractal properties of the attractor. Some properties of the Rössler system can be deduced via linear methods such as eigenvectors, but the main features of the system require non-linear methods such as Poincaré maps and bifurcation diagrams. The original Rössler paper says the Rössler attractor was intended to behave similarly to the Lorenz attractor, but also be easier to analyze qualitatively. An orbit within the attractor follows an outward spiral close to the x, y plane around an unstable fixed point. Once the graph spirals out enough, a second fixed point influences the graph,

causing a rise and twist in the z -dimension. In the time domain, it becomes apparent that although each variable is oscillating within a fixed range of values, the oscillations are chaotic. This attractor has some similarities to the Lorenz attractor, but is simpler and has only one manifold. Otto Rössler designed the Rössler attractor in 1976, but the originally theoretical equations were later found to be useful in modeling equilibrium in chemical reactions. The defining equations are:



The Rössler attractor



$$\frac{dx}{dt} = -y - z$$

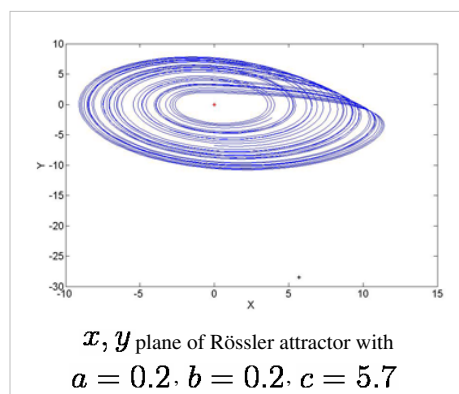
$$\frac{dy}{dt} = x + ay$$

$$\frac{dz}{dt} = b + z(x - c)$$

Rössler studied the chaotic attractor with $a = 0.2$, $b = 0.2$, and $c = 5.7$, though properties of $a = 0.1$, $b = 0.1$, and $c = 14$ have been more commonly used since.

An analysis

Some of the Rössler attractor's elegance is due to two of its equations being linear; setting $z = 0$, allows examination of the behavior on the x, y plane



$$\begin{aligned}\frac{dx}{dt} &= -y \\ \frac{dy}{dt} &= x + ay\end{aligned}$$

The stability in the x, y plane can then be found by calculating the eigenvalues of the Jacobian $\begin{pmatrix} 0 & -1 \\ 1 & a \end{pmatrix}$, which are $(a \pm \sqrt{a^2 - 4})/2$. From this, we can see that when $0 < a < 2$, the eigenvalues are complex and both have a positive real component, making the origin unstable with an outwards spiral on the x, y plane. Now consider the z plane behavior within the context of this range for a . So long as x is smaller than c , the c term will keep the orbit close to the x, y plane. As the orbit approaches x greater than c , the z -values begin to climb. As z climbs, though, the $-z$ in the equation for dx/dt stops the growth in x .

Fixed points

In order to find the fixed points, the three Rössler equations are set to zero and the (x, y, z) coordinates of each fixed point were determined by solving the resulting equations. This yields the general equations of each of the fixed point coordinates:

$$\begin{aligned}x &= \frac{c \pm \sqrt{c^2 - 4ab}}{2} \\ y &= -\left(\frac{c \pm \sqrt{c^2 - 4ab}}{2a}\right) \\ z &= \frac{c \pm \sqrt{c^2 - 4ab}}{2a}\end{aligned}$$

Which in turn can be used to show the actual fixed points for a given set of parameter values:

$$\begin{aligned}&\left(\frac{c + \sqrt{c^2 - 4ab}}{2}, \frac{-c - \sqrt{c^2 - 4ab}}{2a}, \frac{c + \sqrt{c^2 - 4ab}}{2a}\right) \\ &\left(\frac{c - \sqrt{c^2 - 4ab}}{2}, \frac{-c + \sqrt{c^2 - 4ab}}{2a}, \frac{c - \sqrt{c^2 - 4ab}}{2a}\right)\end{aligned}$$

As shown in the general plots of the Rössler Attractor above, one of these fixed points resides in the center of the attractor loop and the other lies comparatively removed from the attractor.

Eigenvalues and eigenvectors

The stability of each of these fixed points can be analyzed by determining their respective eigenvalues and eigenvectors. Beginning with the Jacobian:

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x - c \end{pmatrix}$$

the eigenvalues can be determined by solving the following cubic:

$$-\lambda^3 + \lambda^2(a + x - c) + \lambda(ac - ax - 1 - z) + x - c + az = 0$$

For the centrally located fixed point, Rössler's original parameter values of $a=0.2$, $b=0.2$, and $c=5.7$ yield eigenvalues of:

$$\lambda_1 = 0.0971028 + 0.995786i$$

$$\lambda_2 = 0.0971028 - 0.995786i$$

$$\lambda_3 = -5.68718$$

(Using Mathematica 7)

The magnitude of a negative eigenvalue characterizes the level of attraction along the corresponding eigenvector. Similarly the magnitude of a positive eigenvalue characterizes the level of repulsion along the corresponding eigenvector.

The eigenvectors corresponding to these eigenvalues are:

$$v_1 = \begin{pmatrix} 0.7073 \\ -0.07278 - 0.7032i \\ 0.0042 - 0.0007i \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 0.7073 \\ 0.07278 + 0.7032i \\ 0.0042 + 0.0007i \end{pmatrix}$$

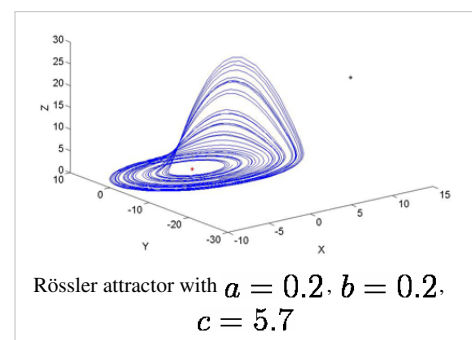
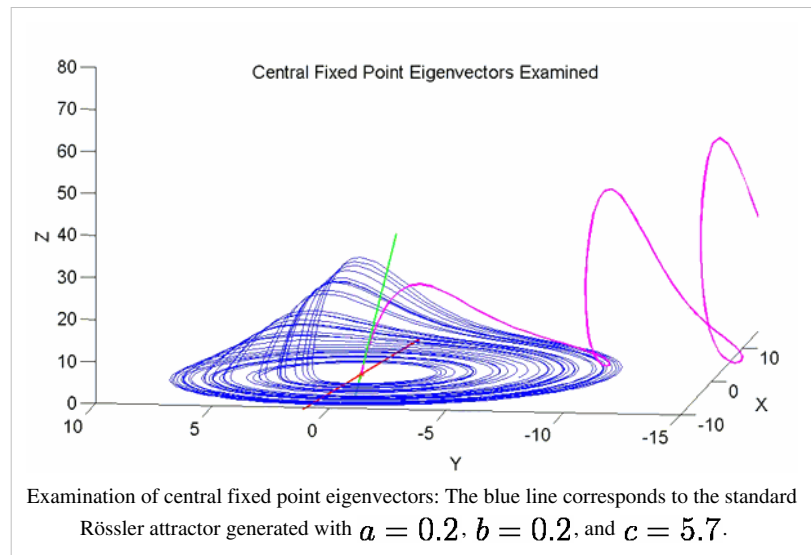
$$v_3 = \begin{pmatrix} 0.1682 \\ -0.0286 \\ 0.9853 \end{pmatrix}$$

These eigenvectors have several interesting implications. First, the two eigenvalue/eigenvector pairs (v_1 and v_2) are responsible for the steady outward slide that occurs in the main disk of the attractor. The last eigenvalue/eigenvector pair is attracting along an axis that runs through the center of the manifold and accounts for the z motion that occurs within the attractor. This effect is roughly demonstrated with the figure below.

The figure examines the central fixed point eigenvectors. The blue line corresponds to the standard Rössler attractor generated with $a = 0.2$, $b = 0.2$, and $c = 5.7$. The red dot in the center of this attractor is FP_1 .

The red line intersecting that fixed point is an illustration of the repulsing plane generated by v_1 and v_2 . The green line is an illustration of the attracting v_3 . The magenta line is generated by stepping backwards through time from a point

on the attracting eigenvector which is slightly above FP_1 – it illustrates the behavior of points that become completely dominated by that vector. Note that the magenta line nearly touches the plane of the attractor before being pulled upwards into the fixed point; this suggests that the general appearance and behavior of the Rössler attractor is largely a product of the interaction between the attracting v_3 and the repelling v_1 and v_2 plane. Specifically it implies that a sequence generated from the Rössler equations will begin to loop around FP_1 , start being pulled upwards into the v_3 vector, creating the upward arm of a curve



that bends slightly inward toward the vector before being pushed outward again as it is pulled back towards the repelling plane.

For the outlier fixed point, Rössler's original parameter values of $a = 0.2$, $b = 0.2$, and $c = 5.7$ yield eigenvalues of:

$$\lambda_1 = -0.0000046 + 5.4280259i$$

$$\lambda_2 = -0.0000046 - 5.4280259i$$

$$\lambda_3 = 0.1929830$$

The eigenvectors corresponding to these eigenvalues are:

$$v_1 = \begin{pmatrix} 0.0002422 + 0.1872055i \\ 0.0344403 - 0.0013136i \\ 0.9817159 \end{pmatrix}$$

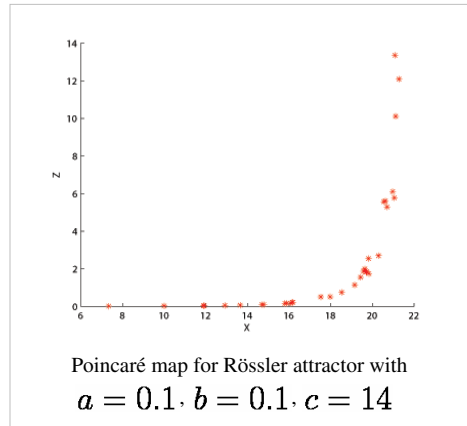
$$v_2 = \begin{pmatrix} 0.0002422 - 0.1872055i \\ 0.0344403 + 0.0013136i \\ 0.9817159 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 0.0049651 \\ -0.7075770 \\ 0.7066188 \end{pmatrix}$$

Although these eigenvalues and eigenvectors exist in the Rössler attractor, their influence is confined to iterations of the Rössler system whose initial conditions are in the general vicinity of this outlier fixed point. Except in those cases where the initial conditions lie on the attracting plane generated by λ_1 and λ_2 , this influence effectively involves pushing the resulting system towards the general Rössler attractor. As the resulting sequence approaches the central fixed point and the attractor itself, the influence of this distant fixed point (and its eigenvectors) will wane.

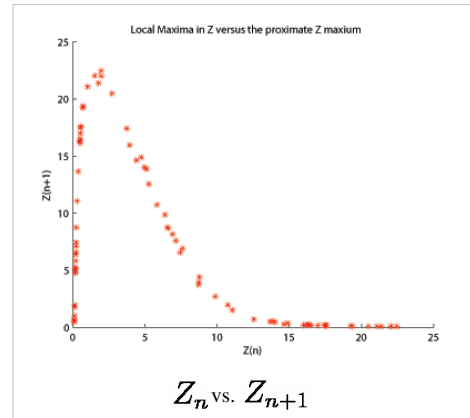
Poincaré map

The Poincaré map is constructed by plotting the value of the function every time it passes through a set plane in a specific direction. An example would be plotting the y, z value every time it passes through the $x = 0$ plane where x is changing from negative to positive, commonly done when studying the Lorenz attractor. In the case of the Rössler attractor, the $x = 0$ plane is uninteresting, as the map always crosses the $x = 0$ plane at $z = 0$ due to the nature of the Rössler equations. In the $x = 0.1$ plane for $a = 0.1$, $b = 0.1$, $c = 14$, the Poincaré map shows the upswing in z values as x increases, as is to be expected due to the upswing and twist section of the Rössler plot. The number of points in this specific Poincaré plot is infinite, but when a different c value is used, the number of points can vary. For example, with a c value of 4, there is only one point on the Poincaré map, because the function yields a periodic orbit of period one, or if the c value is set to 12.8, there would be six points corresponding to a period six orbit.



Mapping local maxima

In the original paper on the Lorenz Attractor, Edward Lorenz analyzed the local maxima of z against the immediately preceding local maxima. When visualized, the plot resembled the tent map, implying that similar analysis can be used between the map and attractor. For the Rössler attractor, when the z_n local maximum is plotted against the next local z maximum, z_{n+1} , the resulting plot (shown here for $a = 0.2$, $b = 0.2$, $c = 5.7$) is unimodal, resembling a skewed Henon map. Knowing that the Rössler attractor can be used to create a pseudo 1-d map, it then follows to use similar analysis methods. The bifurcation diagram is specifically a useful analysis method.



Variation of parameters

Rössler attractor's behavior is largely a factor of the values of its constant parameters a , b , and c . In general, varying each parameter has a comparable effect by causing the system to converge toward a periodic orbit, fixed point, or escape towards infinity, however the specific ranges and behaviors induced vary substantially for each parameter. Periodic orbits, or "unit cycles," of the Rössler system are defined by the number of loops around the central point that occur before the loops series begins to repeat itself.

Bifurcation diagrams are a common tool for analyzing the behavior of dynamical systems, of which the Rössler attractor is one. They are created by running the equations of the system, holding all but one of the variables constant and varying the last one. Then, a graph is plotted of the points that a particular value for the changed variable visits after transient factors have been neutralised. Chaotic regions are indicated by filled-in regions of the plot.

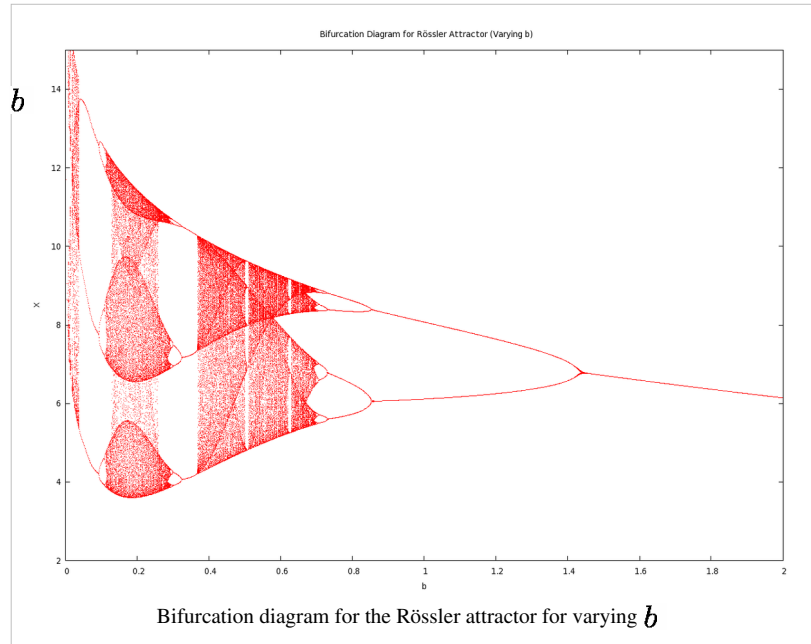
Varying a

Here, b is fixed at 0.2, c is fixed at 5.7 and a changes. Numerical examination of the attractor's behavior over changing a suggests it has a disproportional influence over the attractor's behavior. The results of the analysis are:

- $a \leq 0$: Converges to the centrally located fixed point
- $a = 0.1$: Unit cycle of period 1
- $a = 0.2$: Standard parameter value selected by Rössler, chaotic
- $a = 0.3$: Chaotic attractor, significantly more Möbius strip-like (folding over itself).
- $a = 0.35$: Similar to .3, but increasingly chaotic
- $a = 0.38$: Similar to .35, but increasingly chaotic.

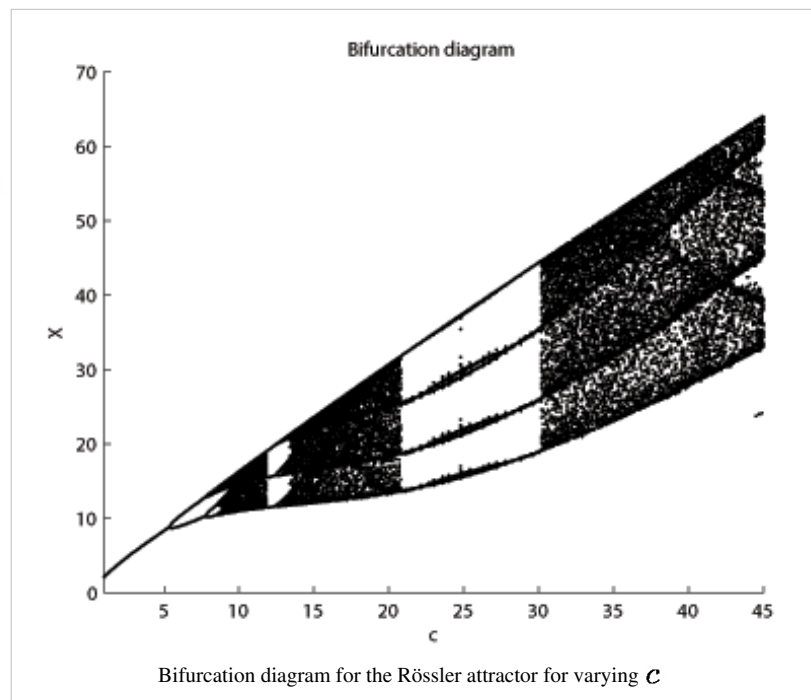
Varying b

Here, a is fixed at 0.2, c is fixed at 5.7 and b changes. As shown in the accompanying diagram, as b approaches 0 the attractor approaches infinity (note the upswing for very small values of b). Comparative to the other parameters, varying b generates a greater range when period-3 and period-6 orbits will occur. In contrast to a and c , higher values of b converge to period-1, not to a chaotic state.

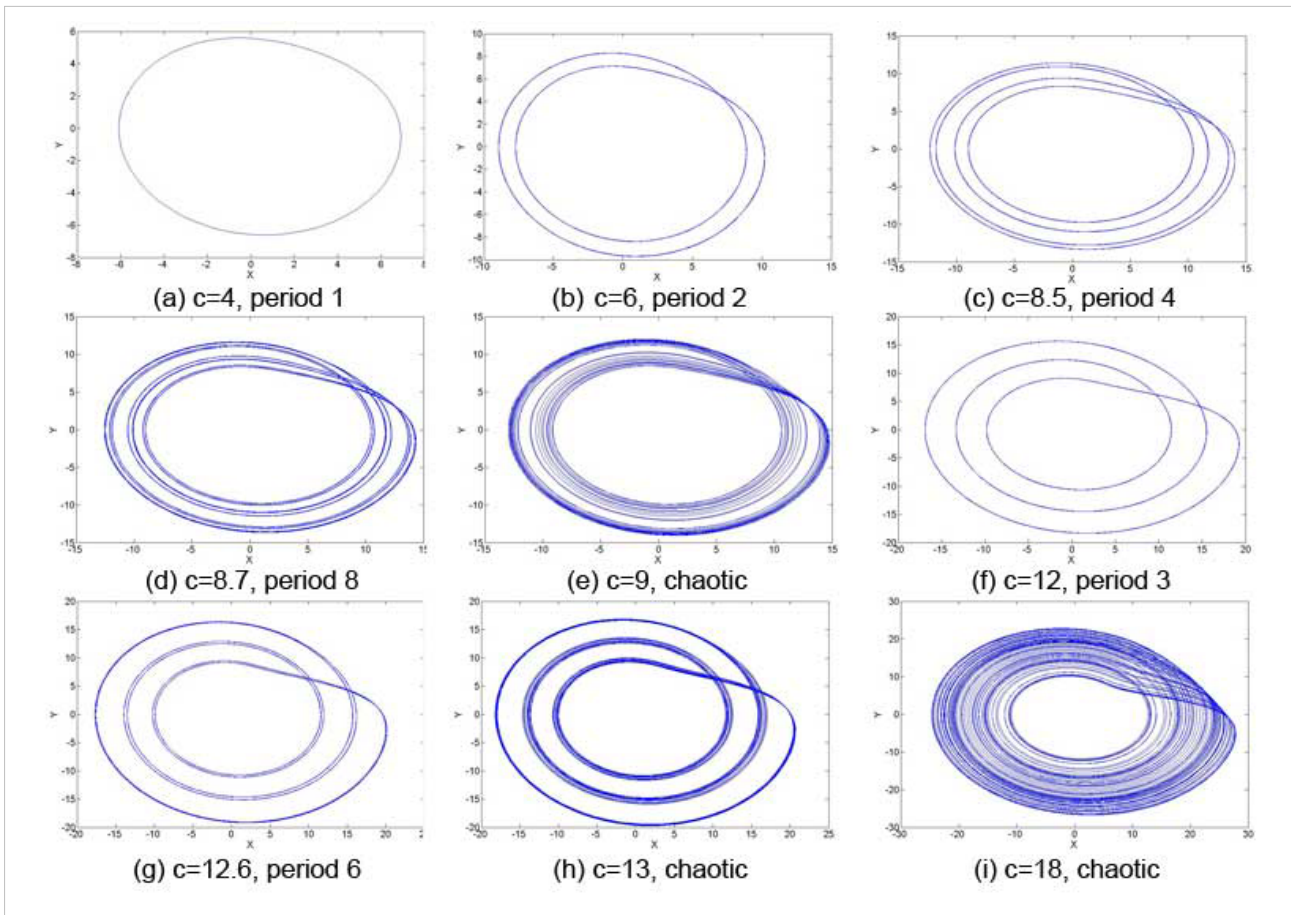


Varying c

Here, $a = b = 0.1$ and c changes. The bifurcation diagram reveals that low values of c are periodic, but quickly become chaotic as c increases. This pattern repeats itself as c increases – there are sections of periodicity interspersed with periods of chaos, and the trend is towards higher-period orbits as c increases. For example, the period one orbit only appears for values of c around 4 and is never found again in the bifurcation diagram. The same phenomena is seen with period three; until $c = 12$, period three orbits can be found, but thereafter, they do not appear.



A graphical illustration of the changing attractor over a range of c values illustrates the general behavior seen for all of these parameter analyses – the frequent transitions between periodicity and aperiodicity.



The above set of images illustrates the variations in the post-transient Rössler system as c is varied over a range of values. These images were generated with $a = b = .1$.

- $c = 4$, period-1 orbit.
- $c = 6$, period-2 orbit.
- $c = 8.5$, period-4 orbit.
- $c = 8.7$, period-8 orbit.
- $c = 9$, sparse chaotic attractor.
- $c = 12$, period-3 orbit.
- $c = 12.6$, period-6 orbit.
- $c = 13$, sparse chaotic attractor.
- $c = 18$, filled-in chaotic attractor.

Links to other topics

The banding evident in the Rössler attractor is similar to a Cantor set rotated about its midpoint. Additionally, the half-twist in the Rössler attractor makes it similar to a Möbius strip.

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External links

- Flash Animation using PovRay ^[1]
- [2]
- Lorenz and Rössler attractors ^[1] – Java animation
- 3D Attractors: Mac program to visualize and explore the Rössler and Lorenz attractors in 3 dimensions ^[11]
- Rössler attractor in Scholarpedia ^[3]

References

- [1] <http://lagrange.physics.drexel.edu/flash/rossray>
- [2] <http://www.soe.ucsc.edu/classes/ams214/Winter09/foundingpapers/Rossler1976.pdf>
- [3] http://scholarpedia.org/article/Rossler_attractor

Synchronizing Chaos

Synchronization of chaos is a phenomenon that may occur when two, or more, chaotic oscillators are coupled, or when a chaotic oscillator drives another chaotic oscillator. Because of the butterfly effect, which causes the exponential divergence of the trajectories of two identical chaotic system started with nearly the same initial conditions, having two chaotic system evolving in synchrony might appear quite surprising. However, synchronization of coupled or driven chaotic oscillators is a phenomenon well established experimentally and reasonably well understood theoretically.

It has been found that chaos synchronization is quite a rich phenomenon that may present a variety of forms. When two chaotic oscillators are considered, these include:

- **Identical synchronization.** This is a straightforward form of synchronization that may occur when two identical chaotic oscillators are mutually coupled, or when one of them drives the other. If (x_1, x_2, \dots, x_n) and $(x'_1, x'_2, \dots, x'_n)$ denote the set of dynamical variables that describe the state of the first and second oscillator, respectively, it is said that identical synchronization occurs when there is a set of initial conditions $[x_1(0), x_2(0), \dots, x_n(0)]$, $[x'_1(0), x'_2(0), \dots, x'_n(0)]$ such that, denoting the time by t , $|x'_i(t) - x_i(t)| \rightarrow 0$, for $i=1, 2, \dots, n$, when $t \rightarrow \infty$. That means that for time large enough the dynamics of the two oscillators verifies $x'_i(t) = x_i(t)$, for $i=1, 2, \dots, n$, in a good approximation. This is called the synchronized state in the sense of identical synchronization.
- **Generalized synchronization.** This type of synchronization occurs mainly when the coupled chaotic oscillators are different, although it has also been reported between identical oscillators. Given the dynamical variables (x_1, x_2, \dots, x_n) and (y_1, y_2, \dots, y_m) that determine the state of the oscillators, generalized synchronization occurs when there is a functional, Φ , such that, after a transitory evolution from appropriate initial conditions, it is $[y_1(t),$

$y_2(t), \dots, y_m(t) = \Phi[x_1(t), x_2(t), \dots, x_n(t)]$. This means that the dynamical state of one of the oscillators is completely determined by the state of the other. When the oscillators are mutually coupled this functional has to be invertible, if there is a drive-response configuration the drive determines the evolution of the response, and Φ does not need to be invertible. Identical synchronization is the particular case of generalized synchronization when Φ is the identity.

- **Phase synchronization.** This form of synchronization, which occurs when the oscillators coupled are not identical, is partial in the sense that, in the synchronized state, the amplitudes of the oscillator remain unsynchronized, and only their phases evolve in synchrony. Observation of phase synchronization requires a previous definition of the phase of a chaotic oscillator. In many practical cases, it is possible to find a plane in phase space in which the projection of the trajectories of the oscillator follows a rotation around a well-defined center. If this is the case, the phase is defined by the angle, $\varphi(t)$, described by the segment joining the center of rotation and the projection of the trajectory point onto the plane. In other cases it is still possible to define a phase by means of techniques provided by the theory of signal processing, such as the Hilbert transform. In any case, if $\varphi_1(t)$ and $\varphi_2(t)$ denote the phases of the two coupled oscillators, synchronization of the phase is given by the relation $n\varphi_1(t) = m\varphi_2(t)$ with m and n whole numbers.
- **Anticipated and lag synchronization.** In these cases the synchronized state is characterized by a time interval τ such that the dynamical variables of the oscillators, (x_1, x_2, \dots, x_n) and $(x'_1, x'_2, \dots, x'_n)$, are related by $x'_i(t) = x_i(t + \tau)$; this means that the dynamics of one of the oscillators follows, or anticipates, the dynamics of the other. **Anticipated synchronization** may occur between chaotic oscillators whose dynamics is described by delay differential equations, coupled in a drive-response configuration. In this case, the response anticipates the dynamics of the drive. **Lag synchronization** may occur when the strength of the coupling between phase-synchronized oscillators is increased.
- **Amplitude envelope synchronization.** This is a mild form of synchronization that may appear between two weakly coupled chaotic oscillators. In this case, there is no correlation between phases nor amplitudes; instead, the oscillations of the two systems develop a periodic envelope that has the same frequency in the two systems. This has the same order of magnitude than the difference between the average frequencies of oscillation of the two chaotic oscillator. Often, amplitude envelope synchronization precedes phase synchronization in the sense that when the strength of the coupling between two amplitude envelope synchronized oscillators is increased, phase synchronization develops.

All these forms of synchronization share the property of asymptotic stability. This means that once the synchronized state has been reached, the effect of a small perturbation that destroys synchronization is rapidly damped, and synchronization is recovered again. Mathematically, asymptotic stability is characterized by a positive Lyapunov exponent of the system composed of the two oscillators, which becomes negative when chaotic synchronization is achieved.

Some chaotic systems allow even stronger control of chaos. Both synchronization of chaos and control of chaos constitute parts of Cybernetical Physics.

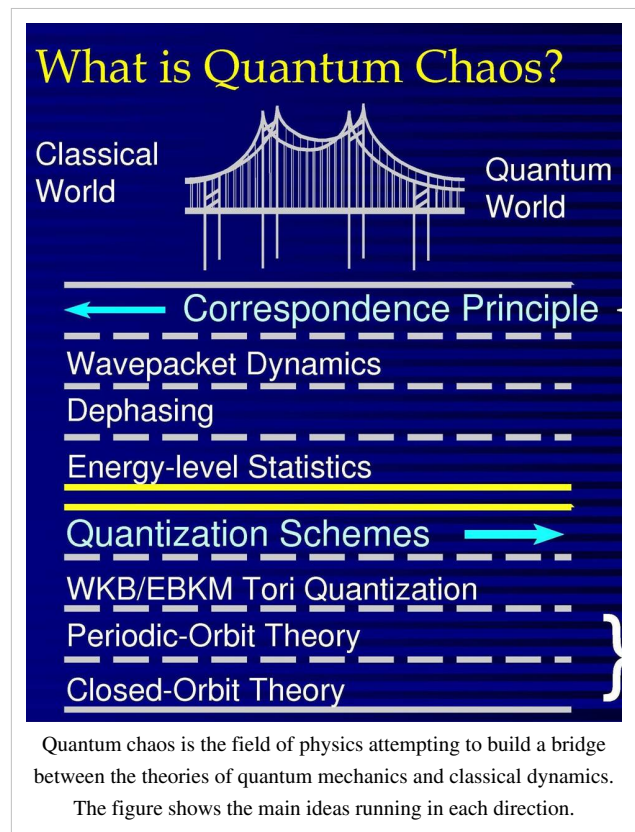
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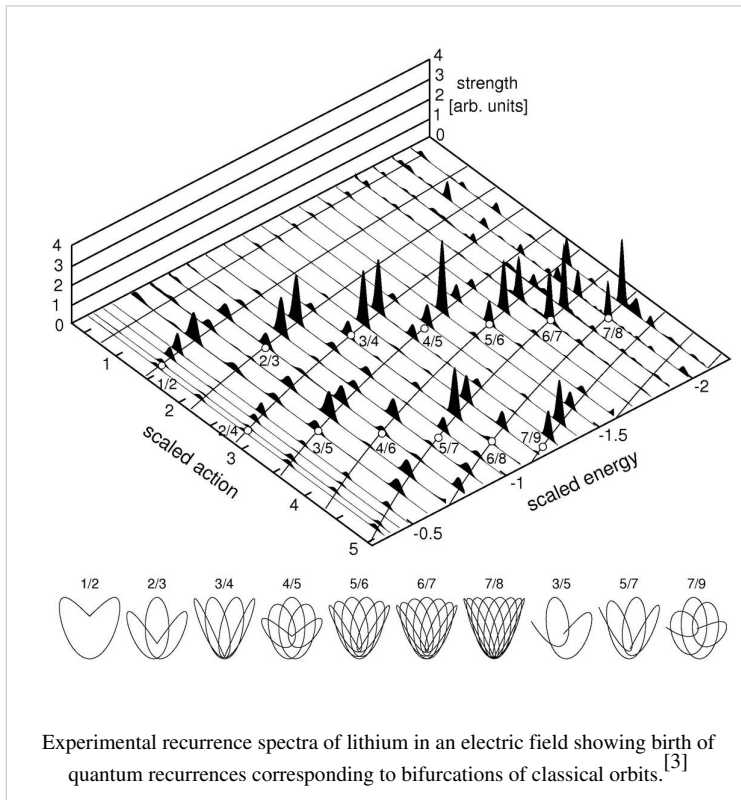
The Possibility of Quantum Chaos ?

Quantum chaos is a branch of physics which studies how chaotic classical dynamical systems can be described in terms of quantum theory. The primary question that quantum chaos seeks to answer is, "What is the relationship between quantum mechanics and classical chaos?" The correspondence principle states that classical mechanics is the classical limit of quantum mechanics. If this is true, then there must be quantum mechanisms underlying classical chaos; although this may not be a fruitful way of examining classical chaos. If quantum mechanics does not demonstrate an exponential sensitivity to initial conditions, how can exponential sensitivity to initial conditions arise in classical chaos, which must be the correspondence principle limit of quantum mechanics? [1] [2] In seeking to address the basic question of quantum chaos, several approaches have been employed:

1. Development of methods for solving quantum problems where the perturbation cannot be considered small in perturbation theory and where quantum numbers are large.
2. Correlating statistical descriptions of eigenvalues (energy levels) with the classical behavior of the same Hamiltonian (system).
3. Semiclassical methods such as periodic-orbit theory connecting the classical trajectories of the dynamical system with quantum features.
4. Direct application of the correspondence principle.



History



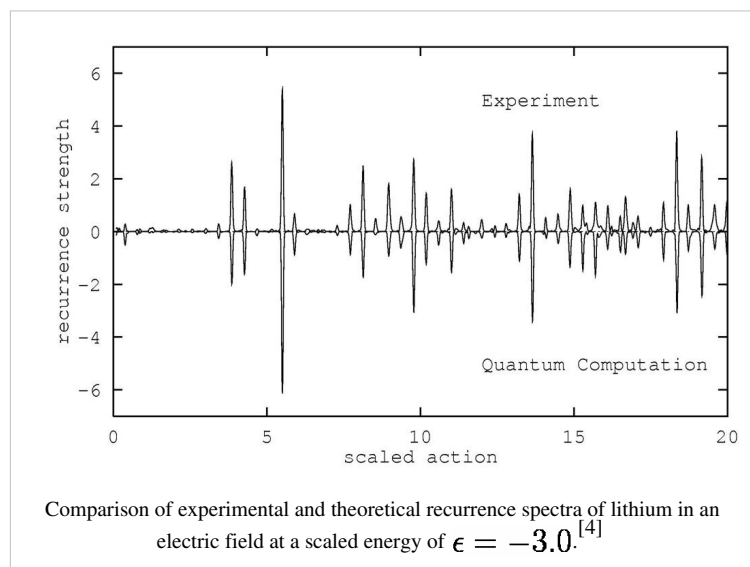
During the first half of the twentieth century, chaotic behavior in mechanics was recognized (as in the three-body problem in celestial mechanics), but not well-understood. The foundations of modern quantum mechanics were laid in that period, essentially leaving aside the issue of the quantum-classical correspondence in systems whose classical limit exhibit chaos.

Approaches

Questions related to the correspondence principle arise in many different branches of physics, ranging from nuclear to atomic, molecular and solid-state physics, and even to acoustics, microwaves and optics. Important observations often associated with classically chaotic quantum systems are spectral level repulsion, dynamical localization in time evolution (e.g. ionization rates of atoms), and enhanced stationary wave intensities in regions of space where classical dynamics exhibits only unstable trajectories (as in scattering).

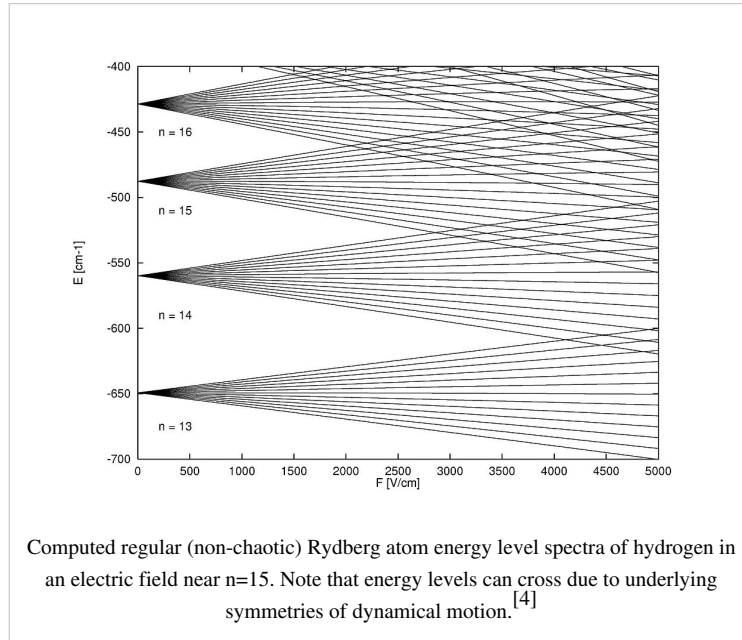
In the semiclassical approach of quantum chaos, phenomena are identified in spectroscopy by analyzing the statistical distribution of spectral lines and by connecting spectral periodicities with classical orbits. Other phenomena show up in the time evolution of a quantum system, or in its response to various types of external forces. In some contexts, such as acoustics or microwaves, wave patterns are directly observable and exhibit irregular amplitude distributions.

Quantum chaos typically deals with systems whose properties need to be calculated using either numerical techniques or approximation schemes (see e.g. Dyson series). Simple and exact solutions are precluded by the fact

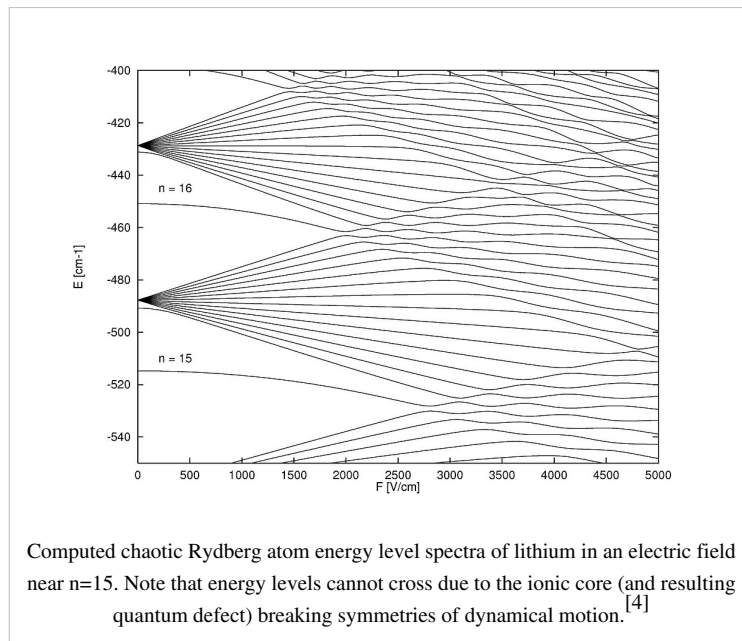


that the system's constituents either influence each other in a complex way, or depend on temporally varying external forces.

Quantum Mechanics in Non-Perturbative Regimes



For conservative systems, the goal of quantum mechanics in non-perturbative regimes is to find the eigenvalues and eigenvectors of a Hamiltonian of the form



$$H = H_s + \epsilon H_{ns},$$

where H_s is separable in some coordinate system, H_{ns} is non-separable in the coordinate system in which H_s is separated, and ϵ is a parameter which cannot be considered small. Physicists have historically approached problems of this nature by trying to find the coordinate system in which the non-separable Hamiltonian is smallest and then treating the non-separable Hamiltonian as a perturbation.

Finding constants of motion so that this separation can be performed can be a difficult (sometimes impossible) analytical task. Solving the classical problem can give valuable insight into solving the quantum problem. If there are

regular classical solutions of the same Hamiltonian, then there are (at least) approximate constants of motion, and by solving the classical problem, we gain clues how to find them.

Other approaches have been developed in recent years. One is to express the Hamiltonian in different coordinate systems in different regions of space, minimizing the non-separable part of the Hamiltonian in each region. Wavefunctions are obtained in these regions, and eigenvalues are obtained by matching boundary conditions.

Another approach is numerical matrix diagonalization. If the Hamiltonian matrix is computed in any complete basis, eigenvalues and eigenvectors are obtained by diagonalizing the matrix. However, all complete basis sets are infinite, and we need to truncate the basis and still obtain accurate results. These techniques boil down to choosing a truncated basis from which accurate wavefunctions can be constructed. The computational time required to diagonalize a matrix scales as N^3 , where N is the dimension of the matrix, so it is important to choose the smallest basis possible from which the relevant wavefunctions can be constructed. It is also convenient to choose a basis in which the matrix is sparse and/or the matrix elements are given by simple algebraic expressions because computing matrix elements can also be a computational burden.

A given Hamiltonian shares the same constants of motion for both classical and quantum dynamics. Quantum systems can also have additional quantum numbers corresponding to discrete symmetries (such as parity conservation from reflection symmetry). However, if we merely find quantum solutions of a Hamiltonian which is not approachable by perturbation theory, we may learn a great deal about quantum solutions, but we have learned little about quantum chaos. Nevertheless, learning how to solve such quantum problems is an important part of answering the question of quantum chaos.

Correlating Statistical Descriptions of Quantum Mechanics with Classical Behavior

Statistical measures of quantum chaos were born out of a desire to quantify spectral features of complex systems. Random matrix theory was developed in an attempt to characterize spectra of complex nuclei. The remarkable result is that the statistical properties of many systems with unknown Hamiltonians can be predicted using random matrices of the proper symmetry class. Furthermore, random matrix theory also correctly predicts statistical properties of the eigenvalues of many chaotic systems with known Hamiltonians. This makes it useful as a tool for characterizing spectra which require large numerical efforts to compute.

A number of statistical measures are available for quantifying spectral features in a simple way. It is of great interest whether or not there are universal statistical behaviors of classically chaotic systems. The statistical tests mentioned here are universal, at least to systems with few degrees of freedom (Berry and Tabor [5] have put forward strong arguments for a Poisson distribution in the case of regular motion and Heusler et al. [6] present a semiclassical explanation of the so-called Bohigas-Giannoni-Schmit conjecture which asserts universality of spectral fluctuations in chaotic dynamics). The nearest-neighbor distribution (NND) of energy levels is relatively simple to interpret and it has been widely used to describe quantum chaos.

Qualitative observations of level repulsions can be quantified and related to the classical dynamics using the NND, which is believed to be an important signature of classical dynamics in quantum systems. It is thought that regular classical dynamics is manifested by a Poisson distribution of energy levels:

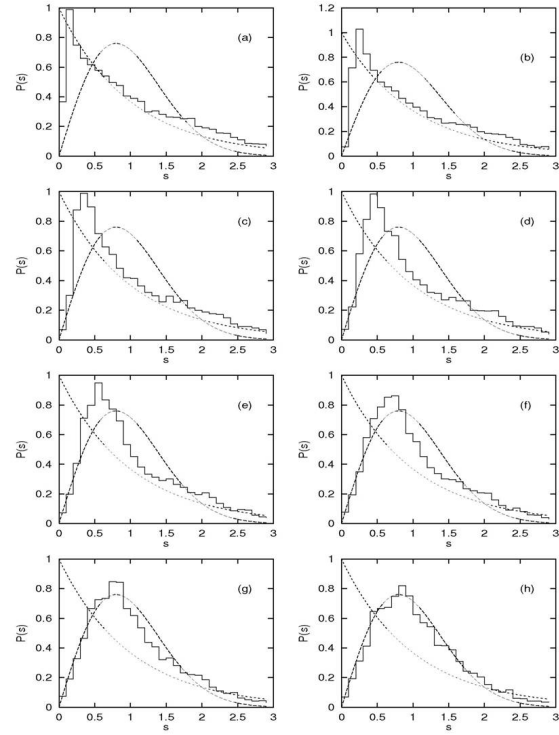
$$P(s) = e^{-s}.$$

In addition, systems which display chaotic classical motion are expected to be characterized by the statistics of random matrix eigenvalue ensembles. For systems invariant under time reversal, the energy-level statistics of a number of chaotic systems have been shown to be in good agreement with the predictions of the Gaussian orthogonal ensemble (GOE) of random matrices, and it has been suggested that this phenomenon is generic for all chaotic systems with this symmetry. If the normalized spacing between two energy levels is s , the normalized distribution of spacings is well approximated by

$$P(s) = \frac{\pi}{2} s e^{-\pi s^2/4},$$

which is the Wigner distribution.

Many Hamiltonian systems which are classically integrable (non-chaotic) have been found to have quantum solutions that yield nearest neighbor distributions which follow the Poisson distributions. Similarly, many systems



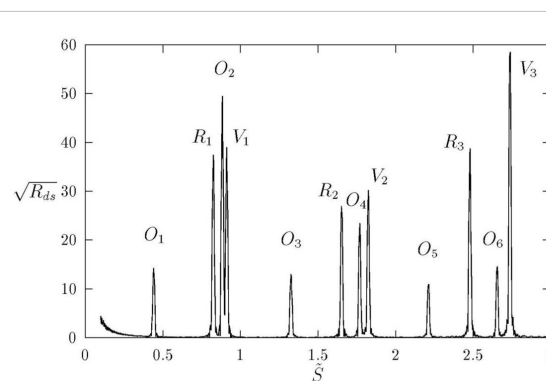
Nearest neighbor distribution for Rydberg atom energy level spectra in an electric field as quantum defect is increased from 0.04 (a) to 0.32 (h). The system becomes more chaotic as dynamical symmetries are broken by increasing the quantum defect; consequently, the distribution evolves from nearly a Poisson distribution (a) to a Wigner distribution (h).

which exhibit classical chaos have been found with quantum solutions yielding a Wigner distribution, thus supporting the ideas above. One notable exception is diamagnetic lithium which, though exhibiting classical chaos, demonstrates Wigner (chaotic) statistics for the even-parity energy levels and nearly Poisson (regular) statistics for the odd-parity energy level distribution.^[7]

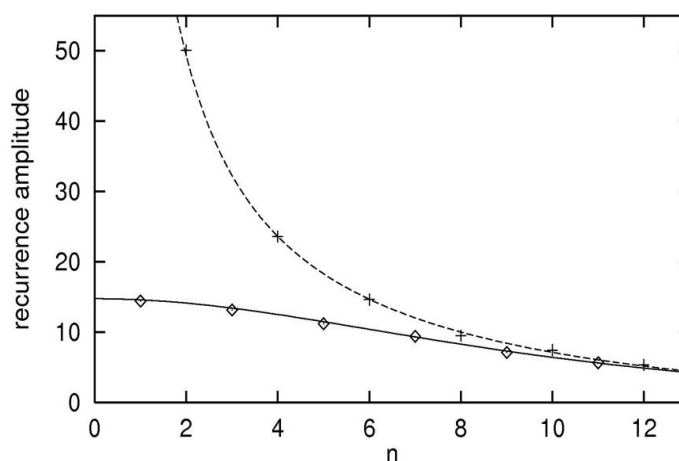
Semiclassical Methods

Periodic Orbit Theory

Periodic-orbit theory gives a recipe for computing spectra from the periodic orbits of a system. In contrast to the Einstein-Brillouin-Keller method of action quantization, which applies only to integrable or near-integrable systems and computes individual eigenvalues from each trajectory, periodic-orbit theory is applicable to both integrable and non-integrable systems and asserts that each periodic orbit produces a sinusoidal fluctuation in the density of states.



Even parity recurrence spectrum (Fourier transform of the density of states) of diamagnetic hydrogen showing peaks corresponding to periodic orbits of the classical system. Spectrum is at a scaled energy of -0.6. Peaks labeled R and V are repetitions of the closed orbit perpendicular and parallel to the field, respectively. Peaks labeled O correspond to the near circular periodic orbit that goes around the nucleus.



Relative recurrence amplitudes of even and odd recurrences of the near circular orbit. Diamonds and plus signs are for odd and even quarter periods, respectively. Solid line is $A/\cosh(nX/8)$. Dashed line is $A/\sinh(nX/8)$ where $A = 14.75$ and $X = 1.18$.

The principal result of this development is an expression for the density of states which is the trace of the semiclassical Green's function and is given by the Gutzwiller trace formula:

$$g_c(E) = \sum_k T_k \sum_{n=1}^{\infty} \frac{1}{2 \sinh(\chi_{nk}/2)} e^{i(nS_k - \alpha_{nk}\pi/2)}.$$

The index k distinguishes the primitive periodic orbits: the shortest period orbits of a given set of initial conditions. T_k is the period of the primitive periodic orbit and S_k is its classical action. Each primitive orbit retraces itself, leading to a new orbit with action nS_k and a period which is an integral multiple n of the primitive period. Hence, every repetition of a periodic orbit is another periodic orbit. These repetitions are separately classified by the intermediate sum over the indices n . α_{nk} is the orbit's Maslov index. The amplitude factor, $1/\sinh(\chi_{nk}/2)$, represents the square root of the density of neighboring orbits. Neighboring trajectories of an unstable periodic orbit diverge exponentially in time from the periodic orbit. The quantity χ_{nk} characterizes the instability of the orbit. A stable orbit moves on a torus in phase space, and neighboring trajectories wind around it. For stable orbits, $\sinh(\chi_{nk}/2)$ becomes $\sin(\chi_{nk}/2)$, where χ_{nk} is the winding number of the periodic orbit. $\chi_{nk} = 2\pi m$, where m is the number of times that neighboring orbits intersect the periodic orbit in one period. This presents a difficulty because $\sin(\chi_{nk}/2) = 0$ at a classical bifurcation. This causes that orbit's contribution to the energy density to diverge. This also occurs in the context of photo-absorption spectrum.

Using the trace formula to compute a spectrum requires summing over all of the periodic orbits of a system. This presents several difficulties for chaotic systems: 1) The number of periodic orbits proliferates exponentially as a function of action. 2) There are an infinite number of periodic orbits, and the convergence properties of periodic-orbit theory are unknown. This difficulty is also present when applying periodic-orbit theory to regular systems. 3) Long-period orbits are difficult to compute because most trajectories are unstable and sensitive to roundoff errors and details of the numerical integration.

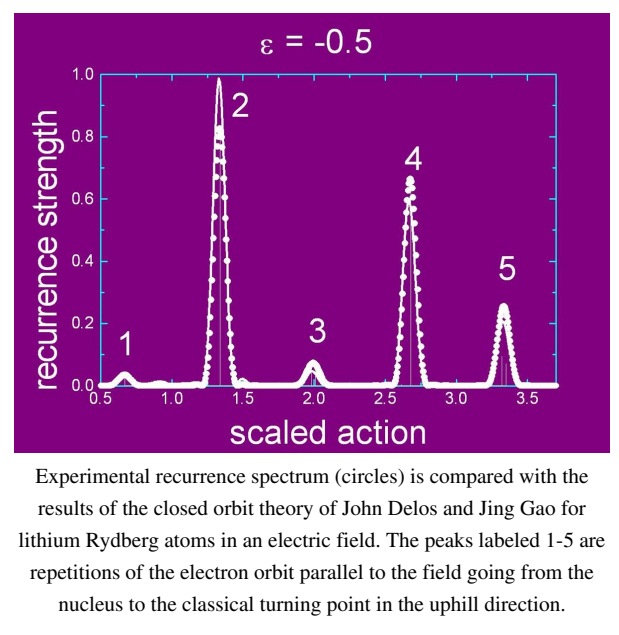
Gutzwiller applied the trace formula to approach the anisotropic Kepler problem (a single particle in a $1/r$ potential with an anisotropic mass tensor) semiclassically. He found agreement with quantum computations for low lying (up to $n = 6$) states for small anisotropies by using only a small set of easily computed periodic orbits, but the agreement was poor for large anisotropies.

The figures above use an inverted approach to testing periodic-orbit theory. The trace formula asserts that each periodic orbit contributes a sinusoidal term to the spectrum. Rather than dealing with the computational difficulties surrounding long-period orbits to try and find the density of states (energy levels), one can use standard quantum mechanical perturbation theory to compute eigenvalues (energy levels) and use the Fourier transform to look for the periodic modulations of the spectrum which are the signature of periodic orbits. Interpreting the spectrum then amounts to finding the orbits which correspond to peaks in the Fourier transform.

Closed Orbit Theory

Closed-orbit theory was developed by J.B. Delos, M.L. Du, J. Gao, and J. Shaw. It is similar to periodic-orbit theory, except that closed-orbit theory is applicable only to atomic and molecular spectra and yields the oscillator strength density (observable photo-absorption spectrum) from a specified initial state whereas periodic-orbit theory yields the density of states.

Only orbits that begin and end at the nucleus are important in closed-orbit theory. Physically, these are associated with the outgoing waves that are generated when a tightly bound electron is excited to a high-lying state. For Rydberg atoms and molecules, every orbit which is closed at the nucleus is also a periodic orbit whose period is equal to either the closure time or twice the closure time.



According to closed-orbit theory, the average oscillator strength density at constant ϵ is given by a smooth background plus an oscillatory sum of the form $f(w) = \sum_k \sum_{n=1}^{\infty} D_{nk}^i \sin(2\pi n w \tilde{S}_k - \phi_{nk})$.

ϕ_{nk} is a phase that depends on the Maslov index and other details of the orbits. D_{nk}^i is the recurrence amplitude of a closed orbit for a given initial state (labeled i). It contains information about the stability of the orbit, its initial and final directions, and the matrix element of the dipole operator between the initial state and a zero-energy Coulomb wave. For scaling systems such as Rydberg atoms in strong fields, the Fourier transform of an oscillator strength spectrum computed at fixed ϵ as a function of w is called a recurrence spectrum, because it gives peaks which correspond to the scaled action of closed orbits and whose heights correspond to D_{nk}^i . Closed-orbit theory has found broad agreement with a number of chaotic systems, including diamagnetic hydrogen, hydrogen in parallel electric and magnetic fields, diamagnetic lithium, lithium in an electric field, the H^- ion in crossed and parallel electric and magnetic fields, barium in an electric field, and helium in an electric field.

Recent directions in quantum chaos

The traditional topics in quantum chaos concerns spectral statistics (universal and non-universal features), and the study of eigenfunctions (Quantum ergodicity, scars) of various chaotic Hamiltonian $H(x, p; R)$.

Further studies concern the parametric (R) dependence of the Hamiltonian, as reflected in e.g. the statistics of avoided crossings, and the associated mixing as reflected in the (parametric) local density of states (LDOS). There is vast literature on wavepacket dynamics, including the study of fluctuations, recurrences, quantum irreversibility issues etc. Special place is reserved to the study of the dynamics of quantized maps: The Standard map and The Kicked Rotator are considered to be prototype problems.

Recent works are also focused in the study of driven chaotic systems,^[8] where the Hamiltonian $H(x, p; R(t))$ is time dependent, in particular in the adiabatic and in the linear response regimes.

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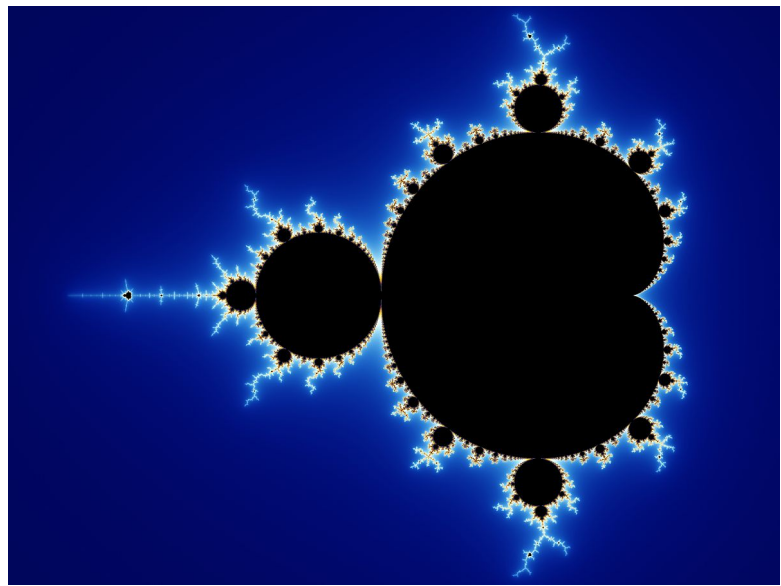
External links

- Quantum Chaos (<http://www.sciam.com/article.cfm?id=quantum-chaos-subatomic-worlds>) by Martin Gutzwiller (1992, *Scientific American*)
- What is... Quantum Chaos (<http://www.ams.org/notices/200801/tx080100032p.pdf>) by Ze'ev Rudnick (January 2008, *Notices of the American Mathematical Society*)
- Brian Hayes, "The Spectrum of Riemannium"; *American Scientist* (<http://www.americanscientist.org/template/AssetDetail/assetid/21879/page/1.jsessionid=aaa-ZYP5NrRxh8>). Discusses relation to the Riemann zeta function.
- Eigenfunctions in chaotic quantum systems (<http://nbn-resolving.de/urn:nbn:de:bsz:14-ds-1213275874643-50420>) by Arnd Bäcker.
- Quantum Chaos at Scholarpedia (http://www.scholarpedia.org/article/Category:Quantum_Chaos)

Fractals and Fractional Dimensions

A **fractal** is "a rough or fragmented geometric shape that can be split into parts, each of which is (at least approximately) a reduced-size copy of the whole,"^[1] a property called self-similarity. Roots of the idea of fractals go back to the 17th century, while mathematically rigorous treatment of fractals can be traced back to functions studied by Karl Weierstrass, Georg Cantor and Felix Hausdorff a century later in studying functions that were continuous but not differentiable; however, the term *fractal* was coined by Benoît Mandelbrot in 1975 and was derived from the Latin *fractus* meaning

"broken" or "fractured." A mathematical fractal is based on an equation that undergoes iteration, a form of feedback based on recursion.^[2] There are several examples of fractals, which are defined as portraying exact self-similarity, quasi self-similarity, or statistical self-similarity. While fractals are a mathematical construct, they are found in nature, which has led to their inclusion in artwork. They are useful in medicine, soil mechanics, seismology, and technical analysis.



The Mandelbrot set is a famous example of a fractal

Characteristics

A fractal often has the following features:^[3]

- It has a fine structure at arbitrarily small scales.
- It is too irregular to be easily described in traditional Euclidean geometric language.
- It is self-similar (at least approximately or stochastically).
- It has a Hausdorff dimension which is greater than its topological dimension (although this requirement is not met by space-filling curves such as the Hilbert curve).^[4]
- It has a simple and recursive definition.

Because they appear similar at all levels of magnification, fractals are often considered to be infinitely complex (in informal terms). Natural objects that are approximated by fractals to a degree include clouds, mountain ranges, lightning bolts, coastlines, snow flakes, various vegetables (cauliflower and broccoli), and animal coloration patterns.

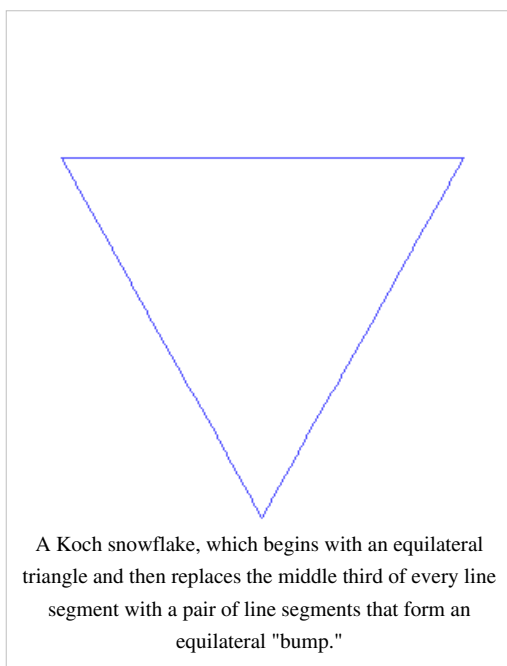


Frost crystals formed naturally on cold glass illustrate fractal process development in a purely physical system

However, not all self-similar objects are fractals—for example, the real line (a straight Euclidean line) is formally self-similar but fails to have other fractal characteristics; for instance, it is regular enough to be described in Euclidean terms.

Images of fractals can be created using fractal-generating software. Images produced by such software are normally referred to as being fractals even if they do not have the above characteristics, such as when it is possible to zoom into a region of the fractal that does not exhibit any fractal properties. Also, these may include calculation or display artifacts which are not characteristics of true fractals.

History



The mathematics behind fractals began to take shape in the 17th century when a mathematician and philosopher Gottfried Leibniz considered recursive self-similarity (although he made the mistake of thinking that only the straight line was self-similar in this sense).

It was not until 1872 that a function appeared whose graph would today be considered fractal, when Karl Weierstrass gave an example of a function with the non-intuitive property of being everywhere continuous but nowhere differentiable. In 1904, Helge von Koch, dissatisfied with Weierstrass's abstract and analytic definition, gave a more geometric definition of a similar function, which is now called the Koch curve.^[5] Waław Sierpiński constructed his triangle in 1915 and, one year later, his carpet. The idea of self-similar curves was taken further by Paul Pierre Lévy, who, in his 1938 paper *Plane or Space Curves and Surfaces Consisting of Parts Similar to the Whole* described a new fractal curve, the Lévy C curve. Georg Cantor also gave examples of

subsets of the real line with unusual properties—these Cantor sets are also now recognized as fractals.

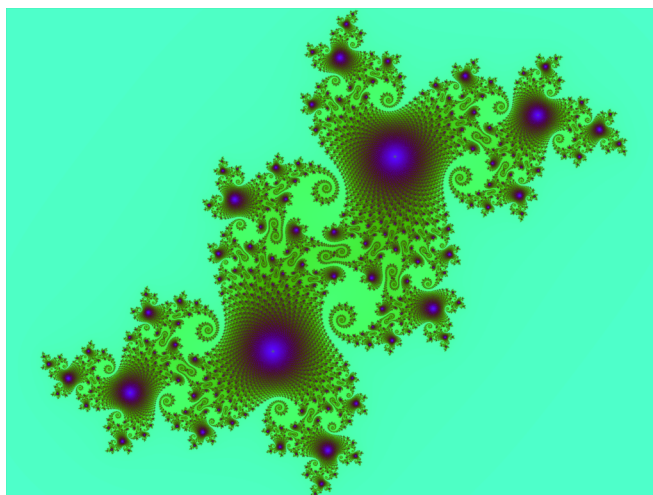
Iterated functions in the complex plane were investigated in the late 19th and early 20th centuries by Henri Poincaré, Felix Klein, Pierre Fatou and Gaston Julia. Without the aid of modern computer graphics, however, they lacked the means to visualize the beauty of many of the objects that they had discovered.

In the 1960s, Benoît Mandelbrot started investigating self-similarity in papers such as *How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension*,^[6] which built on earlier work by Lewis Fry Richardson. Finally, in 1975 Mandelbrot coined the word "fractal" to denote an object whose Hausdorff–Besicovitch dimension is greater than its topological dimension. He illustrated this mathematical definition with striking computer-constructed visualizations. These images captured the popular imagination; many of them were based on recursion, leading to the popular meaning of the term "fractal".^[7]

Examples

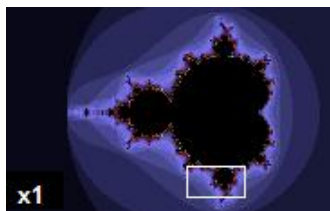
A class of examples is given by the Cantor sets, Sierpinski triangle and carpet, Menger sponge, dragon curve, space-filling curve, and Koch curve. Additional examples of fractals include the Lyapunov fractal and the limit sets of Kleinian groups. Fractals can be deterministic (all the above) or stochastic (that is, non-deterministic). For example, the trajectories of the Brownian motion in the plane have a Hausdorff dimension of 2.

Chaotic dynamical systems are sometimes associated with fractals. Objects in the phase space of a dynamical system can be fractals (see attractor). Objects in the parameter space for a family of systems may be fractal as well. An interesting example is the Mandelbrot set. This set contains whole discs, so it has a Hausdorff dimension equal to its topological dimension of 2—but what is truly surprising is that the boundary of the Mandelbrot set also has a Hausdorff dimension of 2 (while the topological dimension of 1), a result proved by Mitsuhiro Shishikura in 1991. A closely related fractal is the Julia set.



A Julia set, a fractal related to the Mandelbrot set

Generation



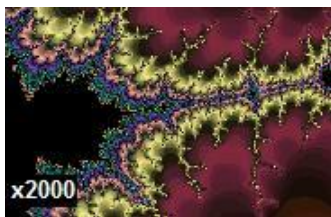
The whole Mandelbrot set



Mandelbrot zoomed 6x



Mandelbrot Zoomed 100x



Even 2000 times magnification of the Mandelbrot set uncovers fine detail resembling the full set

Four common techniques for generating fractals are:

- **Escape-time fractals** – (also known as "orbits" fractals) These are defined by a formula or recurrence relation at each point in a space (such as the complex plane). Examples of this type are the Mandelbrot set, Julia set, the Burning Ship fractal, the Nova fractal and the Lyapunov fractal. The 2d vector fields that are generated by one or two iterations of escape-time formulae also give rise to a fractal form when points (or pixel data) are passed through this field repeatedly.
- **Iterated function systems** – These have a fixed geometric replacement rule. Cantor set, Sierpinski carpet, Sierpinski gasket, Peano curve, Koch snowflake, Harter-Highway dragon curve, T-Square, Menger sponge, are some examples of such fractals.
- **Random fractals** – Generated by stochastic rather than deterministic processes, for example, trajectories of the Brownian motion, Lévy flight, fractal landscapes and the Brownian tree. The latter yields so-called mass- or dendritic fractals, for example, diffusion-limited aggregation or reaction-limited aggregation clusters.
- **Strange attractors** – Generated by iteration of a map or the solution of a system of initial-value differential equations that exhibit chaos.

Classification

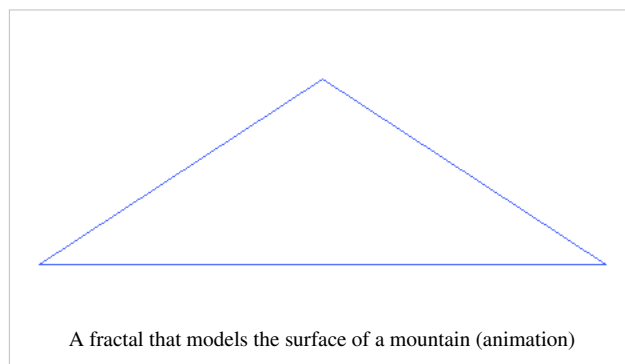
Fractals can also be classified according to their self-similarity. There are three types of self-similarity found in fractals:

- **Exact self-similarity** – This is the strongest type of self-similarity; the fractal appears identical at different scales. Fractals defined by iterated function systems often display exact self-similarity. For example, the Sierpinski triangle and Koch snowflake exhibit exact self-similarity.
- **Quasi-self-similarity** – This is a looser form of self-similarity; the fractal appears approximately (but not exactly) identical at different scales. Quasi-self-similar fractals contain small copies of the entire fractal in distorted and degenerate forms. Fractals defined by recurrence relations are usually quasi-self-similar but not exactly self-similar. The Mandelbrot set is quasi-self-similar, as the satellites are approximations of the entire set, but not exact copies.
- **Statistical self-similarity** – This is the weakest type of self-similarity; the fractal has numerical or statistical measures which are preserved across scales. Most reasonable definitions of "fractal" trivially imply some form of statistical self-similarity. (Fractal dimension itself is a numerical measure which is preserved across scales.) Random fractals are examples of fractals which are statistically self-similar, but neither exactly nor quasi-self-similar. The coastline of Britain is another example; one cannot expect to find microscopic Britains (even distorted ones) by looking at a small section of the coast with a magnifying glass.

Possessing self-similarity is not the sole criterion for an object to be termed a fractal. Examples of self-similar objects that are not fractals include the logarithmic spiral and straight lines, which do contain copies of themselves at increasingly small scales. These do not qualify, since they have the same Hausdorff dimension as topological dimension.

In nature

Approximate fractals are easily found in nature. These objects display self-similar structure over an extended, but finite, scale range. Examples include clouds, river networks, fault lines, mountain ranges, craters,^[8] snow flakes,^[9] crystals,^[10] lightning, cauliflower or broccoli, and systems of blood vessels and pulmonary vessels, and ocean waves^[11]. DNA and heartbeat^[12] can be analyzed as fractals. Even coastlines may be loosely considered fractal in nature.



Trees and ferns are fractal in nature and can be modeled on a computer by using a recursive algorithm. This recursive nature is obvious in these examples—a branch from a tree or a frond from a fern is a miniature replica of the whole: not identical, but similar in nature. The connection between fractals and leaves is currently being used to determine how much carbon is contained in trees.^[13]

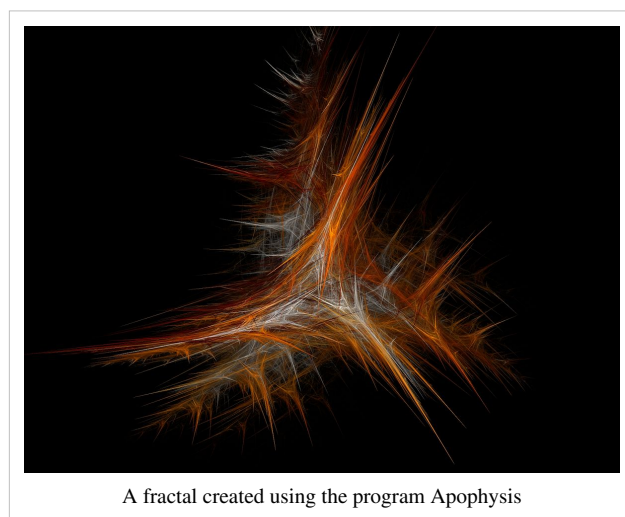
In 1999, certain self similar fractal shapes were shown to have a property of "frequency invariance"—the same electromagnetic properties no matter what the frequency—from Maxwell's equations (see fractal antenna).^[14]

In creative works

Fractal patterns have been found in the paintings of American artist Jackson Pollock. While Pollock's paintings appear to be composed of chaotic dripping and splattering, computer analysis has found fractal patterns in his work.^[15]

Decalcomania, a technique used by artists such as Max Ernst, can produce fractal-like patterns.^[16] It involves pressing paint between two surfaces and pulling them apart.

Cyberneticist Ron Eglash has suggested that fractal-like structures are prevalent in African art and architecture. Circular houses appear in circles of circles, rectangular houses in rectangles of rectangles, and so on. Such scaling patterns can also be found in African textiles, sculpture, and even cornrow hairstyles.^[17] ^[18]

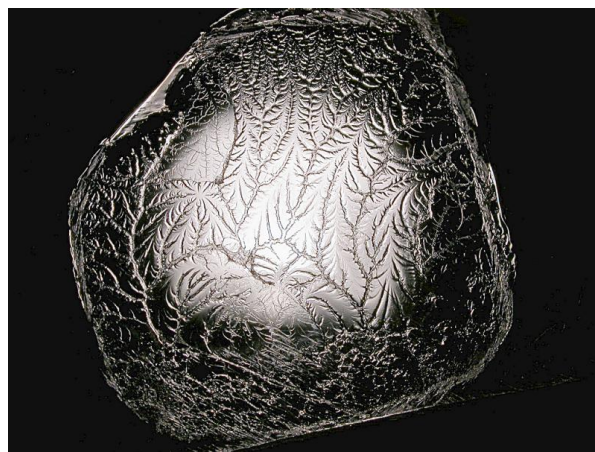


In a 1996 interview with Michael Silverblatt, David Foster Wallace admitted that the structure of the first draft of *Infinite Jest* he gave to his editor Michael Pietsch was inspired by fractals, specifically the Sierpinski triangle (aka Sierpinski gasket) but that the edited novel is "more like a lopsided Sierpinsky Gasket".^[19]

Applications

As described above, random fractals can be used to describe many highly irregular real-world objects. Other applications of fractals include:^[20]

- Classification of histopathology slides in medicine
- Fractal landscape or Coastline complexity
- Enzyme/enzymology (Michaelis-Menten kinetics)
- Generation of new music
- Signal and image compression
- Creation of digital photographic enlargements
- Seismology
- Fractal in soil mechanics
- Computer and video game design, especially computer graphics for organic environments and as part of procedural generation
- Fractography and fracture mechanics
- Fractal antennas – Small size antennas using fractal shapes
- Small angle scattering theory of fractally rough systems
- T-shirts and other fashion
- Generation of patterns for camouflage, such as MARPAT
- Digital sundial
- Technical analysis of price series (see Elliott wave principle)
- Fractals in networks



A fractal is formed when pulling apart two glue-covered acrylic sheets



High voltage breakdown within a 4" block of acrylic creates a fractal Lichtenberg figure

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Nots

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External links

- Fractals (http://www.dmoz.org/Science/Math/Chaos_And_Fractals/) at the Open Directory Project

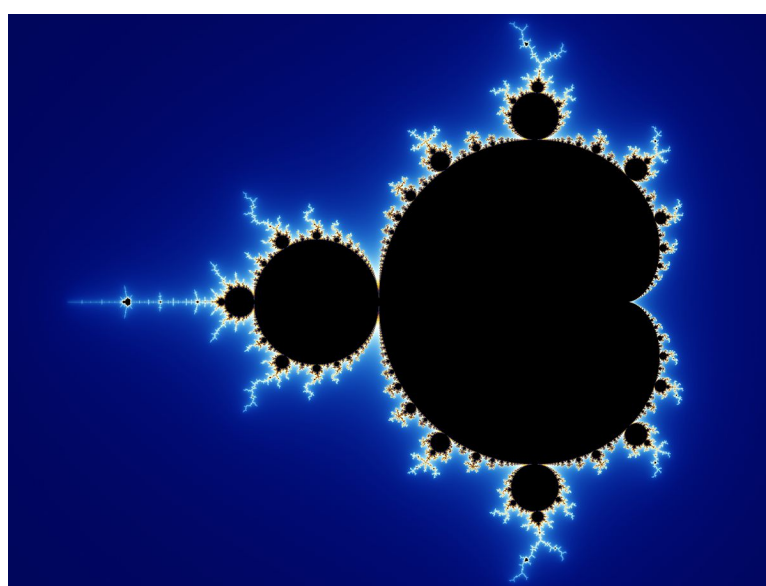
Mandelbrot set

The **Mandelbrot set** is a particular mathematical set of points, whose boundary generates a distinctive and easily recognisable two-dimensional fractal shape. The set is closely related to the Julia set (which generates similarly complex shapes), and is named after the mathematician Benoît Mandelbrot, who studied and popularized it.

More technically, the Mandelbrot set is the set of values of c in the complex plane for which the orbit of 0 under iteration of the complex quadratic polynomial $z_{n+1} = z_n^2 + c$ remains bounded.^[1] That is, a complex number, c , is part of the Mandelbrot set if, when starting with $z_0 = 0$ and applying the iteration repeatedly, the absolute value of z_n never exceeds a certain number (that number depends on c) however large n gets.

For example, letting $c = 1$ gives the sequence 0, 1, 2, 5, 26,..., which tends to infinity. As this sequence is unbounded, 1 is not an element of the Mandelbrot set. On the other hand, $c = i$ (where i is defined as $i^2 = -1$) gives the sequence 0, i , $(-1 + i)$, $-i$, $(-1 + i)$, $-i$, ..., which is bounded and so i belongs to the Mandelbrot set.

Images of the Mandelbrot set display an elaborate boundary that reveals progressively ever-finer recursive detail at increasing magnifications. The "style" of this repeating detail depends on the region of the set being examined. The set's boundary also incorporates smaller versions of the main shape, so the fractal property of self-similarity applies to the whole set, and not just to its parts.

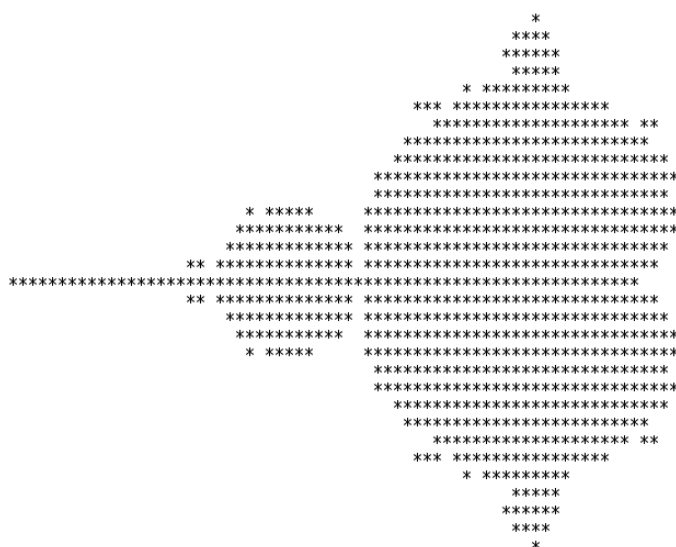


Initial image of a Mandelbrot set zoom sequence with a continuously coloured environment

The Mandelbrot set has become popular outside mathematics both for its aesthetic appeal and as an example of a complex structure arising from the application of simple rules, and is one of the best-known examples of mathematical visualization.

History

The Mandelbrot set has its place in complex dynamics, a field first investigated by the French mathematicians Pierre Fatou and Gaston Julia at the beginning of the 20th century. The first pictures of this fractal were drawn in 1978 by Robert Brooks and Peter Matelski as part of a study of Kleinian groups.^[2] On 1 March 1980, at IBM's Thomas J. Watson Research Center in upstate New York, Benoît Mandelbrot first saw a visualization of the set.^[3]



The first picture of the Mandelbrot set, by Robert Brooks and Peter Matelski in 1978

Mandelbrot studied the parameter space of quadratic polynomials in an article that appeared in 1980.^[4] The mathematical study of the Mandelbrot set really began with work by the mathematicians Adrien Douady and John H. Hubbard,^[5] who established many of its fundamental properties and named the set in honour of Mandelbrot.

The mathematicians Heinz-Otto Peitgen and Peter Richter became well-known for promoting the set with stunning photographs, books,^[6] and an internationally touring exhibit of the German Goethe-Institut.^{[7] [8]}

The cover article of the August 1985 *Scientific American* introduced the algorithm for computing the Mandelbrot set to a wide audience. The cover featured an image created by Peitgen, et al.^{[9] [10]}

The work of Douady and Hubbard coincided with a huge increase in interest in complex dynamics and abstract mathematics, and the study of the Mandelbrot set has been a centerpiece of this field ever since. An exhaustive list of all the mathematicians who have contributed to the understanding of this set since then is beyond the scope of this article, but such a list would notably include Mikhail Lyubich,^{[11] [12]} Curt McMullen, John Milnor, Mitsuhiro Shishikura, and Jean-Christophe Yoccoz.

Formal definition

The Mandelbrot set M is defined by a family of complex quadratic polynomials

$$P_c : \mathbb{C} \rightarrow \mathbb{C}$$

given by

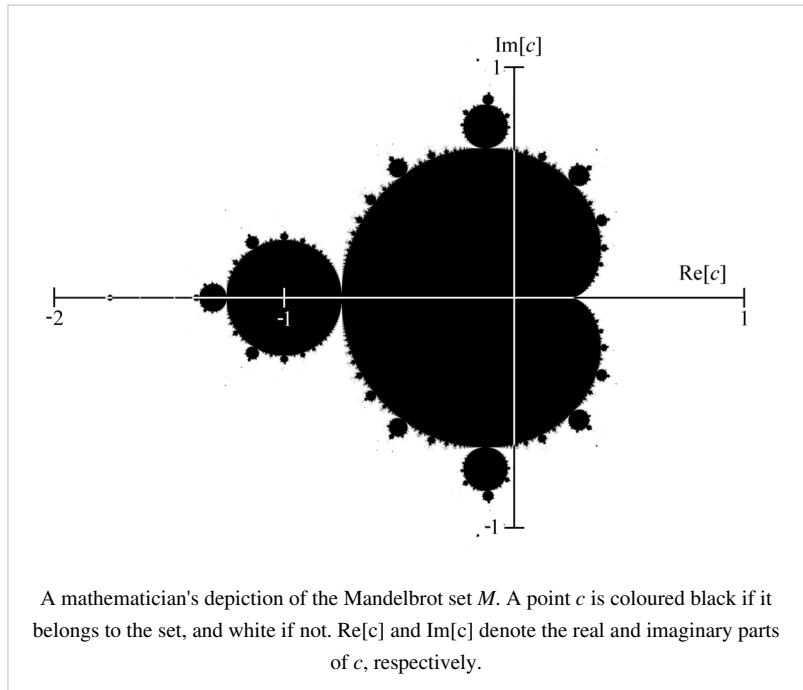
$$P_c : z \mapsto z^2 + c,$$

where c is a complex parameter. For each c , one considers the behavior of the sequence

$$(0, P_c(0), P_c(P_c(0)), P_c(P_c(P_c(0))), \dots)$$

obtained by iterating $P_c(z)$ starting at critical point $z = 0$, which either escapes to infinity or stays within a disk of some finite radius. The Mandelbrot set is defined as the set of all points c such that the above sequence does *not* escape to infinity.

More formally, if $P_c^n(z)$ denotes the n th iterate of $P_c(z)$ (i.e. $P_c(z)$ composed with itself n times), the Mandelbrot set is the subset of the complex plane given by



$$M = \{c \in \mathbb{C} : \exists s \in \mathbb{R}, \forall n \in \mathbb{N}, |P_c^n(0)| \leq s\}.$$

As explained below, it is in fact possible to simplify this definition by taking $s = 2$.

Mathematically, the Mandelbrot set is just a set of complex numbers. A given complex number c either belongs to M or it does not. A picture of the Mandelbrot set can be made by colouring all the points c which belong to M black, and all other points white. The more colourful pictures usually seen are generated by colouring points not in the set according to how quickly or slowly the sequence $|P_c^n(0)|$ diverges to infinity. See the section on computer drawings below for more details.

The Mandelbrot set can also be defined as the connectedness locus of the family of polynomials $P_c(z)$. That is, it is the subset of the complex plane consisting of those parameters c for which the Julia set of P_c is connected.

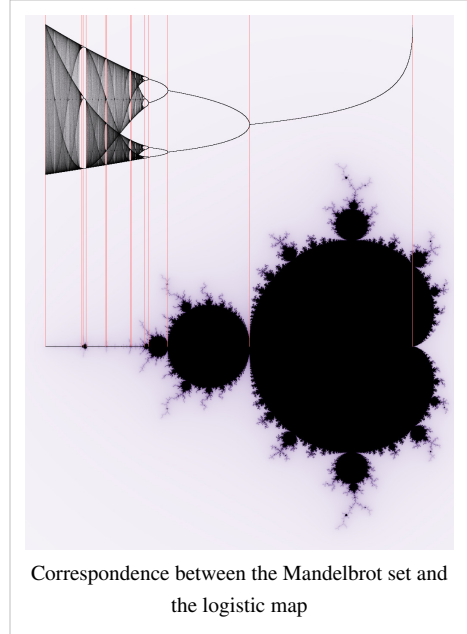
Basic properties

The Mandelbrot set is a compact set, contained in the closed disk of radius 2 around the origin. In fact, a point c belongs to the Mandelbrot set if and only if

$$|P_c^n(0)| \leq 2 \text{ for all } n \geq 0.$$

In other words, if the absolute value of $P_c^n(0)$ ever becomes larger than 2, the sequence will escape to infinity.

The intersection of M with the real axis is precisely the interval $[-2, 0.25]$. The parameters along this interval can be put in one-to-one correspondence with those of the real logistic family,



$$z \mapsto \lambda z(1 - z), \quad \lambda \in [1, 4].$$

The correspondence is given by

$$c = \frac{\lambda}{2} \left(1 - \frac{\lambda}{2} \right).$$

In fact, this gives a correspondence between the entire parameter space of the logistic family and that of the Mandelbrot set.

The area of the Mandelbrot set is estimated to be $1.50659177 \pm 0.00000008$.^[13]

Douady and Hubbard have shown that the Mandelbrot set is connected. In fact, they constructed an explicit conformal isomorphism between the complement of the Mandelbrot set and the complement of the closed unit disk. Mandelbrot had originally conjectured that the Mandelbrot set is disconnected. This conjecture was based on computer pictures generated by programs which are unable to detect the thin filaments connecting different parts of M . Upon further experiments, he revised his conjecture, deciding that M should be connected.

The dynamical formula for the uniformisation of the complement of the Mandelbrot set, arising from Douady and Hubbard's proof of the connectedness of M , gives rise to external rays of the Mandelbrot set. These rays can be used to study the Mandelbrot set in combinatorial terms and form the backbone of the Yoccoz parapuzzle.^[14]

The boundary of the Mandelbrot set is exactly the bifurcation locus of the quadratic family; that is, the set of parameters c for which the dynamics changes abruptly under small changes of c . It can be constructed as the limit set of a sequence of plane algebraic curves, the *Mandelbrot curves*, of the general type known as polynomial lemniscates. The Mandelbrot curves are defined by setting $p_0 = z$, $p_n = p_{n-1}^2 + z$, and then interpreting the set of points $|p_n(z)| = 2$ in the complex plane as a curve in the real Cartesian plane of degree 2^{n+1} in x and y .

Other properties

The main cardioid and period bulbs

Upon looking at a picture of the Mandelbrot set, one immediately notices the large cardioid-shaped region in the center. This *main cardioid* is the region of parameters c for which P_c has an attracting fixed point. It consists of all parameters of the form

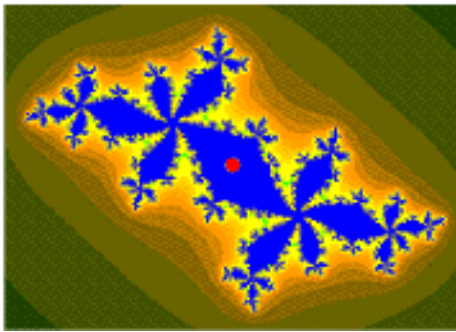
$$c = \frac{\mu}{2} \left(1 - \frac{\mu}{2} \right)$$

for some μ in the open unit disk.

To the left of the main cardioid, attached to it at the point $c = -3/4$, a circular-shaped bulb is visible. This bulb consists of those parameters c for which P_c has an attracting cycle of period 2. This set of parameters is an actual circle, namely that of radius $1/4$ around -1 .

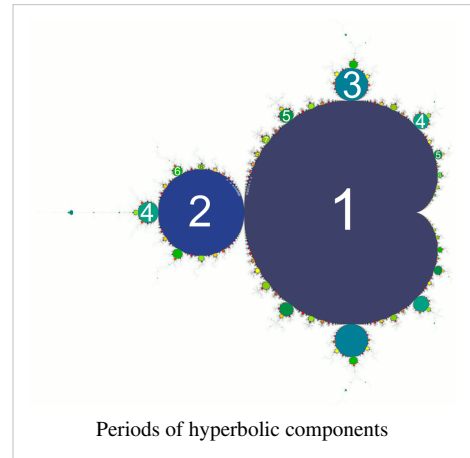
There are infinitely many other bulbs tangent to the main cardioid: for every rational number $\frac{p}{q}$, with p and q coprime, there is such a bulb that is tangent at the parameter

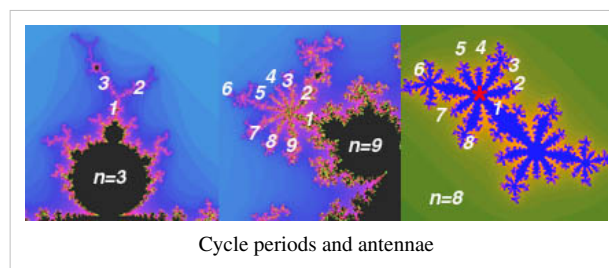
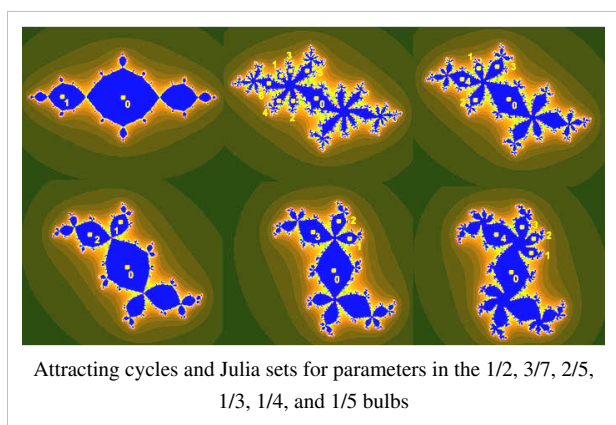
$$c_{\frac{p}{q}} = \frac{e^{2\pi i \frac{p}{q}}}{2} \left(1 - \frac{e^{2\pi i \frac{p}{q}}}{2} \right).$$



Attracting cycle in 2/5-bulb plotted over Julia set
(animation)

This bulb is called the $\frac{p}{q}$ -*bulb* of the Mandelbrot set. It consists of parameters which have an attracting cycle of period q and combinatorial rotation number $\frac{p}{q}$. More precisely, the q periodic Fatou components containing the attracting cycle all touch at a common point (commonly called the α -fixed point). If we label these components U_0, \dots, U_{q-1} in counterclockwise orientation, then P_c maps the component U_j to the component $U_{j+p \pmod{q}}$.





Hyperbolic components

All the bulbs we encountered in the previous section were interior components of the Mandelbrot set in which the maps P_c have an attracting periodic cycle. Such components are called *hyperbolic components*.

It is conjectured that these are the *only* interior regions of \mathcal{M} . This problem, known as *density of hyperbolicity*, may be the most important open problem in the field of complex dynamics. Hypothetical non-hyperbolic components of the Mandelbrot set are often referred to as "queer" components.

For *real* quadratic polynomials, this question was answered positively in the 1990s independently by Lyubich and by Graczyk and Świątek. (Note that hyperbolic components intersecting the real axis correspond exactly to periodic windows in the Feigenbaum diagram. So this result states that such windows exist near every parameter in the diagram.)

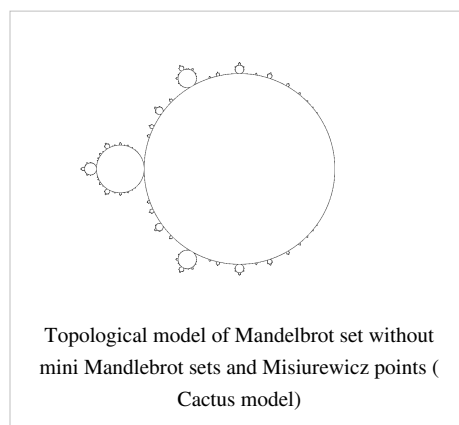
Not every hyperbolic component can be reached by a sequence of direct bifurcations from the main cardioid of the Mandelbrot set. However, such a component *can* be reached by a sequence of direct bifurcations from the main cardioid of a little Mandelbrot copy (see below).

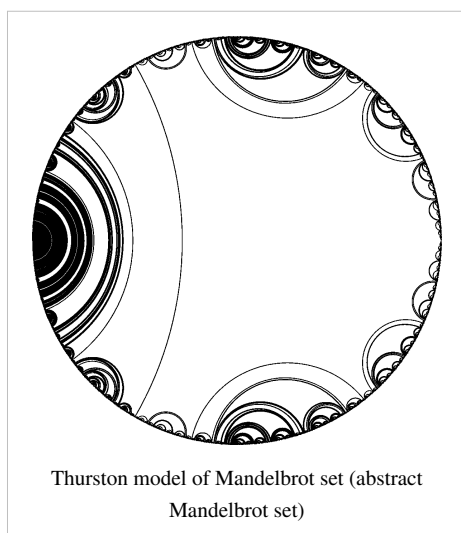
Each of the hyperbolic components has a *centre*, namely the point c such that the inner Fatou domain for $P_c(z)$ has a super-attracting cycle (the attraction is infinite). This means that the cycle contains the critical point 0, so that 0 is iterated back to itself after some iterations. We therefore have that $P_c^n(0) = 0$ for some n . If we call this polynomial $Q^n(c)$, we have that $Q^{n+1}(c) = Q^n(c)^2 + 1$ and that the degree of $Q^n(c)$ is 2^{n-1} . We can therefore construct the centres of the hyperbolic components, by successive solvation of the equations $Q^n(c) = 0, n = 1, 2, 3, \dots$. Note that for each step, we get just as many new centres as we have found so far.

Local connectivity

It is conjectured that the Mandelbrot set is locally connected. This famous conjecture is known as *MLC* (for *Mandelbrot Locally Connected*). By the work of Adrien Douady and John H. Hubbard, this conjecture would result in a simple abstract "pinched disk" model of the Mandelbrot set. In particular, it would imply the important *hyperbolicity conjecture* mentioned above.

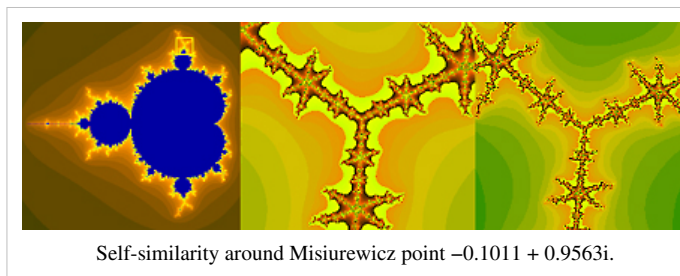
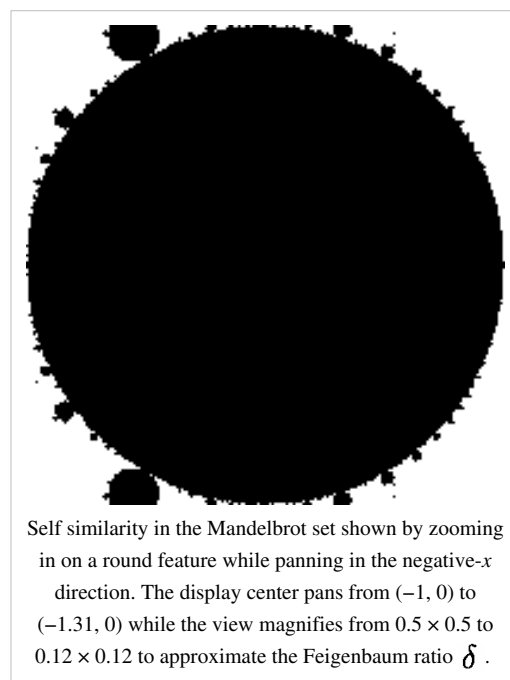
The work of Jean-Christophe Yoccoz established local connectivity of the Mandelbrot set at all finitely-renormalizable parameters; that is, roughly speaking those which are contained only in finitely many small Mandelbrot copies. Since then, local connectivity has been proved at many other points of \mathcal{M} , but the full conjecture is still open.

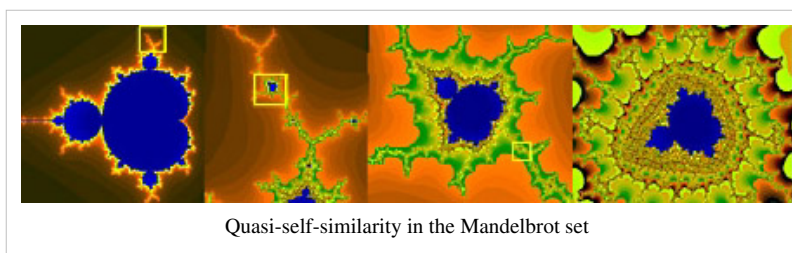




Self-similarity

The Mandelbrot set is self-similar under magnification in the neighborhoods of the Misiurewicz points. It is also conjectured to be self-similar around generalized Feigenbaum points (e.g. -1.401155 or $-0.1528 + 1.0397i$), in the sense of converging to a limit set.^{[15] [16]}





The Mandelbrot set in general is not strictly self-similar but it is quasi-self-similar, as small slightly different versions of itself can be found at arbitrarily small scales.

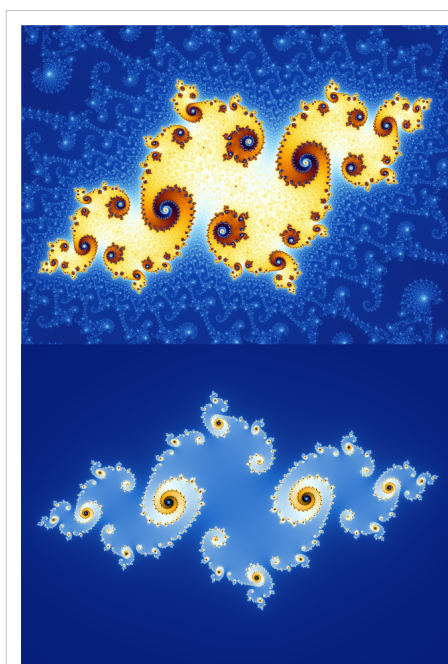
The little copies of the Mandelbrot set are all slightly different, mostly

because of the thin threads connecting them to the main body of the set.

Further results

The Hausdorff dimension of the boundary of the Mandelbrot set equals 2 as determined by a result of Mitsuhiro Shishikura.^[17] It is not known whether the boundary of the Mandelbrot set has positive planar Lebesgue measure.

In the Blum-Shub-Smale model of real computation, the Mandelbrot set is not computable, but its complement is computably enumerable. However, many simple objects (e.g., the graph of exponentiation) are also not computable in the BSS model. At present it is unknown whether the Mandelbrot set is computable in models of real computation based on computable analysis, which correspond more closely to the intuitive notion of "plotting the set by a computer." Hertling has shown that the Mandelbrot set is computable in this model if the hyperbolicity conjecture is true.



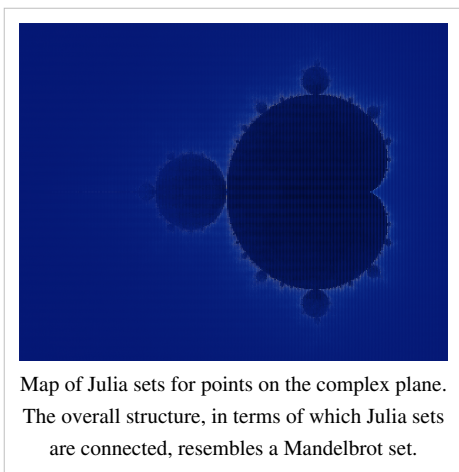
A zoom into the Mandelbrot set illustrating a Julia "island" and the corresponding Julia set of the form $f_c(z) = z^2 + c$, in which c is the centre of the Mandelbrot set zoom in.

Relationship with Julia sets

As a consequence of the definition of the Mandelbrot set, there is a close correspondence between the geometry of the Mandelbrot set at a given point and the structure of the corresponding Julia set.

This principle is exploited in virtually all deep results on the Mandelbrot set. For example, Shishikura proves that, for a dense set of parameters in the boundary of the Mandelbrot set, the Julia set has Hausdorff dimension two, and then transfers this information to the parameter plane. Similarly, Yoccoz first proves the local connectivity of Julia sets, before establishing it for the Mandelbrot set at the corresponding parameters. Adrien Douady phrases this principle as

Plough in the dynamical plane, and harvest in parameter space.



Geometry

Recall that, for every rational number $\frac{p}{q}$, where p and q are relatively prime, there is a hyperbolic component of period q bifurcating from the main cardioid. The part of the Mandelbrot set connected to the main cardioid at this bifurcation point is called the **p/q -limb**. Computer experiments suggest that the diameter of the limb tends to zero like $\frac{1}{q^2}$. The best current estimate known is the famous *Yoccoz-inequality*, which states that the size tends to zero like $\frac{1}{q}$.

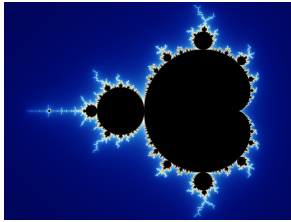
A period- q limb will have $q - 1$ "antennae" at the top of its limb. We can thus determine the period of a given bulb by counting these antennas.

In an attempt to demonstrate that the thickness of the p/q -limb is zero, David Boll carried out a computer experiment in 1991, where he computed the number of iterations required for the series to converge for $z = -\frac{3}{4} + i\epsilon (-\frac{3}{4}$ being the location thereof). As the series doesn't converge for the exact value of $z = -\frac{3}{4}$, the number of iterations required increases with a small ϵ . It turns out that multiplying the value of ϵ with the number of iterations required yields an approximation of π that becomes better the smaller ϵ . For example, for $\epsilon = 0.0000001$ the number of iterations is 31415928 and the product is 3.1415928.^[18]

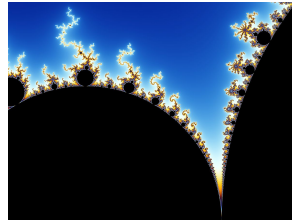
Image gallery of a zoom sequence

The Mandelbrot set shows more intricate detail the closer one looks or magnifies the image, usually called "zooming in". The following example of an image sequence zooming to a selected c value gives an impression of the infinite richness of different geometrical structures, and explains some of their typical rules.

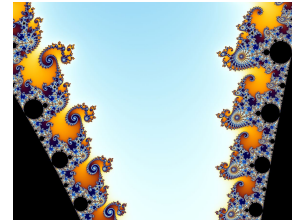
The magnification of the last image relative to the first one is about 10,000,000,000 to 1. Relating to an ordinary monitor, it represents a section of a Mandelbrot set with a diameter of 4 million kilometres. Its border would show an astronomical number of different fractal structures.



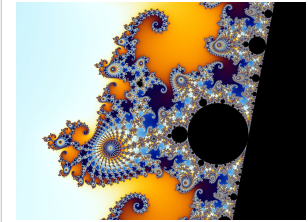
Start. Mandelbrot set with continuously coloured environment.



Gap between the "head" and the "body" also called the "seahorse valley".

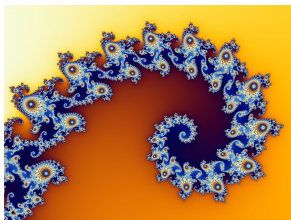


On the left double-spirals, on the right "seahorses".

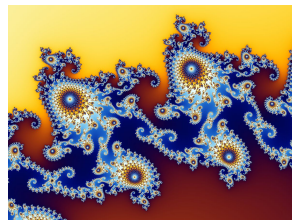


"Seahorse" upside down.

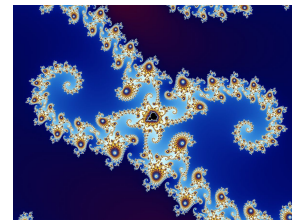
The seahorse "body" is composed by 25 "spokes" consisting of 2 groups of 12 "spokes" each and one "spoke" connecting to the main cardioid. These 2 groups can be attributed by some kind of metamorphosis to the 2 "fingers" of the "upper hand" of the Mandelbrot set, therefore, the number of "spokes" increases from one "seahorse" to the next by 2; the "hub" is a so-called Misiurewicz point. Between the "upper part of the body" and the "tail" a distorted small copy of the Mandelbrot set called satellite may be recognized.



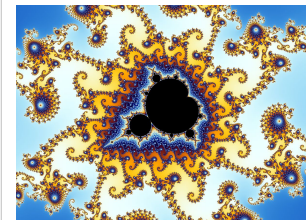
The central endpoint of the "seahorse tail" is also a Misiurewicz point.



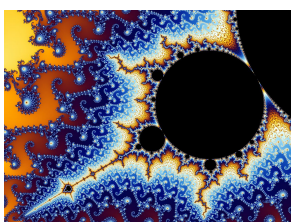
Part of the "tail" — there is only one path consisting of the thin structures that lead through the whole "tail". This zigzag path passes the "hubs" of the large objects with 25 "spokes" at the inner and outer border of the "tail"; thus the Mandelbrot set is a simply connected set, which means there are no islands and no loop roads around a hole.



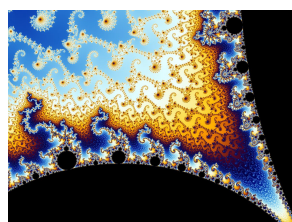
Satellite. The two "seahorse tails" are the beginning of a series of concentric crowns with the satellite in the center. Open this location in an interactive viewer. [19]



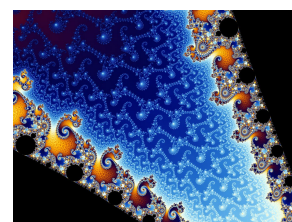
Each of these crowns consists of similar "seahorse tails"; their number increases with powers of 2, a typical phenomenon in the environment of satellites. The unique path to the spiral center passes the satellite from the groove of the cardioid to the top of the "antenna" on the "head".



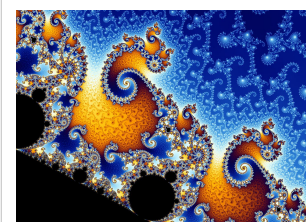
"Antenna" of the satellite. Several satellites of second order may be recognized.



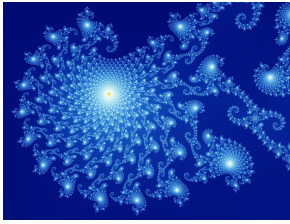
The "seahorse valley" of the satellite. All the structures from the start of the zoom reappear.



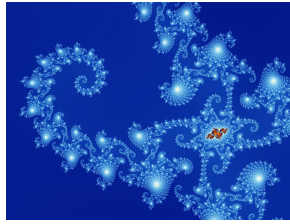
Double-spirals and "seahorses" - unlike the 2nd image from the start they have appendices consisting of structures like "seahorse tails"; this demonstrates the typical linking of $n+1$ different structures in the environment of satellites of the order n , here for the simplest case $n=1$.



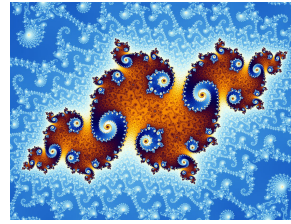
Double-spirals with satellites of second order - analog to the "seahorses" the double-spirals may be interpreted as a metamorphosis of the "antenna".



In the outer part of the appendices islands of structures may be recognized; they have a shape like Julia sets J_c ; the largest of them may be found in the center of the "double-hook" on the right side.



Part of the "double-hook".



Islands, see below.

The islands above seem to consist of infinitely many parts like Cantor sets, as is actually the case for the corresponding Julia set J_c . However they are connected by tiny structures so that the whole represents a simply connected set. The tiny structures meet each other at a satellite in the center that is too small to be recognized at this magnification. The value of c for the corresponding J_c is not that of the image center but, relative to the main body of the Mandelbrot set, has the same position as the center of this image relative to the satellite shown in the 7th zoom step.

Generalizations

Multibrot sets are bounded sets found in the complex plane for members of the general monic univariate polynomial family of recursions

$$z \mapsto z^d + c.$$

For integer d , these sets are connectedness loci for the Julia sets built from the same formula. The full cubic connectedness map has also been studied; here one considers the two-parameter recursion $z \mapsto z^3 + 3kz + c$, whose two critical points are the complex square roots of the parameter k . A point is in the map if either critical point is stable.^[20]

For general families of holomorphic functions, the *boundary* of the Mandelbrot set generalizes to the bifurcation locus, which is a natural object to study even when the connectedness locus is not useful.

Other non-analytic mappings

Of particular interest is the tricorn fractal, the connectedness locus of the anti-holomorphic family

$$z \mapsto \bar{z}^2 + c.$$

The tricorn (also sometimes called the *Mandelbar set*) was encountered by Milnor in his study of parameter slices of real cubic polynomials. It is *not* locally connected. This property is inherited by the connectedness locus of real cubic polynomials.

Another non-analytic generalization is the Burning Ship fractal which is obtained by iterating the mapping

$$z \mapsto (|\Re(z)| + i|\Im(z)|)^2 + c.$$

The Multibrot set is obtained by varying the value of the exponent d . The article has a video that shows the development from $d = 0$ to 7 at which point there are 6 i.e. $(d - 1)$ lobes around the perimeter. A similar development with negative exponents results in $(1 - d)$ clefts on the inside of a ring.

Computer drawings

There are many programs used to generate the Mandelbrot set and other fractals, some of which are described in fractal-generating software. These programs use a variety of algorithms to determine the color of individual pixels and achieve efficient computation.

Escape time algorithm

The simplest algorithm for generating a representation of the Mandelbrot set is known as the "escape time" algorithm. A repeating calculation is performed for each x, y point in the plot area and based on the behaviour of that calculation, a colour is chosen for that pixel.

The x and y locations of each point are used as starting values in a repeating, or iterating calculation (described in detail below). The result of each iteration is used as the starting values for the next. The values are checked during each iteration to see if they have reached a critical 'escape' condition or 'bailout'. If that condition is reached, the calculation is stopped, the pixel is drawn, and the next x, y point is examined. For some starting values, escape occurs quickly, after only a small number of iterations. For starting values very close to but not in the set, it may take hundreds or thousands of iterations to escape. For values within the Mandelbrot set, escape will never occur. The programmer or user must choose how much iteration, or 'depth,' they wish to examine. The higher the maximum number of iterations, the more detail and subtlety emerge in the final image, but the longer time it will take to calculate the fractal image.

Escape conditions can be simple or complex. Because no complex number with a real or imaginary part greater than 2 can be part of the set, a common bailout is to escape when either coefficient exceeds 2. A more computationally complex method, but which detects escapes sooner, is to compute the distance from the origin using the Pythagorean theorem, and if this distance exceeds two, the point has reached escape. More computationally intensive rendering variations such as Buddhabrot detect an escape, then use values iterated along the way.

The colour of each point represents how quickly the values reached the escape point. Often black is used to show values that fail to escape before the iteration limit, and gradually brighter colours are used for points that escape. This gives a visual representation of how many cycles were required before reaching the escape condition.

For programmers

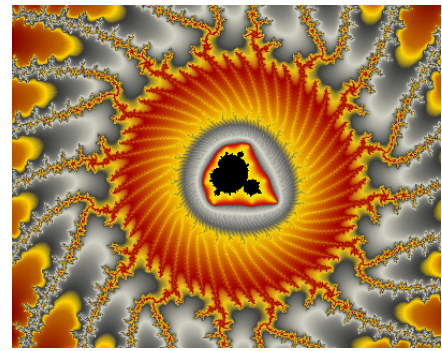
The definition of the Mandelbrot set, together with its basic properties, suggests a simple algorithm for drawing a picture of the Mandelbrot set. The region of the complex plane we are considering is subdivided into a certain number of pixels. To color any such pixel, let c be the midpoint of that pixel. We now iterate the critical point 0 under P_c , checking at each step whether the orbit point has modulus larger than 2.

When this is the case, we know that c does not belong to the Mandelbrot set, and we color our pixel according to the number of iterations used to find out. Otherwise, we keep iterating up to a fixed number of steps, after which we decide that our parameter is "probably" in the Mandelbrot set, or at least very close to it, and color the pixel black.

In pseudocode, this algorithm would look as follows. The algorithm does not use complex numbers, and manually simulates complex number operations using two real numbers, for those who do not have a complex data type. If you have a complex data type in your programming language, using it can simplify your program.

```
For each pixel on the screen do:
```

```
{
    x0 = scaled x co-ordinate of pixel (must be scaled to lie somewhere in the interval (-2.5 to 1))
    y0 = scaled y co-ordinate of pixel (must be scaled to lie somewhere in the interval (-1, 1))
```



Still image of a movie of increasing magnification on $0.001643721971153 + 0.822467633298876i$

```

x = 0
y = 0

iteration = 0
max_iteration = 1000

while ( x*x + y*y <= (2*2)  AND  iteration < max_iteration )
{
    xtemp = x*x - y*y + x0
    y = 2*x*y + y0

    x = xtemp

    iteration = iteration + 1
}

if ( iteration == max_iteration )
then
    color = black
else
    color = iteration

plot(x0,y0,color)
}

```

where, relating the pseudocode to c , z and P_c :

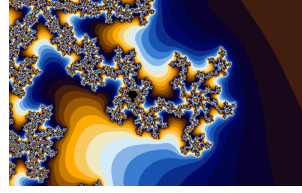
- $z = x + iy$
- $z^2 = x^2 + i2xy - y^2$
- $c = x_0 + iy_0$

and so, as can be seen in the pseudocode in the computation of x and y :

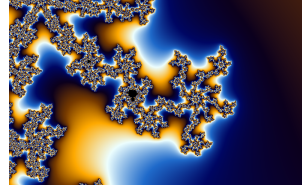
- $x = \operatorname{Re}(z^2 + c) = x^2 - y^2 + x_0$ and $y = \operatorname{Im}(z^2 + c) = 2xy + y_0$.

To get colorful images of the set, the assignment of a color to each value of the number of executed iterations can be made using one of a variety of functions (linear, exponential, etc.). One practical way to do it, without slowing down the calculations, is to use the number of executed iterations as an entry to a look-up color palette table initialized at startup. If the color table has, for instance, 500 entries, then you can use $n \bmod 500$, where n is the number of iterations, to select the color to use. You can initialize the color palette matrix in various different ways, depending on what special feature of the escape behavior you want to emphasize graphically.

Continuous (smooth) coloring



This image was rendered with the Escape Time Algorithm. Notice the very obvious "bands" of color.



This image was rendered with the Normalized Iteration Count Algorithm. Notice the bands of color have been replaced by a smooth gradient. Also, the colors take on the same pattern that would be observed if the Escape Time Algorithm was used.

The Escape Time Algorithm is popular for its simplicity. However, it creates bands of color, which, as a type of aliasing, can detract from an image's aesthetic value. This can be improved using an algorithm known as "Normalized Iteration Count",^{[21] [22]} which provides a smooth transition of colors between iterations. The algorithm associates a real number ν with each value of z by using the connection of the iteration number with the potential function. This function is given by

$$\phi(z) = \lim_{n \rightarrow \infty} (\log |z_n| / P^n),$$

where z_n is the value after n iterations and P is the power for which z is raised to in the Mandelbrot set equation ($z_{n+1} = z_n^P + c$, P is generally 2).

If we choose a large bailout radius N (e.g. 10^{100}), we have that

$$\log |z_n| / P^n = \log(N) / P^{\nu(z)},$$

for some real number $\nu(z)$, and this is

$$\nu(z) = n - \log_P(\log |z_n| / \log(N)),$$

and as n is the first iteration number such that $|z_n| > N$, the number we subtract from n is in the interval $[0, 1)$.

For the colouring we must have a cyclic scale of colours (constructed mathematically, for instance) and containing H colours numbered from 0 to $H - 1$ ($H = 500$, for instance). We multiply the real number $\nu(z)$ by a fixed real number determining the density of the colours in the picture, and take the integral part of this number modulo H .

Distance estimates

One can compute the distance from point c (in exterior or interior) to nearest point on the boundary of Mandelbrot set.^[23]

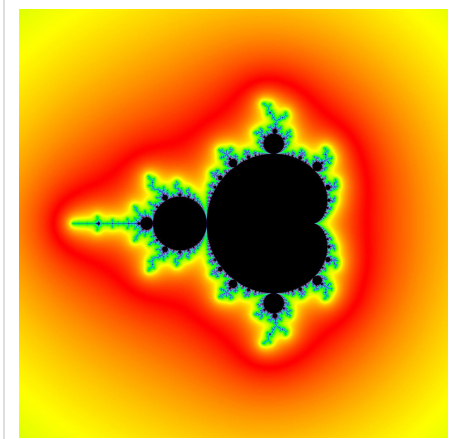
Exterior distance estimation

The proof of the connectedness of the Mandelbrot set in fact gives a formula for the uniformizing map of the complement of M (and the derivative of this map). By the Koebe 1/4 theorem, one can then estimate the distance between the mid-point of our pixel and the Mandelbrot set up to a factor of 4.

In other words, provided that the maximal number of iterations is sufficiently high, one obtains a picture of the Mandelbrot set with the following properties:

1. Every pixel which contains a point of the Mandelbrot set is colored black.
2. Every pixel which is colored black is close to the Mandelbrot set.

The distance estimate b of a pixel c (a complex number) from the Mandelbrot set is given by



Exterior distance estimate may be used to color whole complement of Mandelbrot set

$$b = \lim_{n \rightarrow \infty} 2 \cdot \frac{|P_c^n(c)| \cdot \ln |P_c^n(c)|}{\left| \frac{\partial}{\partial c} P_c^n(c) \right|}$$

where

- $P_c(z)$ stands for complex quadratic polynomial
- $P_c^n(c)$ stands for n iterations of $P_c(z) \rightarrow z$ or $z^2 + c \rightarrow z$, starting with $z = c$: $P_c^0(c) = c$,
 $P_c^{n+1}(c) = P_c^n(c)^2 + c$;
- $\frac{\partial}{\partial c} P_c^n(c)$ is the derivative of $P_c^n(c)$ with respect to c . This derivative can be found by starting with $\frac{\partial}{\partial c} P_c^0(c) = 1$ and then $\frac{\partial}{\partial c} P_c^{n+1}(c) = 2 \cdot P_c^n(c) \cdot \frac{\partial}{\partial c} P_c^n(c) + 1$. This can easily be verified by using the chain rule for the derivative.

The idea behind this formula is simple: When the equipotential lines for the potential function $\phi(z)$ lie close, the number $|\phi'(z)|$ is large, and conversely, therefore the equipotential lines for the function $\phi(z)/|\phi'(z)|$ should lie approximately regularly.

From a mathematician's point of view, this formula only works in limit where n goes to infinity, but very reasonable estimates can be found with just a few additional iterations after the main loop exits.

Once b is found, by the Koebe 1/4-theorem, we know there's no point of the Mandelbrot set with distance from c smaller than $b/4$.

The distance estimation can be used for drawing of the boundary of the Mandelbrot set, see the article Julia set.

Interior distance estimation

It is also possible to estimate the distance of a limitly periodic (i.e., inner) point to the boundary of the Mandelbrot set. The estimate is given by

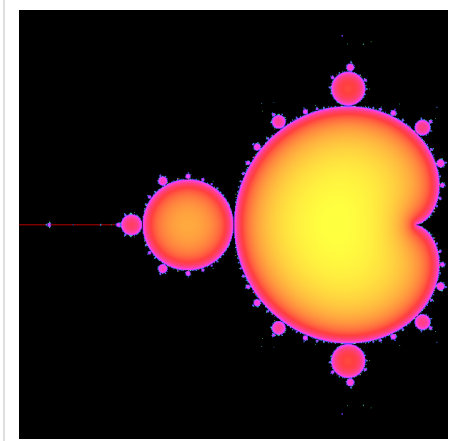
$$b = \frac{1 - \left| \frac{\partial}{\partial z} P_c^p(z_0) \right|^2}{\left| \frac{\partial}{\partial c} \frac{\partial}{\partial z} P_c^p(z_0) + \frac{\partial}{\partial z} \frac{\partial}{\partial z} P_c^p(z_0) \frac{\frac{\partial}{\partial c} P_c^p(z_0)}{1 - \frac{\partial}{\partial z} P_c^p(z_0)} \right|}$$

where

- p is the period,
- c is the point to be estimated,
- $P_c(z)$ is the complex quadratic polynomial $P_c(z) = z^2 + c$
- $P_c^p(z_0)$ is the p -fold iteration of $P_c(z) \rightarrow z$, starting with $P_c^0(z) = z_0$
- z_0 is any of the p points that make the attractor of the iterations of $P_c(z) \rightarrow z$ starting with $P_c^0(z) = c$; z_0 satisfies $z_0 = P_c^p(z_0)$,
- $\frac{\partial}{\partial c} \frac{\partial}{\partial z} P_c^p(z_0)$, $\frac{\partial}{\partial z} \frac{\partial}{\partial z} P_c^p(z_0)$, $\frac{\partial}{\partial c} P_c^p(z_0)$ and $\frac{\partial}{\partial z} P_c^p(z_0)$ are various derivatives of $P_c^p(z)$, evaluated at z_0 .

Analogous to the exterior case, once b is found, we know that all points within the distance of $b/4$ from c are inside the Mandelbrot set.

There are two practical problems with the interior distance estimate: first, we need to find z_0 precisely, and second, we need to find p precisely. The problem with z_0 is that the convergence to z_0 by iterating $P_c(z)$ requires, theoretically, an infinite number of operations. The problem with period is that, sometimes, due to rounding errors, a period is falsely identified to be an integer multiple of the real period (e.g., a period of 86 is detected, while the real period is only $43=86/2$). In such case, the distance is overestimated, i.e., the reported radius could contain points outside the Mandelbrot set.



Pixels colored according to the estimated interior distance

Optimizations

One way to improve calculations is to find out beforehand whether the given point lies within the cardioid or in the period-2 bulb. Before passing the complex value through the escape time algorithm, first check if:

$$p = \sqrt{\left(x - \frac{1}{4}\right)^2 + y^2}$$

$$x < p - 2p^2 + \frac{1}{4}$$

$$(x + 1)^2 + y^2 < \frac{1}{16}$$

where x represents the real value of the point and y the imaginary value. The first two equations determine if the point is within the cardioid, the last the period-2 bulb.

The cardioid test can equivalently be performed without the square root:

$$q = \left(x - \frac{1}{4}\right)^2 + y^2$$

$$q \left(q + \left(x - \frac{1}{4}\right)\right) < \frac{1}{4}y^2.$$

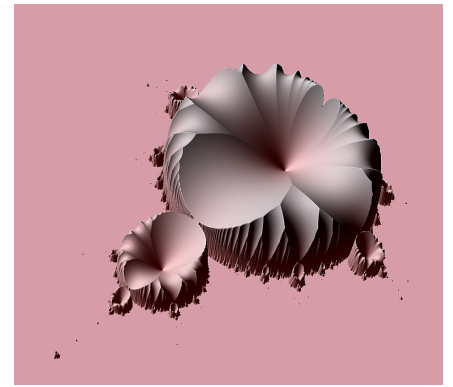
3rd- and higher-order buds do not have equivalent tests, because they are not perfectly circular.^[24]

To prevent having to do huge numbers of iterations for other points in the set, one can do "periodicity checking"; which means check if a point reached in iterating a pixel has been reached before. If so, the pixel cannot diverge, and must be in the set. This is most relevant for fixed-point calculations, where there is a relatively high chance of such periodicity—a full floating-point (or higher-accuracy) implementation would rarely go into such a period.

Periodicity checking is, of course, a trade-off. The need to remember points costs memory and *data management* instructions, whereas it saves *computational* instructions.

Popular culture

- The Mandelbrot set shape was used by Heart as artwork for their 2004 album, *Jupiter's Darling*.
- The Jonathan Coulton song, "Mandelbrot Set", is a tribute to both the fractal itself, and to its father Benoît Mandelbrot. However, the definition given in the song describes the orbit of some arbitrary point on the complex plane, instead of the orbit of 0.
- The second book of the Mode series, *Fractal Mode*, describes an entire *world* that's a perfect 3d model of the set.
- In *The Ghost from the Grand Banks* by Arthur C. Clarke, the Craig family is obsessed with the Mandelbrot set, even having a pond in its shape dug. However Clarke's descriptions of the M-set are incorrect.^[25]
- Fractals are referenced by Blue Man Group in their stage performances and used in song names on their album *Audio* and two of the songs have "Mandelbrot" in the title.



3D view : smallest absolute value of the orbit of the interior points of the Mandelbrot set

- The alternative cover to Outkast's 2000 album *Stankonia* features the Mandelbrot set.
- The movie *Blueberry* employs fractal animations in the title sequence and throughout the movie as scene change intermissions. The movie director Jan Kounen is generally fond of psychedelia.
- The Mandelbrot set is found on the wings of the fictional Quantum Weather Butterfly in Terry Pratchett's Discworld series.

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- [24] "Mandelbrot Bud Maths" (<http://linas.org/art-gallery/bud/bud.html>). .
- [25] Kasman, Alex. "MathFiction: The Ghost from the Grand Banks (Arthur C. Clarke)" (<http://kasmana.people.cofc.edu/MATHFICT/mfview.php?callnumber=mf12>). . Retrieved 2010-10-18.

Further reading

- John W. Milnor, *Dynamics in One Complex Variable* (Third Edition), Annals of Mathematics Studies 160, (Princeton University Press, 2006), ISBN 0-691-12488-4
(First appeared in 1990 as a Stony Brook IMS Preprint (<http://www.math.sunysb.edu/preprints.html>), available as arXiv:math.DS/9201272 (<http://www.arxiv.org/abs/math.DS/9201272>))
- Nigel Lesmoir-Gordon, *The Colours of Infinity: The Beauty, The Power and the Sense of Fractals*, ISBN 1-904555-05-5
(includes a DVD featuring Arthur C. Clarke and David Gilmour)
- Heinz-Otto Peitgen, Hartmut Jürgens, Dietmar Saupe, *Chaos and Fractals - New Frontiers of Science* (Springer, New-York, 1992, 2004), ISBN 0-387-20229-3
- Eric Baird, *Alt.Fractals: A visual guide to fractal geometry and design* (Chocolate Tree Books, Brighton, 2011), ISBN 0955706831

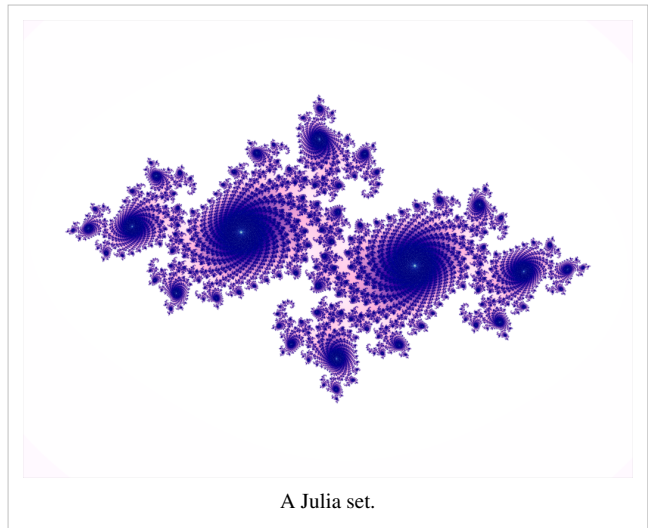
External links

- Chaos and Fractals (http://www.dmoz.org/Science/Math/Chaos_and_Fractals/) at the Open Directory Project
- The Mandelbrot Set and Julia Sets by Michael Frame, Benoit Mandelbrot, and Nial Neger (<http://classes.yale.edu/Fractals/MandelSet/welcome.html>)
- For Fractal Design and Consultancy (<http://www.fractal.org>)
- Mandelbulb/Julia bulb/Julius bulb (<http://www.fractal.org/Mandelbulb.pdf>)
- Video: Mandelbrot fractal zoom to 6.066 e228 (<http://vimeo.com/12185093>)
- मण्डलबेथ (maṇḍalabeth) (<http://www-personal.umich.edu/~bethchen/mandalabeth/>) 3D analog of the mandelbrot set, with various symmetry groups

Julia set

In the context of complex dynamics, a topic of mathematics, the **Julia set** and the **Fatou set** are two complementary sets defined from a function. Informally, the Fatou set of the function consists of values with the property that all nearby values behave similarly under repeated iteration of the function, and the Julia set consists of values such that an arbitrarily small perturbation can cause drastic changes in the sequence of iterated function values. Thus the behavior of the function on the Fatou set is 'regular', while on the Julia set its behavior is 'chaotic'.

The Julia set of a function f is commonly denoted $J(f)$, and the Fatou set is denoted $F(f)$.^[1] These sets are named after the French mathematicians Gaston Julia^[2] and Pierre Fatou,^[3] whose work began the study of complex dynamics during the early 20th century.



A Julia set.

Formal definition

Let $f(z)$ be a complex rational map from the plane into itself, that is, $f(z) = p(z)/q(z)$, where $p(z)$ and $q(z)$ are complex polynomials. Then there are a finite number of open sets $F_i, i = 1, \dots, r$, that are left invariant by $f(z)$ and are such that:

1. the union of the F_i 's is dense in the plane and
2. $f(z)$ behaves in a regular and equal way on each of the sets F_i .

The last statement means that the termini of the sequences of iterations generated by the points of F_i are either precisely the same set, which is then a finite cycle, or they are finite cycles of finite or annular shaped sets that are lying concentrically. In the first case the cycle is *attracting*, in the second it is *neutral*.

These sets F_i are the **Fatou domains** of $f(z)$, and their union is the **Fatou set** $F(f)$ of $f(z)$. Each of the Fatou domains contains at least one critical point of $f(z)$, that is, a (finite) point z satisfying $f'(z) = 0$, or $z = \infty$, if the degree of the numerator $p(z)$ is at least two larger than the degree of the denominator $q(z)$, or if $f(z) = 1/g(z) + c$ for some c and a rational function $g(z)$ satisfying this condition.

The complement of $F(f)$ is the **Julia set** $J(f)$ of $f(z)$. $J(f)$ is a nowhere dense set (it is without interior points) and an uncountable set (of the same cardinality as the real numbers). Like $F(f)$, $J(f)$ is left invariant by $f(z)$, and on this set the iteration is repelling, meaning that $|f(z) - f(w)| > |z - w|$ for all w in a neighbourhood of z (within $J(f)$). This means that $f(z)$ behaves chaotically on the Julia set. Although there are points in the Julia set whose sequence of iterations is finite, there are only a countable number of such points (and they make up an infinitely small part of the Julia set). The sequences generated by points outside this set behave chaotically, a phenomenon called *deterministic chaos*.

There has been extensive research on the Fatou set and Julia set of iterated rational functions, known as rational maps. For example, it is known that the Fatou set of a rational map has either 0, 1, 2 or infinitely many components.^[4] Each component of the Fatou set of a rational map can be classified into one of four different classes.^[5]

Equivalent descriptions of the Julia set

- $J(f)$ is the smallest closed set containing at least three points which is completely invariant under f .
- $J(f)$ is the closure of the set of repelling periodic points.
- For all but at most two points $z \in X$, the Julia set is the set of limit points of the full backwards orbit $\bigcup_n f^{-n}(z)$. (This suggests a simple algorithm for plotting Julia sets, see below.)
- If f is an entire function - in particular, when f is a polynomial, then $J(f)$ is the boundary of the set of points which converge to infinity under iteration.
- If f is a polynomial, then $J(f)$ is the boundary of the filled Julia set; that is, those points whose orbits under iterations of f remain bounded.

Properties of the Julia set and Fatou set

The Julia set and the Fatou set of f are both completely invariant under iterations of the holomorphic function f , i.e.

$$f^{-1}(J(f)) = f(J(f)) = J(f)$$

and

$$f^{-1}(F(f)) = f(F(f)) = F(f).^{[6]}$$

Examples

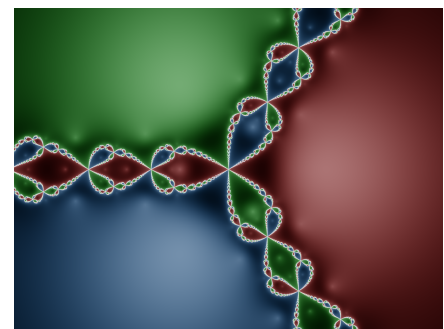
For $f(z) = z^2$ the Julia set is the unit circle and on this the iteration is given by doubling of angles (an operation that is chaotic on the non-rational points). There are two Fatou domains: the interior and the exterior of the circle, with iteration towards 0 and ∞ , respectively.

For $f(z) = z^2 - 2$ the Julia set is the line segment between -2 and 2, and the iteration corresponds to $x \rightarrow 4(x - 0.5)^2$ in the unit interval. This can be used as a method for generating pseudorandom numbers. There is one Fatou domain: the points not on the line segment iterate towards ∞ .

These two functions are of the form $z^2 + c$, where c is a complex number. For such an iteration the Julia set is not in general a simple curve, but is a fractal, and for some values of c it can take surprising shapes. See the pictures below.

For some functions $f(z)$ we can say beforehand that the Julia set is a fractal and not a simple curve. This is because of the following main theorem on the iterations of a rational function:

Each of the Fatou domains has the same boundary, which consequently is the Julia set



Julia set (in white) for the rational function associated to Newton's method for $f: z \rightarrow z^3 - 1$. Coloring of Fatou set according to attractor (the roots of f)

This means that each point of the Julia set is a point of accumulation for each of the Fatou domains. Therefore, if there are more than two Fatou domains, *each* point of the Julia set must have points of more than two different open sets infinitely close, and this means that the Julia set cannot be a simple curve. This phenomenon happens, for instance, when $f(z)$ is the Newton iteration for solving the equation $z^n = 1$ ($n > 2$):

$f(z) = z - f(z)/f'(z) = (1 + (n-1)z^n)/(nz^{n-1})$. The image on the right shows the case $n = 3$.

Quadratic polynomials

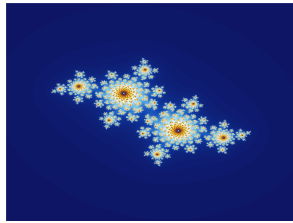
A very popular complex dynamical system is given by the family of quadratic polynomials, a special case of rational maps. The quadratic polynomials can be expressed as

$$f_c(z) = z^2 + c$$

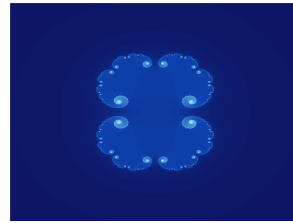
where c is a complex parameter.



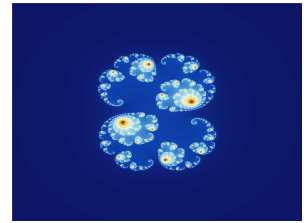
Filled Julia set for f_c , $c=1-\varphi$ where φ is the golden ratio



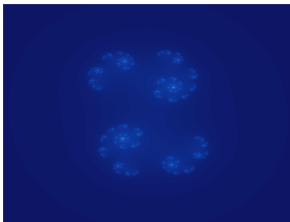
Julia set for f_c , $c=(\varphi-2)+(\varphi-1)i$
 $=-0.4+0.6i$



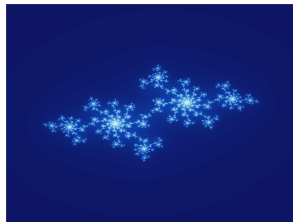
Julia set for f_c , $c=0.285+0i$



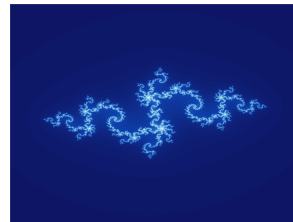
Julia set for f_c , $c=0.285+0.01i$



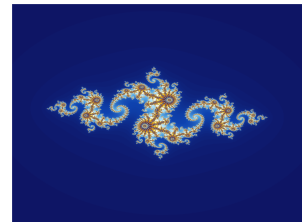
Julia set for f_c , $c=0.45+0.1428i$



Julia set for f_c ,
 $c=-0.70176-0.3842i$



Julia set for f_c , $c=-0.835-0.2321i$



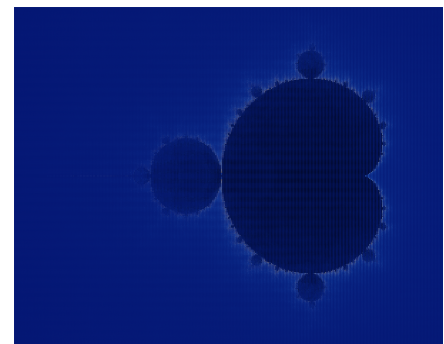
Julia set for f_c , $c=-0.8+0.156i$

The parameter plane of quadratic polynomials - that is, the plane of possible c -values - gives rise to the famous Mandelbrot set. Indeed, the Mandelbrot set is defined as the set of all c such that $J(f_c)$ is connected. For parameters outside the Mandelbrot set, the Julia set is a Cantor set: in this case it is sometimes referred to as **Fatou dust**.

In many cases, the Julia set of c looks like the Mandelbrot set in sufficiently small neighborhoods of c . This is true, in particular, for so-called 'Misiurewicz' parameters, i.e. parameters c for which the critical point is pre-periodic. For instance:

- At $c = i$, the shorter, front toe of the forefoot, the Julia set looks like a branched lightning bolt.
- At $c = -2$, the tip of the long spiky tail, the Julia set is a straight line segment.

In other words the Julia sets $J(f_c)$ are locally similar around Misiurewicz points.^[7]



A Julia set plot showing julia sets for different values of c , the plot resembles the Mandelbrot set

Generalizations

The definition of Julia and Fatou sets easily carries over to the case of certain maps whose image contains their domain; most notably transcendental meromorphic functions and Epstein's 'finite-type maps'.

Julia sets are also commonly defined in the study of dynamics in several complex variables.

The potential function and the real iteration number

The Julia set for $f(z) = z^2$ is the unit circle, and on the outer Fatou domain, the *potential function* $\phi(z)$ is defined by $\phi(z) = \log |z|$. The equipotential lines for this function are concentric circles. As $|f(z)| = |z|^2$ we have $\phi(z) = \lim_{k \rightarrow \infty} \log |z_k|/2^k$, where z_k is the sequence of iteration generated by z . For the more general iteration $f(z) = z^2 + c$, it has been proved that if the Julia set is connected (that is, if c belongs to the (usual) Mandelbrot set), then there exist a biholomorphic map ψ between the outer Fatou domain and the outer of the unit circle such that $|\psi(f(z))| = |\psi(z)|^2$.^[8] This means that the potential function on the outer Fatou domain $\phi(z) = \lim_{k \rightarrow \infty} \log |z_k|/2^k$ is given by:

This formula has meaning also if the Julia set is not connected, so that we for all c can define the potential function on the Fatou domain containing ∞ by this formula. For a general rational function $f(z)$ such that ∞ is a critical point and a fixed point, that is, such that the degree m of the numerator is at least two larger than the degree n of the denominator, we define the *potential function* on the Fatou domain containing ∞ by:

$$\phi(z) = \lim_{k \rightarrow \infty} \log |z_k|/d^k,$$

where $d = m - n$ is the degree of the rational function.^[9]

If N is a very large number (e.g. 10^{100}), and if k is the first iteration number such that $|z_k| > N$, we have that $\log |z_k|/d^k = \log(N)/d^{\nu(z)}$, for some real number $\nu(z)$, which should be regarded as the *real iteration number*, and we have that:

$$\nu(z) = k - \log(\log |z_k| / \log(N)) / \log(d),$$

where the last number is in the interval $[0, 1)$.

For iteration towards a finite attracting cycle of order r , we have that if z^* is a point of the cycle, then $f(f(\dots f(z^*))) = z^*$ (the r -fold composition), and the number $\alpha = 1/|(d(f(f(\dots f(z))))/dz)_{z=z^*}|$ (> 1) is the *attraction* of the cycle. If w is a point very near z^* and w' is w iterated r times, we have that $\alpha = \lim_{k \rightarrow \infty} |w - z^*|/|w' - z^*|$. Therefore the number $|z_{kr} - z^*|/\alpha^k$ is almost independent of k . We define the potential function on the Fatou domain by:

$$\phi(z) = \lim_{k \rightarrow \infty} 1/(|z_{kr} - z^*|/\alpha^k).$$

If ϵ is a very small number and k is the first iteration number such that $|z_k - z^*| < \epsilon$, we have that $\phi(z) = 1/(\epsilon \alpha^{\nu(z)})$ for some real number $\nu(z)$, which should be regarded as the real iteration number, and we have that:

$$\nu(z) = k - \log(\epsilon/|z_k - z^*|) / \log(\alpha).$$

If the attraction is ∞ , meaning that the cycle is *super-attracting*, meaning again that one of the points of the cycle is a critical point, we must replace α by $\alpha = \lim_{k \rightarrow \infty} \log |w' - z^*| / \log |w - z^*|$ (where w' is w iterated r times) and the formula for $\phi(z)$ by:

$$\phi(z) = \lim_{k \rightarrow \infty} \log(1/|z_{kr} - z^*|) / \alpha^k.$$

And now the real iteration number is given by:

$$\nu(z) = k - \log(\log |z_k - z^*| / \log(\epsilon)) / \log(\alpha).$$

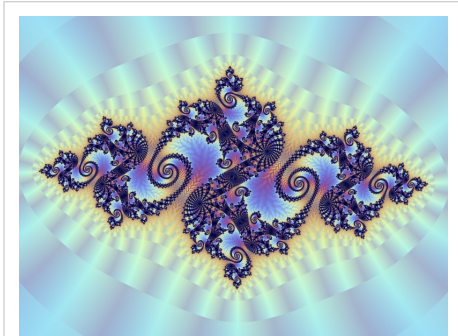
For the colouring we must have a cyclic scale of colours (constructed mathematically, for instance) and containing H colours numbered from 0 to $H-1$ ($H = 500$, for instance). We multiply the real number $\nu(z)$ by a fixed real number determining the density of the colours in the picture, and take the integral part of this number modulo H .

The definition of the potential function and our way of colouring presuppose that the cycle is attracting, that is, not neutral. If the cycle is neutral, we cannot colour the Fatou domain in a natural way. As the terminus of the iteration is a revolving movement, we can, for instance, colour by the minimum distance from the cycle left fixed by the iteration.

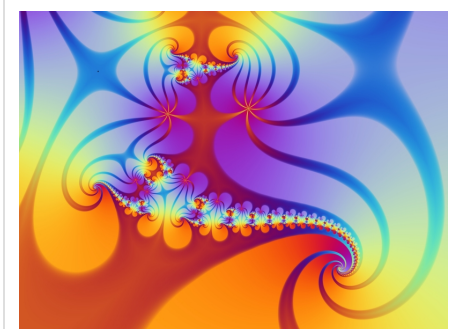
Field lines

In each Fatou domain (that is not neutral) there are two systems of lines orthogonal to each other: the *equipotential lines* (for the potential function or the real iteration number) and the *field lines*.

If we colour the Fatou domain according to the iteration number (and *not* the real iteration number), the bands of iteration show the course of the equipotential lines. If the iteration is towards ∞ (as is the case with the outer Fatou domain for the usual iteration $z^2 + c$), we can easily show the course of the field lines, namely by altering the colour according as the last point in the sequence of iteration is above or below the x-axis (first picture), but in this case (more precisely: when the Fatou domain is super-attracting) we cannot draw the field lines coherently - at least not by the method we describe here. In this case a field line is also called an external ray.



The equipotential lines for iteration towards infinity



Field lines for an iteration of the form $(1 - z^3/6)/(z - z^2/2)^2 + c$

Let z be a point in the attracting Fatou domain. If we iterate z a large number of times, the terminus of the sequence of iteration is a finite cycle C , and the Fatou domain is (by definition) the set of points whose sequence of iteration converges towards C . The field lines issue from the points of C and from the (infinite number of) points that iterate *into* a point of C . And they end on the Julia set in points that are non-chaotic (that is, generating a finite cycle). Let r be the order of the cycle C (its number of points) and let z^* be a point in C . We have $f(f(\dots f(z^*))) = z^*$ (the r -fold composition), and we define the complex number α by

$$\alpha = (d(f(f(\dots f(z))))/dz)_{z=z^*}.$$

If the points of C are $z_i, i = 1, \dots, r (z_1 = z^*)$, α is the product of the r numbers $f'(z_i)$. The real number $1/|\alpha|$ is the *attraction* of the cycle, and our assumption that the cycle is neither neutral nor super-attracting, means that $1 < 1/|\alpha| < \infty$. The point z^* is a fixed point for $f(f(\dots f(z)))$, and near this point the map $f(f(\dots f(z)))$ has (in connection with field lines) character of a rotation with the argument β of α (that is, $\alpha = |\alpha|e^{i\beta}$). In order to colour the Fatou domain, we have chosen a small number ϵ and set the sequences of iteration $z_k (k = 0, 1, 2, \dots, z_0 = z)$ to stop when $|z_k - z^*| < \epsilon$, and we colour the point z according to the number

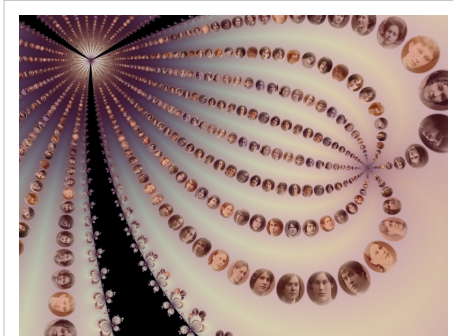
k (or the real iteration number, if we prefer a smooth colouring). If we choose a direction from z^* given by an angle θ , the field line issuing from z^* in this direction consists of the points z such that the argument ψ of the number $z_k - z^*$ satisfies the condition that

$$\psi - k\beta = \theta \pmod{\pi}.$$

For if we pass an iteration band in the direction of the field lines (and away from the cycle), the iteration number k is increased by 1 and the number ψ is increased by β , therefore the number $\psi - k\beta \pmod{\pi}$ is constant along the field line.

A colouring of the field lines of the Fatou domain means that we colour the spaces between pairs of field lines: we choose a number of regularly situated directions issuing from z^* , and in each of these directions we choose two directions around this direction. As it can happen that the two field lines of a pair do not end in the same point of the Julia set, our coloured field lines can ramify (endlessly) in their way towards the Julia set. We can colour on the basis of the distance to the centre line of the field line, and we can mix this colouring with the usual colouring. Such pictures can be very decorative (second picture).

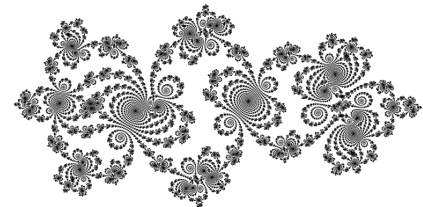
A coloured field line (the domain between two field lines) is divided up by the iteration bands, and such a part can be put into a one-to-one correspondence with the unit square: the one coordinate is (calculated from) the distance from one of the bounding field lines, the other is (calculated from) the distance from the inner of the bounding iteration bands (this number is the non-integral part of the real iteration number). Therefore we can put pictures into the field lines (third picture).



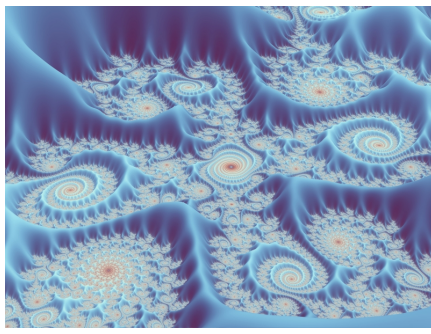
Pictures in the field lines for an iteration of the form $z^2 + c$

Distance estimation

As a Julia set is infinitely thin we cannot draw it effectively by backwards iteration from the pixels. It will appear fragmented because of the impracticality of examining infinitely many startpoints. Since the iteration count changes vigorously near the Julia set, a partial solution is to imply the outline of the set from the nearest color contours, but the set will tend to look muddy.



Julia set drawn by distance estimation, the iteration is of the form $1 - z^2 + z^5 / (2 + 4z) + c$



Three-dimensional rendering of Julia set using distance estimation.

A better way to draw the Julia set in black and white is to estimate the distance of pixels from the set and to color every pixel whose center is close to the set. The formula for the distance estimation is derived from the formula for the potential function $\phi(z)$. When the equipotential lines for $\phi(z)$ lie close, the number $|\phi'(z)|$ is large, and conversely, therefore the equipotential lines for the function $\delta(z) = \phi(z)/|\phi'(z)|$ should lie approximately regularly. It has been proven that the value found by this formula (up to a constant factor) converges towards the true distance for z converging towards the Julia set.^[10]

We assume that $f(z)$ is rational, that is, $f(z) = p(z)/q(z)$ where $p(z)$ and $q(z)$ are complex polynomials of degrees m and n , respectively, and we have to find the derivative of the above expressions for $\phi(z)$. And as it is only z_k that varies, we must calculate the derivative z'_k of z_k with respect to z . But as $z_k = f(f(\dots f(z)))$ (the k -fold composition), z'_k is the product of the numbers $f'(z_k)$, and this sequence can be calculated recursively by $z'_{k+1} = f'(z_k)z'_k$, starting with $z'_0 = 1$ (before the calculation of the next iteration $z_{k+1} = f(z_k)$). For iteration towards ∞ (more precisely when $m \geq n + 2$, so that ∞ is a super-attracting fixed point), we have

$$|\phi'(z)| = \lim_{k \rightarrow \infty} |z'_k|/|z_k|d^k,$$

($d = m - n$) and consequently:

$$\delta(z) = \phi(z)/|\phi'(z)| = \lim_{k \rightarrow \infty} \log |z_k|/|z'_k|.$$

For iteration towards a finite attracting cycle (that is not super-attracting) containing the point z^* and having order r , we have

$$|\phi'(z)| = \lim_{k \rightarrow \infty} |z'_{kr}|/(|z_{kr} - z^*|^2 \alpha^k),$$

and consequently:

$$\delta(z) = \phi(z)/|\phi'(z)| = \lim_{k \rightarrow \infty} |z_{kr} - z^*|/|z'_{kr}|.$$

For a super-attracting cycle, the formula is:

$$\delta(z) = \lim_{k \rightarrow \infty} \log |z_{kr} - z^*|/|z'_{kr}|.$$

We calculate this number when the iteration stops. Note that the distance estimation is independent of the attraction of the cycle. This means that it has meaning for transcendental functions of "degree infinity" (e.g. $\sin(z)$ and $\tan(z)$).

Besides drawing of the boundary, the distance function can be introduced as a 3rd dimension to create a solid fractal landscape.

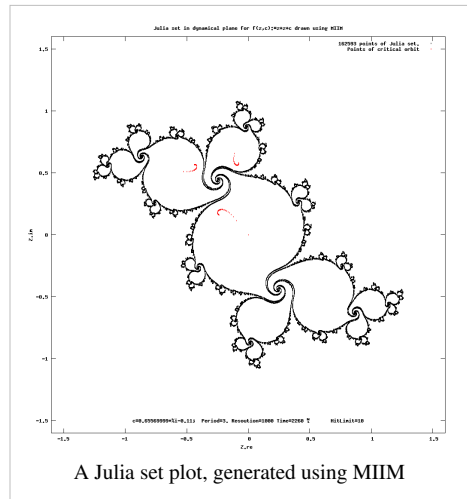
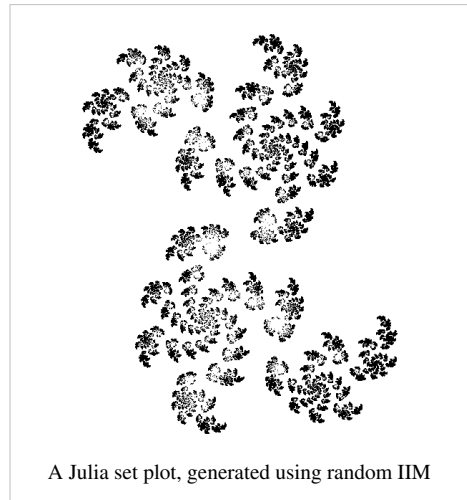
Plotting the Julia set

Using backwards (inverse) iteration (IIM)

As mentioned above, the Julia set can be found as the set of limit points of the set of pre-images of (essentially) any given point. So we can try to plot the Julia set of a given function as follows. Start with any point z we know to be in the Julia set, such as a repelling periodic point, and compute all pre-images of z under some high iterate f^n of f .

Unfortunately, as the number of iterated pre-images grows exponentially, this is not feasible computationally. However, we can adjust this method, in a similar way as the "random game" method for iterated function systems. That is, in each step, we choose at random one of the inverse images of f .

For example, for the quadratic polynomial f_c , the backwards iteration is described by

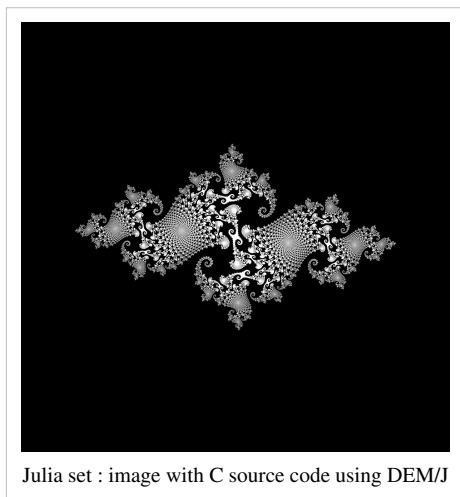


$$z_{n-1} = \sqrt{z_n - c}.$$

At each step, one of the two square roots is selected at random.

Note that certain parts of the Julia set are quite difficult to access with the reverse Julia algorithm. For this reason, one must modify IIM/J (it is called MIIM/J) or use other methods to produce better images.

Using DEM/J



Notes

- [1] Note that for other areas of mathematics the notation $J(f)$ can also represent the Jacobian matrix of a real valued mapping f between smooth manifolds.
- [2] Gaston Julia (1918) "Mémoire sur l'iteration des fonctions rationnelles," *Journal de Mathématiques Pures et Appliquées*, vol. 8, pages 47–245.
- [3] Pierre Fatou (1917) "Sur les substitutions rationnelles," *Comptes Rendus de l'Académie des Sciences de Paris*, vol. 164, pages 806-808 and vol. 165, pages 992–995.
- [4] Beardon, *Iteration of Rational Functions*, Theorem 5.6.2
- [5] Beardon, Theorem 7.1.1
- [6] Beardon, *Iteration of Rational Functions*, Theorem 3.2.4
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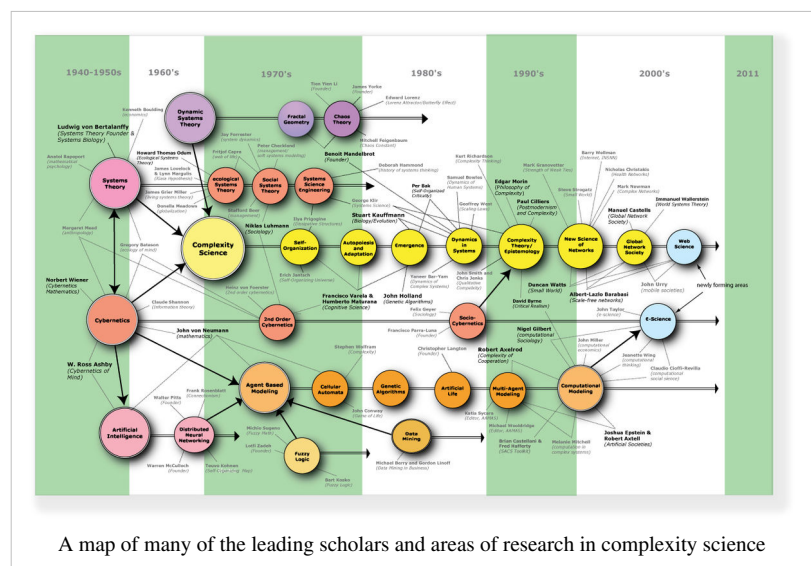
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- A simple program to generate Julia sets (Windows, 370 kb) (<http://www.lizardie.com/links/download/fractal-generator>)
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- Julia meets HTML5 (<http://juliemap.googlelabs.com/>) Google Labs' HTML5 Fractal generator on your browser
- Julia (<http://cran.r-project.org/web/packages/Julia/index.html>) GNU R Package to generate Julia or Mandelbrot set at a given region and resolution.

Complexity

In general usage, **complexity** tends to be used to characterize something with many parts in intricate arrangement. The study of these complex linkages is the main goal of network theory and network science. In science^[1] there are at this time a number of approaches to characterizing complexity, many of which are reflected in this article. In a business context, complexity management is the methodology to minimize value-destroying complexity and efficiently control value-adding complexity in a cross-functional approach.

Overview

Definitions are often tied to the concept of a "system"—a set of parts or elements that have relationships among them differentiated from relationships with other elements outside the relational regime. Many definitions tend to postulate or assume that complexity expresses a condition of numerous elements in a system and numerous forms of relationships among the elements. At the same time, what is complex and what is simple is relative and changes with time.



Some definitions key on the question of the probability of encountering a given condition of a system once characteristics of the system are specified. Warren Weaver has posited that the complexity of a particular system is

the degree of difficulty in predicting the properties of the system, if the properties of the system's parts are given. In Weaver's view, complexity comes in two forms: disorganized complexity, and organized complexity.^[2] Weaver's paper has influenced contemporary thinking about complexity.^[3]

The approaches that embody concepts of systems, multiple elements, multiple relational regimes, and state spaces might be summarized as implying that complexity arises from the number of distinguishable relational regimes (and their associated state spaces) in a defined system.

Some definitions relate to the algorithmic basis for the expression of a complex phenomenon or model or mathematical expression, as is later set out herein.

Disorganized complexity vs. organized complexity

One of the problems in addressing complexity issues has been distinguishing conceptually between the large number of variances in relationships extant in random collections, and the sometimes large, but smaller, number of relationships between elements in systems where constraints (related to correlation of otherwise independent elements) simultaneously reduce the variations from element independence and create distinguishable regimes of more-uniform, or correlated, relationships, or interactions.

Weaver perceived and addressed this problem, in at least a preliminary way, in drawing a distinction between "disorganized complexity" and "organized complexity".

In Weaver's view, disorganized complexity results from the particular system having a very large number of parts, say millions of parts, or many more. Though the interactions of the parts in a "disorganized complexity" situation can be seen as largely random, the properties of the system as a whole can be understood by using probability and statistical methods.

A prime example of disorganized complexity is a gas in a container, with the gas molecules as the parts. Some would suggest that a system of disorganized complexity may be compared, for example, with the (relative) simplicity of the planetary orbits—the latter can be known by applying Newton's laws of motion, though this example involved highly correlated events.

Organized complexity, in Weaver's view, resides in nothing else than the non-random, or correlated, interaction between the parts. These correlated relationships create a differentiated structure that can, as a system, interact with other systems. The coordinated system manifests properties not carried by, or dictated by, individual parts. The organized aspect of this form of complexity vis a vis other systems than the subject system can be said to "emerge," without any "guiding hand".

The number of parts does not have to be very large for a particular system to have emergent properties. A system of organized complexity may be understood in its properties (behavior among the properties) through modeling and simulation, particularly modeling and simulation with computers. An example of organized complexity is a city neighborhood as a living mechanism, with the neighborhood people among the system's parts.^[4]

Sources and factors of complexity

The source of disorganized complexity is the large number of parts in the system of interest, and the lack of correlation between elements in the system.

There is no consensus at present on general rules regarding the sources of organized complexity, though the lack of randomness implies correlations between elements. See e.g. Robert Ulanowicz's treatment of ecosystems.^[5] Consistent with prior statements here, the number of parts (and types of parts) in the system and the number of relations between the parts would have to be non-trivial—however, there is no general rule to separate "trivial" from "non-trivial".

Complexity of an object or system is a relative property. For instance, for many functions (problems), such a computational complexity as time of computation is smaller when multitape Turing machines are used than when Turing machines with one tape are used. Random Access Machines allow one to even more decrease time complexity (Greenlaw and Hoover 1998: 226), while inductive Turing machines can decrease even the complexity class of a function, language or set (Burgin 2005). This shows that tools of activity can be an important factor of complexity.

Specific meanings of complexity

In several scientific fields, "complexity" has a specific meaning :

- In computational complexity theory, the amounts of resources required for the execution of algorithms is studied. The most popular types of computational complexity are the time complexity of a problem equal to the number of steps that it takes to solve an instance of the problem as a function of the size of the input (usually measured in bits), using the most efficient algorithm, and the space complexity of a problem equal to the volume of the memory used by the algorithm (e.g., cells of the tape) that it takes to solve an instance of the problem as a function of the size of the input (usually measured in bits), using the most efficient algorithm. This allows to classify computational problems by complexity class (such as P, NP ...). An axiomatic approach to computational complexity was developed by Manuel Blum. It allows one to deduce many properties of concrete computational complexity measures, such as time complexity or space complexity, from properties of axiomatically defined measures.
- In algorithmic information theory, the *Kolmogorov complexity* (also called *descriptive complexity*, *algorithmic complexity* or *algorithmic entropy*) of a string is the length of the shortest binary program that outputs that string. Different kinds of Kolmogorov complexity are studied: the uniform complexity, prefix complexity, monotone complexity, time-bounded Kolmogorov complexity, and space-bounded Kolmogorov complexity. An axiomatic approach to Kolmogorov complexity based on Blum axioms (Blum 1967) was introduced by Mark Burgin in the paper presented for publication by Andrey Kolmogorov (Burgin 1982). The axiomatic approach encompasses other approaches to Kolmogorov complexity. It is possible to treat different kinds of Kolmogorov complexity as particular cases of axiomatically defined generalized Kolmogorov complexity. Instead, of proving similar theorems, such as the basic invariance theorem, for each particular measure, it is possible to easily deduce all such results from one corresponding theorem proved in the axiomatic setting. This is a general advantage of the axiomatic approach in mathematics. The axiomatic approach to Kolmogorov complexity was further developed in the book (Burgin 2005) and applied to software metrics (Burgin and Debnath, 2003; Debnath and Burgin, 2003).
- In information processing, complexity is a measure of the total number of properties transmitted by an object and detected by an observer. Such a collection of properties is often referred to as a state.
- In business, complexity describes the variances and their consequences in various fields such as product portfolio, technologies, markets and market segments, locations, manufacturing network, customer portfolio, IT systems, organization, processes etc.
- In physical systems, complexity is a measure of the probability of the state vector of the system. This should not be confused with entropy; it is a distinct mathematical measure, one in which two distinct states are never conflated and considered equal, as is done for the notion of entropy statistical mechanics.
- In mathematics, Krohn-Rhodes complexity is an important topic in the study of finite semigroups and automata.
- In software engineering, programming complexity is a measure of the interactions of the various elements of the software. This differs from the computational complexity described above in that it is a measure of the design of the software.

There are different specific forms of complexity:

- In the sense of how complicated a problem is from the perspective of the person trying to solve it, limits of complexity are measured using a term from cognitive psychology, namely the hrait limit.
- Complex adaptive system denotes systems that have some or all of the following attributes^[6]
 - The number of parts (and types of parts) in the system and the number of relations between the parts is non-trivial – however, there is no general rule to separate "trivial" from "non-trivial";
 - The system has memory or includes feedback;
 - The system can adapt itself according to its history or feedback;
 - The relations between the system and its environment are non-trivial or non-linear;
 - The system can be influenced by, or can adapt itself to, its environment; and
 - The system is highly sensitive to initial conditions.

Study of complexity

Complexity has always been a part of our environment, and therefore many scientific fields have dealt with complex systems and phenomena. From one perspective, that which is somehow complex—displaying variation without being random—is most worthy of interest given the rewards found in the depths of exploration.

The use of the term complex is often confused with the term complicated. In today's systems, this is the difference between myriad connecting "stovepipes" and effective "integrated" solutions.^[7] This means that complex is the opposite of independent, while complicated is the opposite of simple.

While this has led some fields to come up with specific definitions of complexity, there is a more recent movement to regroup observations from different fields to study complexity in itself, whether it appears in anthills, human brains, or stock markets. One such interdisciplinary group of fields is relational order theories.

Complexity topics

Complex behaviour

The behavior of a complex system is often said to be due to emergence and self-organization. Chaos theory has investigated the sensitivity of systems to variations in initial conditions as one cause of complex behaviour.

Complex mechanisms

Recent developments around artificial life, evolutionary computation and genetic algorithms have led to an increasing emphasis on complexity and complex adaptive systems.

Complex simulations

In social science, the study on the emergence of macro-properties from the micro-properties, also known as macro-micro view in sociology. The topic is commonly recognized as social complexity that is often related to the use of computer simulation in social science, i.e.: computational sociology.

Complex systems

Systems theory has long been concerned with the study of complex systems (In recent times, *complexity theory* and *complex systems* have also been used as names of the field). These systems can be biological, economic, technological, etc. Recently, complexity is a natural domain of interest of the real world socio-cognitive systems and emerging systemics research. Complex systems tend to be high-dimensional, non-linear and hard to model. In specific circumstances they may exhibit low dimensional behaviour.

Complexity in data

In information theory, algorithmic information theory is concerned with the complexity of strings of data.

Complex strings are harder to compress. While intuition tells us that this may depend on the codec used to compress a string (a codec could be theoretically created in any arbitrary language, including one in which the very small command "X" could cause the computer to output a very complicated string like "18995316"), any two Turing-complete languages can be implemented in each other, meaning that the length of two encodings in different languages will vary by at most the length of the "translation" language—which will end up being negligible for sufficiently large data strings.

These algorithmic measures of complexity tend to assign high values to random noise. However, those studying complex systems would not consider randomness as complexity.

Information entropy is also sometimes used in information theory as indicative of complexity.

Applications of complexity

Computational complexity theory is the study of the complexity of problems—that is, the difficulty of solving them. Problems can be classified by complexity class according to the time it takes for an algorithm—usually a computer program—to solve them as a function of the problem size. Some problems are difficult to solve, while others are easy. For example, some difficult problems need algorithms that take an exponential amount of time in terms of the size of the problem to solve. Take the travelling salesman problem, for example. It can be solved in time $O(n^2 2^n)$ (where n is the size of the network to visit—let's say the number of cities the travelling salesman must visit exactly once). As the size of the network of cities grows, the time needed to find the route grows (more than) exponentially. Even though a problem may be computationally solvable in principle, in actual practice it may not be that simple. These problems might require large amounts of time or an inordinate amount of space. Computational complexity may be approached from many different aspects. Computational complexity can be investigated on the basis of time, memory or other resources used to solve the problem. Time and space are two of the most important and popular considerations when problems of complexity are analyzed.

There exist a certain class of problems that although they are solvable in principle they require so much time or space that it is not practical to attempt to solve them. These problems are called intractable.

There is another form of complexity called hierarchical complexity. It is orthogonal to the forms of complexity discussed so far, which are called horizontal complexity

Bejan and Lorente showed that complexity is modest (not maximum, not increasing), and is a feature of the natural phenomenon of design generation in nature, which is predicted by the Constructal law. ^[8]

Bejan and Lorente also showed that all the optimality (max,min) statements have limited ad-hoc applicability, and are unified under the Constructal law of design and evolution in nature. ^{[9] [10]}

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- Complexity Measures (<http://cscs.umich.edu/~crshalizi/notebooks/complexity-measures.html>) – an article about the abundance of not-that-useful complexity measures.
- Exploring Complexity in Science and Technology (<http://web.cecs.pdx.edu/~mm/ExploringComplexityFall2009/index.html>) – introductory complex system course by Melanie Mitchell
- Quantifying Complexity Theory (<http://www.calresco.org/lucas/quantify.htm>) – classification of complex systems
- Santa Fe Institute (<http://www.santafe.edu/>) focusing on the study of complexity science: Lecture Videos (<http://www.santafe.edu/research/videos/catalog/>)
- UC Four Campus Complexity Videoconferences (<http://eclectic.ss.uci.edu/~drwhite/center/cac.html>) – Human Sciences and Complexity

At the Edge of Chaos

The phrase *edge of chaos* was coined by mathematician Doyne Farmer to describe the transition phenomenon discovered by computer scientist Christopher Langton. The phrase originally refers to an area in the range of a variable, λ (lambda), which was varied while examining the behavior of a cellular automaton (CA). As λ varied, the behavior of the CA went through a phase transition of behaviors. Langton found a small area conducive to produce CAs capable of universal computation. At around the same time physicist James P. Crutchfield and others used the phrase *onset of chaos* to describe more or less the same concept.

In the sciences in general, the phrase has come to refer to a metaphor that some physical, biological, economic and social systems operate in a region between order and either complete randomness or chaos, where the complexity is maximal. The generality and significance of the idea, however, has since been called into question by Melanie Mitchell and others. The phrase has also been borrowed by the business community and is sometimes used inappropriately and in contexts that are far from the original scope of the meaning of the term.

Stuart Kauffman has studied mathematical models of evolving systems in which the rate of evolution is maximized near the edge of chaos.

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- *Origins of Order: Self-Organization and Selection in Evolution* by Stuart Kauffman

External links

- "The Edge of Chaos" ^[3] - a criticism of the idea's prevalence.

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[3] <http://cscs.umich.edu/~crshalizi/notebooks/edge-of-chaos.html>

Chaos control

In chaos theory, **control of chaos** is based on the fact that any chaotic attractor contains an infinite number of unstable periodic orbits. Chaotic dynamics then consists of a motion where the system state moves in the neighborhood of one of these orbits for a while, then falls close to a different unstable periodic orbit where it remains for a limited time, and so forth. This results in a complicated and unpredictable wandering over longer periods of time.

Control of chaos is the stabilization, by means of small system perturbations, of one of these unstable periodic orbits. The result is to render an otherwise chaotic motion more stable and predictable, which is often an advantage. The perturbation must be tiny, to avoid significant modification of the system's natural dynamics.

Several techniques have been devised for chaos control, but most are developments of two basic approaches: the OGY (Ott, Grebogi and Yorke) method, and Pyragas continuous control. Both methods require a previous determination of the unstable periodic orbits of the chaotic system before the controlling algorithm can be designed.

In the OGY method, small, wisely chosen, swift kicks are applied to the system once per cycle, to maintain it near the desired unstable periodic orbit. In the Pyragas method, an appropriate continuous controlling signal is injected into the system, whose intensity is practically zero as the system evolves close to the desired periodic orbit but increases when it drifts away from the desired orbit.

Experimental control of chaos by one or both of these methods has been achieved in a variety of systems, including turbulent fluids, oscillating chemical reactions, magneto-mechanical oscillators, and cardiac tissues. Sarnobat et al. (2000) attempt the control of chaotic bubbling with the OGY method and using electrostatic potential as the primary control variable.

The number of publications devoted to control of chaos is huge, see e.g. Chaos control bibliography (1997-2000) ^[1]

Forcing two systems into the same state is not the only way to achieve synchronization of chaos. Both control of chaos and synchronization constitute parts of Cybernetical Physics. Cybernetical physics is a research area on the border between Physics and Control Theory.

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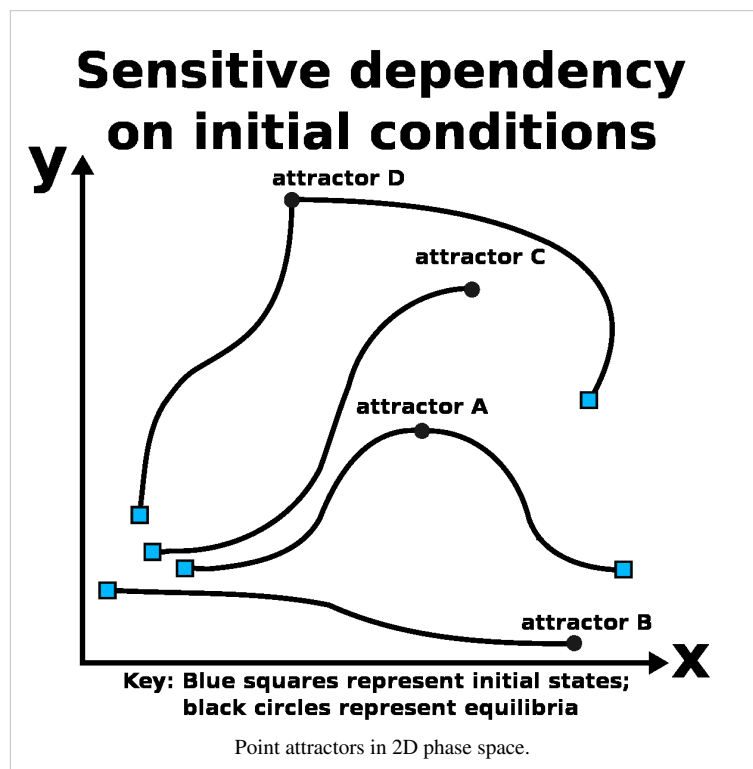
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Butterfly effect

In chaos theory, the **butterfly effect** is the *sensitive dependence on initial conditions*; where a small change at one place in a nonlinear system can result in large differences to a later state. For example, the presence or absence of a butterfly flapping its wings could lead to creation or absence of a hurricane.

Although the butterfly effect may appear to be an esoteric and unusual behavior, it is exhibited by very simple systems: for example, a ball placed at the crest of a hill might roll into any of several valleys depending on slight differences in initial position.

The butterfly effect is a common trope in fiction when presenting scenarios involving time travel and with "what if" cases where one storyline diverges at the moment of a seemingly minor event resulting in two significantly different outcomes.



Theory

Recurrence, the approximate return of a system towards its initial conditions, together with sensitive dependence on initial conditions, are the two main ingredients for chaotic motion. They have the practical consequence of making complex systems, such as the weather, difficult to predict past a certain time range (approximately a week in the case of weather), since it is impossible to measure the starting atmospheric conditions completely accurately.

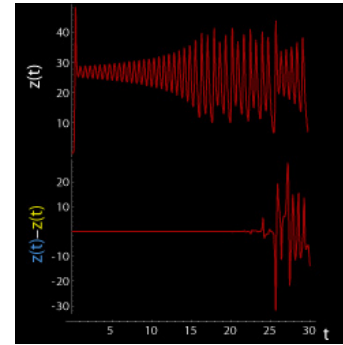
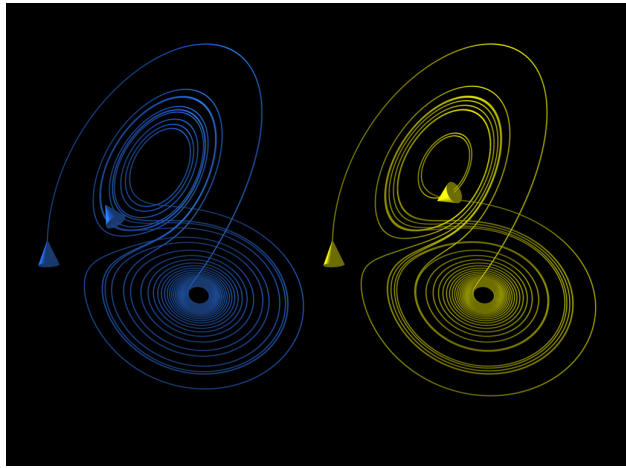
Origin of the concept and the term

The term "butterfly effect" itself is related to the work of Edward Lorenz, and it is based in chaos theory and sensitive dependence on initial conditions, already described in the literature in a particular case of the three-body problem by Henri Poincaré in 1890.^[1] He later proposed that such phenomena could be common, say in meteorology. In 1898,^[1] Jacques Hadamard noted general divergence of trajectories in spaces of negative curvature, and Pierre Duhem discussed the possible general significance of this in 1908.^[1] The idea that one butterfly could eventually have a far-reaching ripple effect on subsequent historic events seems first to have appeared in "A Sound of Thunder", a 1952 short story by Ray Bradbury about time travel (see Literature and print here) although Lorenz made the term popular. In 1961, Lorenz was using a numerical computer model to rerun a weather prediction, when, as a shortcut on a number in the sequence, he entered the decimal .506 instead of entering the full .506127. The result was a completely different weather scenario.^[2] Lorenz published his findings in a 1963 paper^[3] for the New York Academy of Sciences noting that "One meteorologist remarked that if the theory were correct, one flap of a seagull's wings could change the course of weather forever." Later speeches and papers by Lorenz used the more poetic butterfly. According to Lorenz, when Lorenz failed to provide a title for a talk he was to present at the 139th meeting of the American Association for the Advancement of Science in 1972, Philip Merilees concocted *Does the flap of a butterfly's wings in Brazil set off a tornado in Texas?* as a title. Although a butterfly flapping its wings has remained constant in the expression of this concept, the location of the butterfly, the consequences, and the location of the consequences have varied widely.^[4]

The phrase refers to the idea that a butterfly's wings might create tiny changes in the atmosphere that may ultimately alter the path of a tornado or delay, accelerate or even prevent the occurrence of a tornado in another location. The flapping wing represents a small change in the initial condition of the system, which causes a chain of events leading to large-scale alterations of events (compare: domino effect). Had the butterfly not flapped its wings, the trajectory of the system might have been vastly different. While the butterfly does not "cause" the tornado in the sense of providing the energy for the tornado, it does "cause" it in the sense that the flap of its wings is an essential part of the initial conditions resulting in a tornado, and without that flap that particular tornado would not have existed.

Illustration

The butterfly effect in the Lorenz attractor	
time $0 \leq t \leq 30$ (larger)	z coordinate (larger)



These figures show two segments of the three-dimensional evolution of two trajectories (one in blue, the other in yellow) for the same period of time in the Lorenz attractor starting at two initial points that differ only by 10^{-5} in the x-coordinate. Initially, the two trajectories seem coincident, as indicated by the small difference between the z coordinate of the blue and yellow trajectories, but for $t > 23$ the difference is as large as the value of the trajectory. The final position of the cones indicates that the two trajectories are no longer coincident at $t=30$.

A Java animation of the Lorenz attractor ^[1] shows the continuous evolution.

Mathematical definition

A dynamical system with evolution map f^t displays sensitive dependence on initial conditions if points arbitrarily close together become separate with increasing t at an exponential rate. The definition is not topological, but essentially metrical.

If M is the state space for the map f^t , then f^t displays sensitive dependence to initial conditions if for any x in M and any $\delta > 0$, there are y in M , with $0 < d(x, y) < \delta$ such that

$$d(f^t(x), f^t(y)) > \exp(at)d(x, y).$$

The definition does not require that all points from a neighborhood separate from the base point x , but it requires one positive Lyapunov exponent.

Examples

The butterfly effect is most familiar in terms of weather; it can easily be demonstrated in standard weather prediction models, for example.^[5]

The potential for sensitive dependence on initial conditions (the butterfly effect) has been studied in a number of cases in semiclassical and quantum physics including atoms in strong fields and the anisotropic Kepler problem.^{[6] [7]} Some authors have argued that extreme (exponential) dependence on initial conditions is not expected in pure quantum treatments,^{[8] [9]} however, the sensitive dependence on initial conditions demonstrated in classical motion is included in the semiclassical treatments developed by Martin Gutzwiller^[10] and Delos and co-workers.^[11]

Other authors suggest that the butterfly effect can be observed in quantum systems. Karkuszewski et al. consider the time evolution of quantum systems which have slightly different Hamiltonians. They investigate the level of sensitivity of quantum systems to small changes in their given Hamiltonians.^[12] Poulin et al. present a quantum algorithm to measure fidelity decay, which “measures the rate at which identical initial states diverge when subjected to slightly different dynamics.” They consider fidelity decay to be “the closest quantum analog to the (purely classical) butterfly effect.”^[13] Whereas the classical butterfly effect considers the effect of a small change in the position and/or velocity of an object in a given Hamiltonian system, the quantum butterfly effect considers the effect of a small change in the Hamiltonian system with a given initial position and velocity.^{[14] [15]} This quantum butterfly

effect has been demonstrated experimentally.^[16] Quantum and semiclassical treatments of system sensitivity to initial conditions are known as quantum chaos.^[8] ^[14]

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External links

- The meaning of the butterfly: Why pop culture loves the 'butterfly effect,' and gets it totally wrong (http://www.boston.com/bostonglobe/ideas/articles/2008/06/08/the_meaning_of_the_butterfly/?page=full), Peter Dizikes, *Boston Globe*, June 8, 2008
- From butterfly wings to single e-mail (<http://www.news.cornell.edu/releases/Feb04/AAAS.Kleinberg.ws.html>) (Cornell University)
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Applications

Data storage

Data storage can refer to:

- Computer data storage; memory, components, devices and media that retain digital computer data used for computing for some interval of time.
- Any data storage device; that records (stores) or retrieves (reads) information (data) from any medium, including the medium itself.

Online data storage solutions

Today, more and more people are mobile: they are working on similar tasks at home, office, etc.. Thus, the concept of storing data online is becoming increasingly common. ContactOffice offers its users through its virtual office a document storage space accessible online via their collaboration suite's Web interface, but also via the WebDAV protocol as a network folder on the user's computer.

Data transmission

Data transmission, digital transmission, or digital communications is the physical transfer of data (a digital bit stream) over a point-to-point or point-to-multipoint communication channel. Examples of such channels are copper wires, optical fibres, wireless communication channels, and storage media. The data is represented as an electromagnetic signal, such as an electrical voltage, radiowave, microwave, or infrared signal.

While analog communications is the transfer of continuously varying information signal, digital communications is the transfer of discrete messages. The messages are either represented by a sequence of pulses by means of a line code (*baseband transmission*), or by a limited set of continuously varying wave forms (*passband transmission*), using a digital modulation method. The passband modulation and corresponding demodulation (also known as detection) is carried out by modem equipment. According to the most common definition of digital signal, both baseband and passband signals representing bit-streams are considered as digital transmission, while an alternative definition only considers the baseband signal as digital, and passband transmission of digital data as a form of digital-to-analog conversion.

Data transmitted may be digital messages originating from a data source, for example a computer or a keyboard. It may also be an analog signal such as a phone call or a video signal, digitized into a bit-stream for example using pulse-code modulation (PCM) or more advanced source coding (analog-to-digital conversion and data compression) schemes. This source coding and decoding is carried out by codec equipment.

Distinction between related subjects

Courses and textbooks in the field of *data transmission*^[1] as well as *digital transmission*^{[2] [3]} and *digital communications*^{[4] [5]} have similar content.

Digital transmission or data transmission traditionally belongs to telecommunications and electrical engineering. Basic principles of data transmission may also be covered within the computer science/computer engineering topic of data communications, which also includes computer networking or computer communication applications and networking protocols, for example routing, switching and process-to-process communication. Although the Transmission control protocol (TCP) involves the term "transmission", TCP and other transport layer protocols are typically *not* discussed in a textbook or course about data transmission, but in computer networking.

The term tele transmission involves the analog as well as digital communication. In most textbooks, the term analog transmission only refers to the transmission of an analog message signal (without digitization) by means of an analog signal, either as a non-modulated baseband signal, or as a passband signal using an analog modulation method such as AM or FM. It may also include analog-over-analog pulse modulated baseband signals such as pulse-width modulation. In a few books within the computer networking tradition, "analog transmission" also refers to passband transmission of bit-streams using digital modulation methods such as FSK, PSK and ASK. Note that these methods are covered in textbooks named digital transmission or data transmission, for example.^[1]

The theoretical aspects of data transmission are covered by information theory and coding theory.

Protocol layers and sub-topics

Courses and textbooks in the field of data transmission typically deal with the following OSI model protocol layers and topics:

- Layer 1, the physical layer:
 - Channel coding including
 - Digital modulation schemes
 - Line coding schemes
 - Forward error correction (FEC) codes
 - Bit synchronization
 - Multiplexing
 - Equalization
 - Channel models
- Layer 2, the data link layer:
 - Channel access schemes, media access control (MAC)
 - Packet mode communication and Frame synchronization
 - Error detection and automatic repeat request (ARQ)
 - Flow control
- Layer 6, the presentation layer:
 - Source coding (digitization and data compression), and information theory.
 - Cryptography (may occur at any layer)

Applications and history

Data (mainly but not exclusively informational) has been sent via non-electronic (e.g. optical, acoustic, mechanical) means since the advent of communication. Analog signal data has been sent electronically since the advent of the telephone. However, the first data electromagnetic transmission applications in modern time were telegraphy (1809) and teletypewriters (1906), which are both digital signals. The fundamental theoretical work in data transmission and information theory by Harry Nyquist, Ralph Hartley, Claude Shannon and others during the early 20th century, was done with these applications in mind.

Data transmission is utilized in computers in computer buses and for communication with peripheral equipment via parallel ports and serial ports such as RS-232 (1969), Firewire (1995) and USB (1996). The principles of data transmission is also utilized in storage media for Error detection and correction since 1951.

Data transmission is utilized in computer networking equipment such as modems (1940), local area networks (LAN) adapters (1964), repeaters, hubs, microwave links, wireless network access points (1997), etc.

In telephone networks, digital communication is utilized for transferring many phone calls over the same copper cable or fiber cable by means of Pulse code modulation (PCM), i.e. sampling and digitization, in combination with Time division multiplexing (TDM) (1962). Telephone exchanges have become digital and software controlled, facilitating many value added services. For example the first AXE telephone exchange was presented in 1976. Since late 1980th, digital communication to the end user has been possible using Integrated Services Digital Network (ISDN) services. Since the end of 1990th, broadband access techniques such as ADSL, Cable modems, fiber-to-the-building (FTTB) and fiber-to-the-home (FTTH) have become wide spread to small offices and homes. The current tendency is to replace traditional telecommunication services by packet mode communication such as IP telephony and IPTV.

Transmitting analog signals digitally allows for greater signal processing capability. The ability to process a communications signal means that errors caused by random processes can be detected and corrected. Digital signals can also be sampled instead of continuously monitored. The multiplexing of multiple digital signals is much simpler to the multiplexing of analog signals.

Because of all these advantages, and because recent advances in wideband communication channels and solid-state electronics have allowed scientists to fully realize these advantages, digital communications has grown quickly. Digital communications is quickly edging out analog communication because of the vast demand to transmit computer data and the ability of digital communications to do so.

The digital revolution has also resulted in many digital telecommunication applications where the principles of data transmission are applied. Examples are second-generation (1991) and later cellular telephony, video conferencing, digital TV (1998), digital radio (1999), telemetry, etc.

Baseband or passband transmission

The physically transmitted signal may be one of the following:

1. **A baseband signal** ("digital-over-digital" transmission): A sequence of electrical pulses or light pulses produced by means of a line coding scheme such as Manchester coding. This is typically used in serial cables, wired local area networks such as Ethernet, and in optical fiber communication. It results in a pulse amplitude modulated signal, also known as a pulse train.
2. **A passband signal** ("digital-over-analog" transmission): A modulated sine wave signal representing a digital bit-stream. Note that this is in some textbooks considered as analog transmission, but in most books as digital transmission. The signal is produced by means of a digital modulation method such as PSK, QAM or FSK. The modulation and demodulation is carried out by modem equipment. This is used in wireless communication, and over telephone network local-loop and cable-TV networks.

Serial and parallel transmission

In telecommunications, serial transmission is the sequential transmission of signal elements of a group representing a character or other entity of data. Digital serial transmissions are bits sent over a single wire, frequency or optical path sequentially. Because it requires less signal processing and less chances for error than parallel transmission, the transfer rate of each individual path may be faster. This can be used over longer distances as a check digit or parity bit can be sent along it easily.

In telecommunications, parallel transmission is the simultaneous transmission of the signal elements of a character or other entity of data. In digital communications, parallel transmission is the simultaneous transmission of related signal elements over two or more separate paths. Multiple electrical wires are used which can transmit multiple bits simultaneously, which allows for higher data transfer rates than can be achieved with serial transmission. This method is used internally within the computer, for example the internal buses, and sometimes externally for such things as printers. The major issue with this is "skewing" because the wires in parallel data transmission have slightly different properties (not intentionally) so some bits may arrive before others, which may corrupt the message. A parity bit can help to reduce this. However, electrical wire parallel data transmission is therefore less reliable for long distances because corrupt transmissions are far more likely.

Types of communication channels

- Simplex
- Half-duplex
- Full-duplex
- Point-to-point
- Multi-drop:
 - Bus network
 - Ring network
 - Star network
 - Mesh network
 - Wireless network

Asynchronous and synchronous data transmission

Asynchronous transmission uses start and stop bits to signify the beginning bit ASCII character would actually be transmitted using 10 bits e.g.: A "0100 0001" would become "**1** 0100 0001 **0**". The extra one (or zero depending on parity bit) at the start and end of the transmission tells the receiver first that a character is coming and secondly that the character has ended. This method of transmission is used when data is sent intermittently as opposed to in a solid stream. In the previous example the start and stop bits are in bold. The start and stop bits must be of opposite polarity. This allows the receiver to recognize when the second packet of information is being sent.

Synchronous transmission uses no start and stop bits but instead synchronizes transmission speeds at both the receiving and sending end of the transmission using clock signal(s) built into each component. A continual stream of data is then sent between the two nodes. Due to there being no start and stop bits the data transfer rate is quicker although more errors will occur, as the clocks will eventually get out of sync, and the receiving device would have the wrong time that had been agreed in the protocol for sending/receiving data, so some bytes could become corrupted (by losing bits). Ways to get around this problem include re-synchronization of the clocks and use of check digits to ensure the byte is correctly interpreted and received

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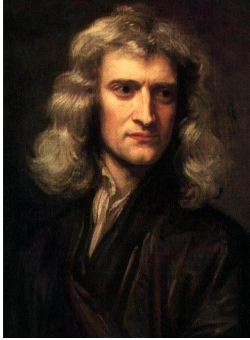
External links

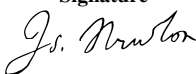
- Asynchronous serial data example (<http://halowave.webs.com>)

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Related Biographies

Isaac Newton

Sir Isaac Newton	
 <p>Godfrey Kneller's 1689 portrait of Isaac Newton (age 46)</p>	
Born	4 January 1643 [OS: 25 December 1642] ^[1] Woolsthorpe-by-Colsterworth Lincolnshire, England
Died	31 March 1727 (aged 84) [OS: 20 March 1726] ^[1] Kensington, Middlesex, England
Residence	England
Nationality	English
Fields	physics, mathematics, astronomy, natural philosophy, alchemy, Christian theology
Institutions	University of Cambridge Royal Society Royal Mint
Alma mater	Trinity College, Cambridge
Academic advisors	Isaac Barrow ^[2] Benjamin Pulleyn ^[3] ^[4]
Notable students	Roger Cotes William Whiston
Known for	Newtonian mechanics Universal gravitation Infinitesimal calculus Optics Binomial series Newton's method Philosophiæ Naturalis Principia Mathematica
Influences	Henry More ^[5] Polish Brethren ^[6]
Influenced	Nicolas Fatio de Duillier John Keill

<p style="text-align: center;">Signature</p> 
<p>Notes</p> <p>His mother was Hannah Ayscough. His half-niece was Catherine Barton.</p>

Sir Isaac Newton PRS (4 January 1643 – 31 March 1727 [OS: 25 December 1642 – 20 March 1727])^[1] was an English physicist, mathematician, astronomer, natural philosopher, alchemist, and theologian. His monograph *Philosophiæ Naturalis Principia Mathematica*, published in 1687, lays the foundations for most of classical mechanics and is one of the most important scientific books ever written. In this work, Newton described universal gravitation and the three laws of motion, which dominated the scientific view of the physical universe for the next three centuries. Newton showed that the motions of objects on Earth and of celestial bodies are governed by the same set of natural laws, by demonstrating the consistency between Kepler's laws of planetary motion and his theory of gravitation; thus removing the last doubts about heliocentrism and advancing the Scientific Revolution.

Newton built the first practical reflecting telescope^[7] and developed a theory of colour based on the observation that a prism decomposes white light into the many colours that form the visible spectrum. He also formulated an empirical law of cooling and studied the speed of sound.

In mathematics, Newton shares the credit with Gottfried Leibniz for the development of differential and integral calculus. He also demonstrated the generalised binomial theorem, developed Newton's method for approximating the roots of a function, and contributed to the study of power series.

Newton was also highly religious. He was an unorthodox Christian, and during his lifetime actually wrote more on Biblical hermeneutics and occult studies than on science and mathematics, the subjects he is mainly associated with. Newton secretly rejected Trinitarianism, fearing to be accused of refusing holy orders. Later studies on his biography claimed that he was influenced by Muslim Arab Scholars, too.^[8]

Newton is considered by many scholars and members of the general public to be one of the most influential people in human history.

Life

Early life

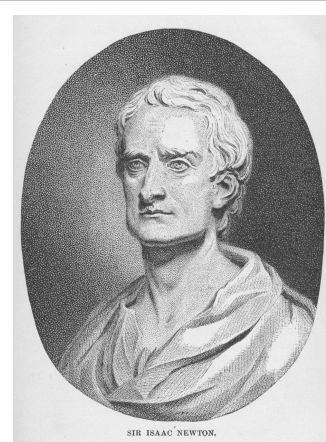
Isaac Newton was born on 4 January 1643 [OS: 25 December 1642]^[1] at Woolsthorpe Manor in Woolsthorpe-by-Colsterworth, a hamlet in the county of Lincolnshire. At the time of Newton's birth, England had not adopted the Gregorian calendar and therefore his date of birth was recorded as Christmas Day, 25 December 1642. Newton was born three months after the death of his father, a prosperous farmer also named Isaac Newton. Born prematurely, he was a small child; his mother Hannah Ayscough reportedly said that he could have fit inside a quart mug (≈ 1.1 litres). When Newton was three, his mother remarried and went to live with her new husband, the Reverend Barnabus Smith, leaving her son in the care of his maternal grandmother, Margery Ayscough. The young Isaac disliked his stepfather and held some enmity towards his mother for marrying him, as revealed by this entry in a list of sins committed up to the age of 19: "Threatening my father and mother Smith to burn them and the house over them."^[9] While Newton was once engaged in his late teens to a Miss Storey, he never married, being highly engrossed in his studies and work.^{[10] [11] [12]}

From the age of about twelve until he was seventeen, Newton was educated at The King's School, Grantham (where his alleged signature can still be seen upon a library window sill)^[13]. He was removed from school, and by October 1659, he was to be found at Woolsthorpe-by-Colsterworth, where his mother, widowed by now for a second time, attempted to make a farmer of him. He hated farming.^[14] Henry Stokes, master at the King's School, persuaded his mother to send him back to school so that he might complete his education. Motivated partly by a desire for revenge against a schoolyard bully, he became the top-ranked student.^[15]

In June 1661, he was admitted to Trinity College, Cambridge as a sizar — a sort of work-study role.^[16] At that time, the college's teachings were based on those of Aristotle, but Newton preferred to read the more advanced ideas of modern philosophers, such as Descartes, and of astronomers such as Copernicus, Galileo, and Kepler. In 1665, he discovered the generalised binomial theorem and began to develop a mathematical theory that would later become infinitesimal calculus. Soon after Newton had obtained his degree in August 1665, the university temporarily closed as a precaution against the Great Plague. Although he had been undistinguished as a Cambridge student,^[17] Newton's private studies at his home in Woolsthorpe over the subsequent two years saw the development of his theories on calculus, optics and the law of gravitation. In 1667, he returned to Cambridge as a fellow of Trinity.^[18] Fellows were required to become ordained priests, something Newton desired to avoid due to his unorthodox views. Luckily for Newton, there was no specific deadline for ordination and it could be postponed indefinitely. The problem became more severe later when Newton was elected for the prestigious Lucasian Chair. For such a significant appointment, ordaining normally could not be dodged. Nevertheless, Newton managed to avoid it by means of a special permission from Charles II (see "Middle years" section below).



Newton in a 1702 portrait by
Godfrey Kneller



Isaac Newton (*Bolton, Sarah K.*
Famous Men of Science. NY: Thomas
Y. Crowell & Co., 1889)

Middle years

Mathematics

Newton's work has been said "to distinctly advance every branch of mathematics then studied".^[19]

His work on the subject usually referred to as fluxions or calculus is seen, for example, in a manuscript of October 1666, now published among Newton's mathematical papers.^[20] A related subject was infinite series. Newton's manuscript "De analysi per aequationes numero terminorum infinitas" ("On analysis by equations infinite in number of terms") was sent by Isaac Barrow to John Collins in June 1669: in August 1669 Barrow identified its author to Collins as "Mr Newton, a fellow of our College, and very young ... but of an extraordinary genius and proficiency in these things".^[21]

Newton later became involved in a dispute with Leibniz over priority in the development of infinitesimal calculus. Most modern historians believe that Newton and Leibniz developed infinitesimal calculus independently, although with very different notations. Occasionally it has been suggested that Newton published almost nothing about it until 1693, and did not give a full account until 1704, while Leibniz began publishing a full account of his methods in 1684. (Leibniz's notation and "differential Method", nowadays recognised as much more convenient notations, were adopted by continental European mathematicians, and after 1820 or so, also by British mathematicians.) Such a suggestion, however, fails to notice the content of calculus which critics of Newton's time and modern times have

pointed out in Book 1 of Newton's *Principia* itself (published 1687) and in its forerunner manuscripts, such as *De motu corporum in gyrum* ("On the motion of bodies in orbit"), of 1684. The *Principia* is not written in the language of calculus either as we know it or as Newton's (later) 'dot' notation would write it. But his work extensively uses an infinitesimal calculus in geometric form, based on limiting values of the ratios of vanishing small quantities: in the *Principia* itself Newton gave demonstration of this under the name of 'the method of first and last ratios'^[22] and explained why he put his expositions in this form,^[23] remarking also that 'hereby the same thing is performed as by the method of indivisibles'.

Because of this, the *Principia* has been called "a book dense with the theory and application of the infinitesimal calculus" in modern times^[24] and "lequel est presque tout de ce calcul" ('nearly all of it is of this calculus') in Newton's time.^[25] His use of methods involving "one or more orders of the infinitesimally small" is present in his *De motu corporum in gyrum* of 1684^[26] and in his papers on motion "during the two decades preceding 1684".^[27]

Newton had been reluctant to publish his calculus because he feared controversy and criticism.^[28] He had a very close relationship with Swiss mathematician Nicolas Fatio de Duillier, who from the beginning was impressed by Newton's gravitational theory. In 1691, Duillier planned to prepare a new version of Newton's *Principia*, but never finished it. However, in 1693 the relationship between the two men changed. At the time, Duillier had also exchanged several letters with Leibniz.^[29]

Starting in 1699, other members of the Royal Society (of which Newton was a member) accused Leibniz of plagiarism, and the dispute broke out in full force in 1711. The Royal Society proclaimed in a study that it was Newton who was the true discoverer and labelled Leibniz a fraud. This study was cast into doubt when it was later found that Newton himself wrote the study's concluding remarks on Leibniz. Thus began the bitter controversy which marred the lives of both Newton and Leibniz until the latter's death in 1716.^[30]

Newton is generally credited with the generalised binomial theorem, valid for any exponent. He discovered Newton's identities, Newton's method, classified cubic plane curves (polynomials of degree three in two variables), made substantial contributions to the theory of finite differences, and was the first to use fractional indices and to employ coordinate geometry to derive solutions to Diophantine equations. He approximated partial sums of the harmonic series by logarithms (a precursor to Euler's summation formula), and was the first to use power series with confidence and to revert power series.

He was appointed Lucasian Professor of Mathematics in 1669 on Barrow's recommendation. In that day, any fellow of Cambridge or Oxford was required to become an ordained Anglican priest. However, the terms of the Lucasian professorship required that the holder *not* be active in the church (presumably so as to have more time for science). Newton argued that this should exempt him from the ordination requirement, and Charles II, whose permission was needed, accepted this argument. Thus a conflict between Newton's religious views and Anglican orthodoxy was averted.^[31]

Optics

From 1670 to 1672, Newton lectured on optics. During this period he investigated the refraction of light, demonstrating that a prism could decompose white light into a spectrum of colours, and that a lens and a second prism could recombine the multicoloured spectrum into white light.^[33]

He also showed that the coloured light does not change its properties by separating out a coloured beam and shining it on various objects. Newton noted that regardless of whether it was reflected or scattered or transmitted, it stayed the same colour. Thus, he observed that colour is the result of objects interacting with already-coloured light rather than objects generating the colour themselves. This is known as Newton's theory of colour.^[34]

From this work, he concluded that the lens of any refracting telescope would suffer from the dispersion of light into colours (chromatic aberration). As a proof of the concept, he constructed a telescope using a mirror as the objective to bypass that problem.^[35] Building the design, the first known functional reflecting telescope, today known as a Newtonian telescope,^[35] involved solving the problem of a suitable mirror material and shaping technique. Newton ground his own mirrors out of a custom composition of highly reflective speculum metal, using Newton's rings to judge the quality of the optics for his telescopes. In late 1668^[36] he was able to produce this first *reflecting telescope*. In 1671, the Royal Society asked for a demonstration of his reflecting telescope.^[37] Their interest encouraged him to publish his notes *On Colour*, which he later expanded into his *Opticks*. When Robert Hooke criticised some of Newton's ideas, Newton was so offended that he withdrew from public debate. Newton and Hooke had brief exchanges in 1679-80, when Hooke, appointed to manage the Royal Society's correspondence, opened up a correspondence intended to elicit contributions from Newton to Royal Society transactions,^[38] which had the effect of stimulating Newton to work out a proof that the elliptical form of planetary orbits would result from a centripetal force inversely proportional to the square of the radius vector (see Newton's law of universal gravitation - History and *De motu corporum in gyrum*). But the two men remained generally on poor terms until Hooke's death.^[39]

Newton argued that light is composed of particles or corpuscles, which were refracted by accelerating into a denser medium. He verged on soundlike waves to explain the repeated pattern of reflection and transmission by thin films (*Opticks* Bk.II, Props. 12), but still retained his theory of 'fits' that disposed corpuscles to be reflected or transmitted (Props.13). Later physicists instead favoured a purely wavelike explanation of light to account for the interference patterns, and the general phenomenon of diffraction. Today's quantum mechanics, photons and the idea of wave-particle duality bear only a minor resemblance to Newton's understanding of light.

In his *Hypothesis of Light* of 1675, Newton posited the existence of the ether to transmit forces between particles. The contact with the theosophist Henry More, revived his interest in alchemy. He replaced the ether with occult forces based on Hermetic ideas of attraction and repulsion between particles. John Maynard Keynes, who acquired many of Newton's writings on alchemy, stated that "Newton was not the first of the age of reason: He was the last of the magicians."^[40] Newton's interest in alchemy cannot be isolated from his contributions to science; however, he did apparently abandon his alchemical researches.^[5] (This was at a time when there was no clear distinction between alchemy and science.) Had he not relied on the occult idea of action at a distance, across a vacuum, he might not have developed his theory of gravity. (See also Isaac Newton's occult studies.)

In 1704, Newton published *Opticks*, in which he expounded his corpuscular theory of light. He considered light to be made up of extremely subtle corpuscles, that ordinary matter was made of grosser corpuscles and speculated that through a kind of alchemical transmutation "Are not gross Bodies and Light convertible into one another, ...and may



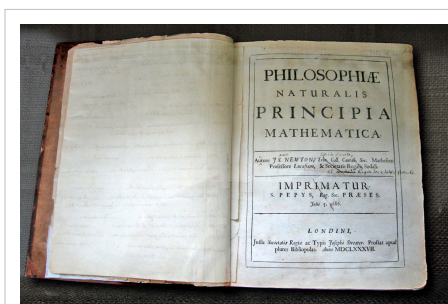
A replica of Newton's second Reflecting telescope that he presented to the Royal Society in 1672^[32]

not Bodies receive much of their Activity from the Particles of Light which enter their Composition?"^[41] Newton also constructed a primitive form of a frictional electrostatic generator, using a glass globe (Optics, 8th Query).

In an article entitled "Newton, prisms, and the 'opticks' of tunable lasers"^[42] it is indicated that Newton in his book *Opticks* was the first to show a diagram using a prism as a beam expander. In the same book he describes, via diagrams, the use of multiple-prism arrays. Some 278 years after Newton's discussion, multiple-prism expanders became central to the development of narrow-linewidth tunable lasers. Also, the use of these prismatic beam expanders led to the multiple-prism dispersion theory.^[42]

Mechanics and gravitation

In 1679, Newton returned to his work on (celestial) mechanics, i.e., gravitation and its effect on the orbits of planets, with reference to Kepler's laws of planetary motion. This followed stimulation by a brief exchange of letters in 1679-80 with Hooke, who had been appointed to manage the Royal Society's correspondence, and who opened a correspondence intended to elicit contributions from Newton to Royal Society transactions.^[38] Newton's reawakening interest in astronomical matters received further stimulus by the appearance of a comet in the winter of 1680-1681, on which he corresponded with John Flamsteed.^[43] After the exchanges with Hooke, Newton worked out a proof that the elliptical form of planetary orbits would result from a centripetal force inversely proportional to the square of the radius vector (see Newton's law of universal gravitation - History and *De motu corporum in gyrum*). Newton communicated his results to Edmond Halley and to the Royal Society in *De motu corporum in gyrum*, a tract written on about 9 sheets which was copied into the Royal Society's Register Book in December 1684.^[44] This tract contained the nucleus that Newton developed and expanded to form the *Principia*.



Newton's own copy of his *Principia*, with hand-written corrections for the second edition

The *Principia* was published on 5 July 1687 with encouragement and financial help from Edmond Halley. In this work, Newton stated the three universal laws of motion that enabled many of the advances of the Industrial Revolution which soon followed and were not to be improved upon for more than 200 years, and are still the underpinnings of the non-relativistic technologies of the modern world. He used the Latin word *gravitas* (weight) for the effect that would become known as gravity, and defined the law of universal gravitation.

In the same work, Newton presented a calculus-like method of geometrical analysis by 'first and last ratios', gave the first analytical determination (based on Boyle's law) of the speed of sound in air, inferred the oblateness of the spheroidal figure of the Earth, accounted for the precession of the equinoxes as a result of the Moon's gravitational attraction on the Earth's oblateness, initiated the gravitational study of the irregularities in the motion of the moon, provided a theory for the determination of the orbits of comets, and much more.

Newton made clear his heliocentric view of the solar system – developed in a somewhat modern way, because already in the mid-1680s he recognised the "deviation of the Sun" from the centre of gravity of the solar system.^[45] For Newton, it was not precisely the centre of the Sun or any other body that could be considered at rest, but rather "the common centre of gravity of the Earth, the Sun and all the Planets is to be esteem'd the Centre of the World", and this centre of gravity "either is at rest or moves uniformly forward in a right line" (Newton adopted the "at rest" alternative in view of common consent that the centre, wherever it was, was at rest).^[46]

Newton's postulate of an invisible force able to act over vast distances led to him being criticised for introducing "occult agencies" into science.^[47] Later, in the second edition of the *Principia* (1713), Newton firmly rejected such criticisms in a concluding General Scholium, writing that it was enough that the phenomena implied a gravitational attraction, as they did; but they did not so far indicate its cause, and it was both unnecessary and improper to frame hypotheses of things that were not implied by the phenomena. (Here Newton used what became his famous

expression *Hypotheses non fingo*).

With the *Principia*, Newton became internationally recognised.^[48] He acquired a circle of admirers, including the Swiss-born mathematician Nicolas Fatio de Duillier, with whom he formed an intense relationship that lasted until 1693, when it abruptly ended, at the same time that Newton suffered a nervous breakdown.^[49]

Later life

In the 1690s, Newton wrote a number of religious tracts dealing with the literal interpretation of the Bible. Henry More's belief in the Universe and rejection of Cartesian dualism may have influenced Newton's religious ideas. A manuscript he sent to John Locke in which he disputed the existence of the Trinity was never published. Later works – *The Chronology of Ancient Kingdoms Amended* (1728) and *Observations Upon the Prophecies of Daniel and the Apocalypse of St. John* (1733) – were published after his death. He also devoted a great deal of time to alchemy (see above).

Newton was also a member of the Parliament of England from 1689 to 1690 and in 1701, but according to some accounts his only comments were to complain about a cold draught in the chamber and request that the window be closed.^[51]

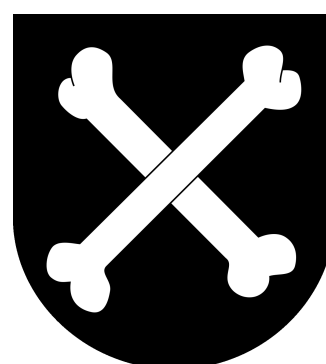
Newton moved to London to take up the post of warden of the Royal Mint in 1696, a position that he had obtained through the patronage of Charles Montagu, 1st Earl of Halifax, then Chancellor of the Exchequer. He took charge of England's great recoinage, somewhat treading on the toes of Lord Lucas, Governor of the Tower (and securing the job of deputy comptroller of the temporary Chester branch for Edmond Halley). Newton became perhaps the best-known Master of the Mint upon the death of Thomas Neale in 1699, a position Newton held until his death. These appointments were intended as sinecures, but Newton took them seriously, retiring from his Cambridge duties in 1701, and exercising his power to reform the currency and punish clippers and counterfeiters. As Master of the Mint in 1717 in the "Law of Queen Anne" Newton moved the Pound Sterling *de facto* from the silver standard to the gold standard by setting the bimetallic relationship between gold coins and the silver penny in favour of gold. This caused silver sterling coin to be melted and shipped out of Britain. Newton was made President of the Royal Society in 1703 and an associate of the French Académie des Sciences. In his position at the Royal Society, Newton made an enemy of John Flamsteed, the Astronomer Royal, by prematurely publishing Flamsteed's *Historia Coelestis Britannica*, which Newton had used in his studies.^[52]

In April 1705, Queen Anne knighted Newton during a royal visit to Trinity College, Cambridge. The knighthood is likely to have been motivated by political considerations connected with the Parliamentary election in May 1705, rather than any recognition of Newton's scientific work or services as Master of the Mint.^[53] Newton was the second scientist to be knighted, after Sir Francis Bacon.

Towards the end of his life, Newton took up residence at Cranbury Park, near Winchester with his niece and her husband, until his death in 1727.^[54] Newton died in his sleep in London on 31 March 1727 [OS: 20 March 1726],^[1] and was buried in Westminster Abbey. His half-niece, Catherine Barton Conduitt,^[55] served as his hostess in social affairs at his house on Jermyn Street in London; he was her "very loving Uncle,"^[56] according to his letter to her when she was recovering from smallpox. Newton, a bachelor, had divested much of his estate to relatives during his last years, and died intestate.



Isaac Newton in old age in 1712, portrait by Sir James Thornhill



Personal coat of arms of Sir Isaac Newton^[50]

After his death, Newton's body was discovered to have had massive amounts of mercury in it, probably resulting from his alchemical pursuits. Mercury poisoning could explain Newton's eccentricity in late life.^[57]

After death

Fame

French mathematician Joseph-Louis Lagrange often said that Newton was the greatest genius who ever lived, and once added that Newton was also "the most fortunate, for we cannot find more than once a system of the world to establish."^[58] English poet Alexander Pope was moved by Newton's accomplishments to write the famous epitaph:

Nature and nature's laws lay hid in night;
God said "Let Newton be" and all was light.

Newton himself had been rather more modest of his own achievements, famously writing in a letter to Robert Hooke in February 1676:

If I have seen further it is by standing on the shoulders of giants.^[59] ^[60]

Two writers think that the above quote, written at a time when Newton and Hooke were in dispute over optical discoveries, was an oblique attack on Hooke (said to have been short and hunchbacked), rather than – or in addition to – a statement of modesty.^[61] ^[62] On the other hand, the widely known proverb about standing on the shoulders of giants published among others by 17th-century poet George Herbert (a former orator of the University of Cambridge and fellow of Trinity College) in his *Jacula Prudentum* (1651), had as its main point that "a dwarf on a giant's shoulders sees farther of the two", and so its effect as an analogy would place Newton himself rather than Hooke as the 'dwarf'.

In a later memoir, Newton wrote:

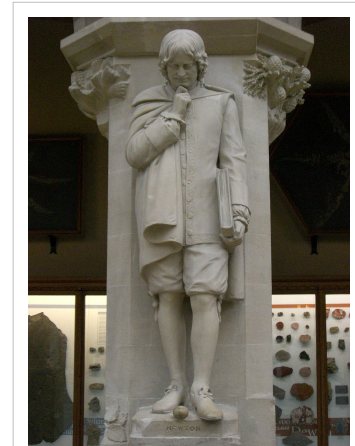
I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.^[63]

Newton remains influential to scientists, as demonstrated by a 2005 survey of members of Britain's Royal Society (formerly headed by Newton) asking who had the greater effect on the history of science, Newton or Albert Einstein. Royal Society scientists deemed Newton to have made the greater overall contribution.^[64] In 1999, an opinion poll of 100 of today's leading physicists voted Einstein the "greatest physicist ever;" with Newton the runner-up, while a parallel survey of rank-and-file physicists by the site PhysicsWeb gave the top spot to Newton.^[65] Charles Murray quantitatively ranked great innovators in his book *Human Accomplishment* and found Newton to be the second most important person in mathematics, to be one of the two most important persons in physics, and to be the most important person in all of science combined.

Commemorations

Newton's monument (1731) can be seen in Westminster Abbey, at the north of the entrance to the choir against the choir screen, near his tomb. It was executed by the sculptor Michael Rysbrack (1694–1770) in white and grey marble with design by the architect William Kent. The monument features a figure of Newton reclining on top of a sarcophagus, his right elbow resting on several of his great books and his left hand pointing to a scroll with a mathematical design. Above him is a pyramid and a celestial globe showing the signs of the Zodiac and the path of the comet of 1680. A relief panel depicts putti using instruments such as a telescope and prism.^[66] The Latin inscription on the base translates as:

Here is buried Isaac Newton, Knight, who by a strength of mind almost divine, and mathematical principles peculiarly his own, explored the course and figures of the planets, the paths of comets, the tides of the sea, the dissimilarities in rays of light, and, what no other scholar has previously imagined, the properties of the colours thus produced. Diligent, sagacious and faithful, in his expositions of nature, antiquity and the holy Scriptures, he vindicated by his philosophy the majesty of God mighty and good, and expressed the simplicity of the Gospel in his manners. Mortals rejoice that there has existed such and so great an ornament of the human race! He was born on 25 December 1642, and died on 20 March 1726/7. — Translation from G.L. Smyth, *The Monuments and Genii of St. Paul's Cathedral, and of Westminster Abbey* (1826), ii, 703–4.^[66]



Newton statue on display at the Oxford University Museum of Natural History

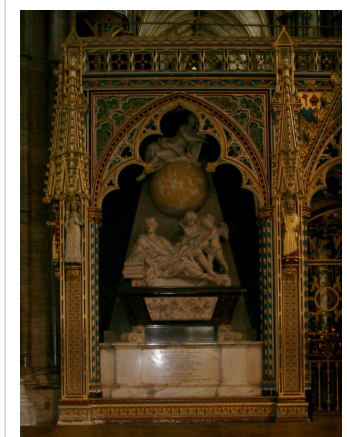
From 1978 until 1988, an image of Newton designed by Harry Ecclestone appeared on Series D £1 banknotes issued by the Bank of England (the last £1 notes to be issued by the Bank of England). Newton was shown on the reverse of the notes holding a book and accompanied by a telescope, a prism and a map of the Solar System.^[67]

A statue of Isaac Newton, standing over an apple, can be seen at the Oxford University Museum of Natural History.

Religious views

According to most scholars, Newton was a monotheist who believed in biblical prophecies but was Antitrinitarian.^[6] ^[68] 'In Newton's eyes, worshipping Christ as God was idolatry, to him the fundamental sin'.^[69] Historian Stephen D. Snobelen says of Newton, "Isaac Newton was a heretic. But ... he never made a public declaration of his private faith — which the orthodox would have deemed extremely radical. He hid his faith so well that scholars are still unravelling his personal beliefs."^[6] Snobelen concludes that Newton was at least a Socinian sympathiser (he owned and had thoroughly read at least eight Socinian books), possibly an Arian and almost certainly an antitrinitarian.^[6] In an age notable for its religious intolerance, there are few public expressions of Newton's radical views, most notably his refusal to take holy orders and his refusal, on his death bed, to take the sacrament when it was offered to him.^[6]

In a view disputed by Snobelen,^[6] T.C. Pfizenmaier argues that Newton held the Arian view of the Trinity rather than the Western one held by Roman Catholics, Anglicans, and most Protestants.^[70] In his own day, he was also accused of being a Rosicrucian (as were many in the Royal Society and in the court of Charles II).^[71]



Newton's tomb in Westminster Abbey

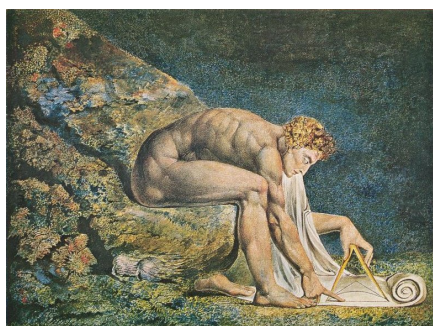
Although the laws of motion and universal gravitation became Newton's best-known discoveries, he warned against using them to view the Universe as a mere machine, as if akin to a great clock. He said, "Gravity explains the motions of the planets, but it cannot explain who set the planets in motion. God governs all things and knows all that is or can be done."^[72]

His scientific fame notwithstanding, Newton's studies of the Bible and of the early Church Fathers were also noteworthy. Newton wrote works on textual criticism, most notably *An Historical Account of Two Notable Corruptions of Scripture*. He also placed the crucifixion of Jesus Christ at 3 April, AD 33, which agrees with one traditionally accepted date.^[73] He also tried, unsuccessfully, to find hidden messages within the Bible.

Newton wrote more on religion than he did on natural science. He believed in a rationally immanent world, but he rejected the hylozoism implicit in Leibniz and Baruch Spinoza. Thus, the ordered and dynamically informed Universe could be understood, and must be understood, by an active reason. In his correspondence, Newton claimed that in writing the *Principia* "I had an eye upon such Principles as might work with considering men for the belief of a Deity".^[74] He saw evidence of design in the system of the world: "Such a wonderful uniformity in the planetary system must be allowed the effect of choice". But Newton insisted that divine intervention would eventually be required to reform the system, due to the slow growth of instabilities.^[75] For this, Leibniz lampooned him: "God Almighty wants to wind up his watch from time to time: otherwise it would cease to move. He had not, it seems, sufficient foresight to make it a perpetual motion."^[76] Newton's position was vigorously defended by his follower Samuel Clarke in a famous correspondence.

Effect on religious thought

Newton and Robert Boyle's mechanical philosophy was promoted by rationalist pamphleteers as a viable alternative to the pantheists and enthusiasts, and was accepted hesitantly by orthodox preachers as well as dissident preachers like the latitudinarians.^[77] Thus, the clarity and simplicity of science was seen as a way to combat the emotional and metaphysical superlatives of both superstitious enthusiasm and the threat of atheism,^[78] and, at the same time, the second wave of English deists used Newton's discoveries to demonstrate the possibility of a "Natural Religion".



"Newton", by William Blake; here, Newton is depicted critically as a "divine geometer".

The attacks made against pre-Enlightenment "magical thinking", and the mystical elements of Christianity, were given their foundation with Boyle's mechanical conception of the Universe. Newton gave Boyle's ideas their completion through mathematical proofs and, perhaps more importantly, was very successful in popularising them.^[79] Newton refashioned the world governed by an interventionist God into a world crafted by a God that designs along rational and universal principles.^[80] These principles were available for all people to discover, allowed people to pursue their own aims fruitfully in this life, not the next, and to perfect themselves with their own rational powers.^[81]

Newton saw God as the master creator whose existence could not be denied in the face of the grandeur of all creation.^[82] ^[83] ^[84] His spokesman, Clarke, rejected Leibniz' theodicy which cleared God from the responsibility for *l'origine du mal* by making God removed from participation in his creation, since as Clarke pointed out, such a deity would be a king in name only, and but one step away from atheism.^[85] But the unforeseen theological consequence of the success of Newton's system over the next century was to reinforce the deist position advocated by Leibniz.^[86] The understanding of the world was now brought down to the level of simple human reason, and humans, as Odo Marquard argued, became responsible for the correction and elimination of evil.^[87]

On the other hand, latitudinarian and Newtonian ideas taken too far resulted in the millenarians, a religious faction dedicated to the concept of a mechanical Universe, but finding in it the same enthusiasm and mysticism that the Enlightenment had fought so hard to extinguish.^[88]

Views of the end of the world

In a manuscript he wrote in 1704 in which he describes his attempts to extract scientific information from the Bible, he estimated that the world would end no earlier than 2060. In predicting this he said, "This I mention not to assert when the time of the end shall be, but to put a stop to the rash conjectures of fanciful men who are frequently predicting the time of the end, and by doing so bring the sacred propheties into discredit as often as their predictions fail."^[89]

Enlightenment philosophers

Enlightenment philosophers chose a short history of scientific predecessors — Galileo, Boyle, and Newton principally — as the guides and guarantors of their applications of the singular concept of Nature and Natural Law to every physical and social field of the day. In this respect, the lessons of history and the social structures built upon it could be discarded.^[90]

It was Newton's conception of the Universe based upon Natural and rationally understandable laws that became one of the seeds for Enlightenment ideology.^[91] Locke and Voltaire applied concepts of Natural Law to political systems advocating intrinsic rights; the physiocrats and Adam Smith applied Natural conceptions of psychology and self-interest to economic systems; and sociologists criticised the current social order for trying to fit history into Natural models of progress. Monboddo and Samuel Clarke resisted elements of Newton's work, but eventually rationalised it to conform with their strong religious views of nature.

Counterfeiters

As warden of the Royal Mint, Newton estimated that 20 percent of the coins taken in during The Great Recoinage of 1696 were counterfeit. Counterfeiting was high treason, punishable by the felon's being hanged, drawn and quartered. Despite this, convicting the most flagrant criminals could be extremely difficult. However, Newton proved to be equal to the task.^[92] Disguised as a habitué of bars and taverns, he gathered much of that evidence himself.^[93] For all the barriers placed to prosecution, and separating the branches of government, English law still had ancient and formidable customs of authority. Newton had himself made a justice of the peace in all the home counties. Then he conducted more than 100 cross-examinations of witnesses, informers, and suspects between June 1698 and Christmas 1699. Newton successfully prosecuted 28 coiners.^[94]

One of Newton's cases as the King's attorney was against William Chaloner.^[95] Chaloner's schemes included setting up phony conspiracies of Catholics and then turning in the hapless conspirators whom he had entrapped. Chaloner made himself rich enough to posture as a gentleman. Petitioning Parliament, Chaloner accused the Mint of providing tools to counterfeiters (a charge also made by others). He proposed that he be allowed to inspect the Mint's processes in order to improve them. He petitioned Parliament to adopt his plans for a coinage that could not be counterfeited, while at the same time striking false coins.^[96] Newton put Chaloner on trial for counterfeiting and had him sent to Newgate Prison in September 1697. But Chaloner had friends in high places, who helped him secure an acquittal and his release.^[95] Newton put him on trial a second time with conclusive evidence. Chaloner was convicted of high treason and hanged, drawn and quartered on 23 March 1699 at Tyburn gallows.^[97]

Laws of motion

The famous three laws of motion (stated in modernised form): *Newton's First Law* (also known as the Law of Inertia) states that an object at rest tends to stay at rest and that an object in uniform motion tends to stay in uniform motion unless acted upon by a net external force.

Newton's Second Law states that an applied force, \vec{F} , on an object equals the rate of change of its momentum, \vec{p} , with time. Mathematically, this is expressed as

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = \vec{v} \frac{dm}{dt} + m \frac{d\vec{v}}{dt}.$$

If applied to an object with constant mass ($dm/dt = 0$), the first term vanishes, and by substitution using the definition of acceleration, the equation can be written in the iconic form

$$\vec{F} = m \vec{a}.$$

The first and second laws represent a break with the physics of Aristotle, in which it was believed that a force was necessary in order to maintain motion. They state that a force is only needed in order to *change* an object's state of motion. The SI unit of force is the newton, named in Newton's honour.

Newton's Third Law states that for every action there is an equal and opposite reaction. This means that any force exerted onto an object has a counterpart force that is exerted in the opposite direction back onto the first object. A common example is of two ice skaters pushing against each other and sliding apart in opposite directions. Another example is the recoil of a firearm, in which the force propelling the bullet is exerted equally back onto the gun and is felt by the shooter. Since the objects in question do not necessarily have the same mass, the resulting acceleration of the two objects can be different (as in the case of firearm recoil).

Unlike Aristotle's, Newton's physics is meant to be universal. For example, the second law applies both to a planet and to a falling stone.

The vector nature of the second law addresses the geometrical relationship between the direction of the force and the manner in which the object's momentum changes. Before Newton, it had typically been assumed that a planet orbiting the sun would need a forward force to keep it moving. Newton showed instead that all that was needed was an inward attraction from the sun. Even many decades after the publication of the *Principia*, this counterintuitive idea was not universally accepted, and many scientists preferred Descartes' theory of vortices.^[98]

Apple analogy



Reputed descendants of Newton's apple tree, at the Cambridge University Botanic Garden and the Instituto Balseiro library garden

Newton himself often told the story that he was inspired to formulate his theory of gravitation by watching the fall of an apple from a tree.^[99]

Cartoons have gone further to suggest the apple actually hit Newton's head, and that its impact somehow made him aware of the force of gravity, though this is not reported in the biographical manuscript by William Stukeley, published in 1752, and made available by the Royal Society.^[100] It is known from his notebooks that Newton was grappling in the late 1660s with the idea that terrestrial gravity extends, in an inverse-square proportion, to the Moon; however it took him two decades to develop the full-fledged theory.^[101] John Conduitt, Newton's assistant at the Royal Mint and husband of Newton's niece, described the event when he wrote about Newton's life:

In the year 1666 he retired again from Cambridge to his mother in Lincolnshire. Whilst he was pensively meandering in a garden it came into his thought that the power of gravity (which brought an apple from a tree to the ground) was not limited to a certain distance from earth, but that this power must extend much further than was usually thought. Why not as high as the Moon said he to himself & if so, that must influence her motion & perhaps retain her in her orbit, whereupon he fell a calculating what would be the effect of that supposition.^[102]

The question was not whether gravity existed, but whether it extended so far from Earth that it could also be the force holding the moon to its orbit. Newton showed that if the force decreased as the inverse square of the distance, one could indeed calculate the Moon's orbital period, and get good agreement. He guessed the same force was responsible for other orbital motions, and hence named it "universal gravitation".

Stukeley recorded in his *Memoirs of Sir Isaac Newton's Life* a conversation with Newton in Kensington on 15 April 1726, in which Newton recalled:

when formerly, the notion of gravitation came into his mind. It was occasioned by the fall of an apple, as he sat in contemplative mood. Why should that apple always descend perpendicularly to the ground, thought he to himself. Why should it not go sideways or upwards, but constantly to the Earth's centre? Assuredly the reason is, that the Earth draws it. There must be a drawing power in matter. And the sum of the drawing power in the matter of the Earth must be in the Earth's centre, not in any side of the Earth. Therefore does this apple fall perpendicularly or towards the centre? If matter thus draws matter; it must be proportion of its quantity. Therefore the apple draws the Earth, as well as the Earth draws the apple."^[103]

In similar terms, Voltaire wrote in his *Essay on Epic Poetry* (1727), "Sir Isaac Newton walking in his gardens, had the first thought of his system of gravitation, upon seeing an apple falling from a tree."

Various trees are claimed to be "the" apple tree which Newton describes. The King's School, Grantham, claims that the tree was purchased by the school, uprooted and transported to the headmaster's garden some years later. The staff of the [now] National Trust-owned Woolsthorpe Manor dispute this, and claim that a tree present in their gardens is the one described by Newton. A descendant of the original tree can be seen growing outside the main gate of Trinity College, Cambridge, below the room Newton lived in when he studied there. The National Fruit Collection at Brogdale^[104] can supply grafts from their tree, which appears identical to Flower of Kent, a coarse-fleshed cooking variety.^[105]

Writings

- *Method of Fluxions* (1671)
- *Of Natures Obvious Laws & Processes in Vegetation* (unpublished, c. 1671–75)^[106]
- *De motu corporum in gyrum* (1684)
- *Philosophiæ Naturalis Principia Mathematica* (1687)
- *Opticks* (1704)
- *Reports as Master of the Mint*^[107] (1701–25)
- *Arithmetica Universalis* (1707)
- *The System of the World, Optical Lectures, The Chronology of Ancient Kingdoms, (Amended) and De mundi systemate* (published posthumously in 1728)
- *Observations on Daniel and The Apocalypse of St. John* (1733)
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- [1] During Newton's lifetime, two calendars were in use in Europe: the Julian or 'Old Style' in Britain and parts of northern Europe (Protestant and eastern Europe, and the Gregorian or 'New Style', in use in Roman Catholic Europe and elsewhere. At Newton's birth, Gregorian dates were ten days ahead of Julian dates: thus Newton was born on Christmas Day, 25 December 1642 by the Julian calendar, but on 4 January 1643 by the Gregorian. By the time he died, the difference between the calendars had increased to eleven days. Moreover, prior to the adoption of the Gregorian calendar in the UK in 1752, the English new year began (for legal and some other civil purposes) on 25 March ('Lady Day', i.e. the feast of the Annunciation: sometimes called 'Annunciation Style') rather than on 1 January (sometimes called 'Circumcision Style'). Unless otherwise noted, the remainder of the dates in this article follow the Julian Calendar.
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Bernhard Riemann

Bernhard Riemann	
<div></div> <div>Bernhard Riemann, 1863</div>	
Born	September 17, 1826Breselenz, Kingdom of Hanover (modern-day Germany)
Died	July 20, 1866 (aged 39)Selasca, Kingdom of Italy
Residence	Kingdom of Hanover
Nationality	German
Fields	Mathematics
Institutions	Georg-August University of Göttingen
Alma mater	Georg-August University of Göttingen Berlin University
Doctoral advisor	Carl Friedrich Gauss
Other academic advisors	Gotthold Eisenstein Moritz Abraham Stern
Notable students	Gustav Roch
Known for	See list
Influences	Johann Peter Gustav Lejeune Dirichlet

Georg Friedrich Bernhard Riemann (German pronunciation: [ˈʁiːman]; September 17, 1826 – July 20, 1866) was an influential German mathematician who made lasting contributions to analysis and differential geometry, some of them enabling the later development of general relativity.

Biography

Early years

Riemann was born in Breselenz, a village near Dannenberg in the Kingdom of Hanover in what is the Federal Republic of Germany today. His father, Friedrich Bernhard Riemann, was a poor Lutheran pastor in Breselenz who fought in the Napoleonic Wars. His mother, Charlotte Ebell, died before her children had reached adulthood. Riemann was the second of six children, shy, and suffered from numerous nervous breakdowns. Riemann exhibited exceptional mathematical skills, such as fantastic calculation abilities, from an early age but suffered from timidity and a fear of speaking in public.

Education

During 1840, Riemann went to Hanover to live with his grandmother and attend lyceum (middle school). After the death of his grandmother in 1842, he attended high school at the Johanneum Lüneburg. In high school, Riemann studied the Bible intensively, but he was often distracted by mathematics. To this end, he even tried to prove mathematically the correctness of the Book of Genesis. His teachers were amazed by his adept ability to solve complicated mathematical operations, in which he often outstripped his instructor's knowledge. In 1846, at the age of 19, he started studying philology and theology in order to become a priest and help with his family's finances.

During the spring of 1846, his father (Friedrich Riemann), after gathering enough money to send Riemann to university, allowed him to stop studying theology and start studying mathematics. He was sent to the renowned University of Göttingen, where he first met Carl Friedrich Gauss, and attended his lectures on the method of least squares.

In 1847, Riemann moved to Berlin, where Jacobi, Dirichlet, Steiner, and Eisenstein were teaching. He stayed in Berlin for two years and returned to Göttingen in 1849.

Academia

Bernhard Riemann held his first lectures in 1854, which founded the field of Riemannian geometry and thereby set the stage for Einstein's general theory of relativity. In 1857, there was an attempt to promote Riemann to extraordinary professor status at the University of Göttingen. Although this attempt failed, it did result in Riemann finally being granted a regular salary. In 1859, following Dirichlet's death, he was promoted to head the mathematics department at Göttingen. He was also the first to suggest using dimensions higher than merely three or four in order to describe physical reality^[1]—an idea that was ultimately vindicated with Einstein's contribution in the early 20th century. In 1862 he married Elise Koch and had a daughter.

Austro-Prussian War

Riemann fled Göttingen when the armies of Hanover and Prussia clashed there in 1866.^[2] He died of tuberculosis during his third journey to Italy in Selasca (now a hamlet of Verbania on Lake Maggiore) where he was buried in the cemetery in Biganzolo (Verbania). Meanwhile, in Göttingen his housekeeper tidied up some of the mess in his office, including much unpublished work. Riemann refused to publish incomplete work and some deep insights may have been lost forever.^[2]

Influence

Riemann's published works opened up research areas combining analysis with geometry. These would subsequently become major parts of the theories of Riemannian geometry, algebraic geometry, and complex manifold theory. The theory of Riemann surfaces was elaborated by Felix Klein and particularly Adolf Hurwitz. This area of mathematics is part of the foundation of topology, and is still being applied in novel ways to mathematical physics.

Riemann made major contributions to real analysis. He defined the Riemann integral by means of Riemann sums, developed a theory of trigonometric series that are not Fourier series—a first step in generalized function theory—and studied the Riemann–Liouville differintegral.

He made some famous contributions to modern analytic number theory. In a single short paper (the only one he published on the subject of number theory), he introduced the Riemann zeta function and established its importance for understanding the distribution of prime numbers. He made a series of conjectures about properties of the zeta function, one of which is the well-known Riemann hypothesis.

He applied the Dirichlet principle from variational calculus to great effect; this was later seen to be a powerful heuristic rather than a rigorous method. Its justification took at least a generation. His work on monodromy and the hypergeometric function in the complex domain made a great impression, and established a basic way of working

with functions by *consideration only of their singularities*.

Euclidean geometry versus Riemannian geometry

In 1853, Gauss asked his student Riemann to prepare a *Habilitationsschrift* on the foundations of geometry. Over many months, Riemann developed his theory of higher dimensions. When he finally delivered his lecture at Göttingen in 1854, the mathematical public received it with enthusiasm, and it is one of the most important works in geometry. It was titled *Über die Hypothesen welche der Geometrie zu Grunde liegen* ("On the hypotheses which underlie geometry"), and was published in 1868.

The subject founded by this work is Riemannian geometry. Riemann found the correct way to extend into n dimensions the differential geometry of surfaces, which Gauss himself proved in his *theorema egregium*. The fundamental object is called the Riemann curvature tensor. For the surface case, this can be reduced to a number (scalar), positive, negative or zero; the non-zero and constant cases being models of the known non-Euclidean geometries.

Higher dimensions

Riemann's idea was to introduce a collection of numbers at every point in space (i.e., a tensor) which would describe how much it was bent or curved. Riemann found that in four spatial dimensions, one needs a collection of ten numbers at each point to describe the properties of a manifold, no matter how distorted it is. This is the famous construction central to his geometry, known now as a Riemannian metric.

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Jean Dieudonné

Jean Alexandre Eugène Dieudonné	
 <p>Jean Alexandre Eugène Dieudonné</p>	
Born	1 July 1906Lille, France
Died	29 November 1992 (aged 86)Paris, France
Nationality	 France
Fields	Mathematics
Institutions	University of São Paulo University of Nancy University of Michigan Northwestern University Institut des Hautes Études Scientifiques University of Nice
Alma mater	École Normale Supérieure
Doctoral advisor	Paul Montel
Doctoral students	Edmond Fedida Alexander Grothendieck Kishore Marathe
Known for	Cartan–Dieudonné theorem

Jean Alexandre Eugène Dieudonné (1 July 1906 – 29 November 1992) was a French mathematician, notable for research in abstract algebra and functional analysis, for close involvement with the Nicolas Bourbaki pseudonymous group and the *Éléments de géométrie algébrique* project of Alexander Grothendieck, and as a historian of mathematics, particularly in the fields of functional analysis and algebraic topology. His work on the classical groups (the book *La Géométrie des groupes classiques* was published in 1955), and on formal groups, introducing what now are called Dieudonné modules, had a major effect on those fields.

He was born and brought up in Lille, with a formative stay in England where he was introduced to algebra. In 1924 he was accepted for the École Normale Supérieure, where André Weil was a contemporary. He began working, conventionally enough, in complex analysis. In 1934 he was one of the group of *normaliens* convened by Weil, which would become 'Bourbaki'.

Education and teaching

He served in the French Army in World War II, and then taught in Clermont-Ferrand until the liberation of France. After holding professorships at the University of São Paulo (1946–47), the University of Nancy (1948–1952) and the University of Michigan (1952–53), he joined the Department of Mathematics at Northwestern University in 1953, before returning to France as a founding member of the Institut des Hautes Études Scientifiques. He moved to the University of Nice to found the Department of Mathematics in 1964, and retired in 1970. He was elected as a member of the Académie des Sciences in 1968.

Career

He drafted much of the Bourbaki series of texts, the many volumes of the EGA algebraic geometry series, and nine volumes of his own *Traité d'Analyse*. The first volume of the *Traité* is a French translation of the book *Foundations of Modern Analysis* (1960), which had become a graduate textbook on functional analysis.

He also wrote individual monographs on *Infinitesimal Calculus*, *Linear Algebra and Elementary Geometry*, invariant theory, commutative algebra, algebraic geometry, and formal groups.

With Laurent Schwartz he supervised the early research of Alexander Grothendieck; later from 1959 to 1964 he was at IHÉS alongside Grothendieck, and collaborating on the expository work needed to support the project of refounding algebraic geometry on the new basis of schemes.

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Alexander Grothendieck

Alexander Grothendieck	
<div></div> <div>Alexander Grothendieck in Montreal, 1970</div>	
Born	28 March 1928Berlin, Germany
Residence	France
Nationality	None (Stateless)
Fields	Mathematics
Institutions	Institut des Hautes Études Scientifiques
Alma mater	University of Montpellier University of Nancy
Doctoral advisor	Laurent Schwartz
Doctoral students	Pierre Berthelot Pierre Deligne Michel Demazure Jean Giraud Luc Illusie Michel Raynaud Jean-Louis Verdier
Notable awards	Fields Medal (1966) Crafoord Prize (1988, declined)

Alexander Grothendieck (born 28 March 1928) is a mathematician and the central figure behind the creation of the modern theory of algebraic geometry. His research program vastly extended the scope of the field, incorporating major elements of commutative algebra, homological algebra, sheaf theory, and category theory into its foundations. This new perspective led to revolutionary advances across many areas of pure mathematics.

Within algebraic geometry itself, his theory of schemes has become the universally accepted language for all further technical work. His generalization of the classical Riemann-Roch theorem launched the study of algebraic and topological K-theory. His construction of new cohomology theories has left deep consequences for algebraic number theory, algebraic topology, and representation theory. His creation of topos theory has had an impact on set theory and logic.

One of his most celebrated achievements is the discovery of the first arithmetic Weil cohomology theory: the ℓ -adic étale cohomology. This key result opened the way for a proof of the Weil conjectures, ultimately completed by his student Pierre Deligne. To this day, ℓ -adic cohomology remains a fundamental tool for number theorists, with important applications to the Langlands program.

Grothendieck's way of thinking has influenced generations of mathematicians long after his departure from mathematics. His emphasis on the role of universal properties brought category theory into the mainstream as an

important organizing principle. His notion of abelian category is now the basic object of study in homological algebra. His conjectural theory of motives has been a driving force behind modern developments in algebraic K-theory, motivic homotopy theory, and motivic integration.

Driven by deep personal and political convictions, Grothendieck left the Institut des Hautes Études Scientifiques, where he had been appointed professor and accomplished his greatest work, after a dispute over military funding in 1970. His mathematical activity essentially ceased after this, and he devoted his energies to political causes. He formally retired in 1988 and within a few years moved to the Pyrenees, where he currently lives in isolation from human society.

Mathematical achievements

Grothendieck's early mathematical work was in functional analysis. Between 1949 and 1953 he worked on his doctoral thesis in this subject at Nancy, supervised by Jean Dieudonné and Laurent Schwartz. His key contributions include topological tensor products of topological vector spaces, the theory of nuclear spaces as foundational for Schwartz distributions, and the application of L^p spaces in studying linear maps between topological vector spaces. In a few years, he had turned himself into a leading authority on this area of functional analysis — to the extent that Dieudonné compares his impact in this field to that of Banach.^[1]

It is, however, in algebraic geometry and related fields where Grothendieck did his most important and influential work. From about 1955 he started to work on sheaf theory and homological algebra, producing the influential "Tôhoku paper" (*Sur quelques points d'algèbre homologique*, published in 1957) where he introduced Abelian categories and applied their theory to show that sheaf cohomology can be defined as certain derived functors in this context.

Homological methods and sheaf theory had already been introduced in algebraic geometry by Jean-Pierre Serre and others, after sheaves had been defined by Jean Leray. Grothendieck took them to a higher level of abstraction and turned them into a key organising principle of his theory. He shifted attention from the study of individual varieties to the *relative point of view* (pairs of varieties related by a morphism), allowing a broad generalization of many classical theorems. The first major application was the relative version of Serre's theorem showing that the cohomology of a coherent sheaf on a complete variety is finite dimensional; Grothendieck's theorem shows that the higher direct images of coherent sheaves under a proper map are coherent; this reduces to Serre's theorem over a one-point space.

In 1956, he applied the same thinking to the Riemann–Roch theorem, which had already recently been generalized to any dimension by Hirzebruch. The Grothendieck–Riemann–Roch theorem was announced by Grothendieck at the initial Mathematische Arbeitstagung in Bonn, in 1957. It appeared in print in a paper written by Armand Borel with Serre. This result was his first major achievement in algebraic geometry. He went on to plan and execute a major foundational programme for rebuilding the foundations of algebraic geometry, which were then in a state of flux and under discussion in Claude Chevalley's seminar; he outlined his programme in his talk at the 1958 International Congress of Mathematicians.

His foundational work on algebraic geometry is at a higher level of abstraction than all prior versions. He adapted the use of non-closed generic points, which led to the theory of schemes. He also pioneered the systematic use of nilpotents. As 'functions' these can take only the value 0, but they carry infinitesimal information, in purely algebraic settings. His *theory of schemes* has become established as the best universal foundation for this major field, because of its great expressive power as well as technical depth. In that setting one can use birational geometry, techniques from number theory, Galois theory and commutative algebra, and close analogues of the methods of algebraic topology, all in an integrated way.^{[2] [3] [4]}

He is also noted for his mastery of abstract approaches to mathematics and his perfectionism in matters of formulation and presentation. Relatively little of his work after 1960 was published by the conventional route of the learned journal, circulating initially in duplicated volumes of seminar notes; his influence was to a considerable

extent personal. His influence spilled over into many other branches of mathematics, for example the contemporary theory of D-modules. (It also provoked adverse reactions, with many mathematicians seeking out more concrete areas and problems.)^[5] ^[6]

EGA and SGA

The bulk of Grothendieck's published work is collected in the monumental, and yet incomplete, *Éléments de géométrie algébrique* (EGA) and *Séminaire de géométrie algébrique* (SGA). The collection *Fondements de la Géométrie Algébrique* (FGA), which gathers together talks given in the Séminaire Bourbaki, also contains important material.

Perhaps Grothendieck's deepest single accomplishment is the invention of the étale and l-adic cohomology theories, which explain an observation of André Weil's that there is a deep connection between the topological characteristics of a variety and its diophantine (number theoretic) properties. For example, the number of solutions of an equation over a finite field reflects the topological nature of its solutions over the complex numbers. Weil realized that to prove such a connection one needed a new cohomology theory, but neither he nor any other expert saw how to do this until such a theory was found by Grothendieck.

This program culminated in the proofs of the Weil conjectures, the last of which was settled by Grothendieck's student Pierre Deligne in the early 1970s after Grothendieck had largely withdrawn from mathematics.

Major mathematical topics (from *Récoltes et Semailles*)

He wrote a retrospective assessment of his mathematical work (see the external link *La Vision* below). As his main mathematical achievements ("maître-thèmes"), he chose this collection of 12 topics (his chronological order):

1. Topological tensor products and nuclear spaces
2. "Continuous" and "discrete" duality (derived categories and "six operations").
3. *Yoga* of the Grothendieck–Riemann–Roch theorem (K-theory, relation with intersection theory).
4. Schemes.
5. Topoi.
6. Étale cohomology including l-adic cohomology.
7. Motives and the motivic Galois group (and Grothendieck categories)
8. Crystals and crystalline cohomology, *yoga* of De Rham and Hodge coefficients.
9. Topological algebra, infinity-stacks, 'dérivateurs', cohomological formalism of toposes as an inspiration for a new homotopic algebra
10. Tame topology.
11. *Yoga* of anabelian geometry and Galois–Teichmüller theory.
12. Schematic point of view, or "arithmetics" for regular polyhedra and regular configurations of all sorts.

He wrote that the central theme of the topics above is that of topos theory, while the first and last were of the least importance to him.

Here the term *yoga* denotes a kind of "meta-theory" that can be used heuristically; Michel Raynaud writes the other terms "Ariadne's thread" and "philosophy" as effective equivalents.^[7]

Life

Family and early life

Alexander Grothendieck was born in Berlin to anarchist parents: a Ukrainian father from an ultimately Hassidic family, Alexander "Sascha" Shapiro aka Tanaroff, and a mother from a German Protestant family, Johanna "Hanka" Grothendieck; both of his parents had broken away from their early backgrounds in their teens.^[8] At the time of his birth Grothendieck's mother was married to Johannes Raddatz, a German journalist, and his birthname was initially recorded as *Alexander Raddatz*. The marriage was dissolved in 1929 and Shapiro/Tanaroff acknowledged his paternity, but never married Hanka Grothendieck.^[8] Grothendieck lived with his parents until 1933 in Berlin. At the end of that year, Shapiro moved to Paris, and Hanka followed him the next year. They left Grothendieck in the care of Wilhelm Heydorn, a Lutheran Pastor and teacher^[9] in Hamburg where he went to school. During this time, his parents fought in the Spanish Civil War.

During WWII

In 1939 Grothendieck came to France and lived in various camps for displaced persons with his mother, first at the Camp de Rieucros, and subsequently lived for the remainder of the war in the village of Le Chambon-sur-Lignon, where he was sheltered and hidden in local boarding-houses or pensions. His father was sent via Drancy to Auschwitz where he died in 1942. While Grothendieck lived in Chambon, he attended the Collège Cévenol (now known as the Le Collège-Lycée Cévenol International), a unique secondary school founded in 1938 by local Protestant pacifists and anti-war activists. Many of the refugee children being hidden in Chambon attended Cévenol and it was at this school that Grothendieck apparently first became fascinated with mathematics.

Studies and contact with research mathematics

After the war, the young Grothendieck studied mathematics in France, initially at the University of Montpellier. After three years of increasingly independent studies there he got a scholarship to go to continue his studies in Paris in 1948^[10]

Initially, Grothendieck attended Henri Cartan's Seminar at École Normale Supérieure, but lacked the necessary background to follow the high-powered seminar. On the advice of Cartan and Weil, he moved to the University of Nancy where he wrote his dissertation under Laurent Schwartz in functional analysis, from 1950 to 1953. At this time he was a leading expert in the theory of topological vector spaces. By 1957, he set this subject aside in order to work in algebraic geometry and homological algebra.

The IHÉS years

Installed at the Institut des Hautes Études Scientifiques (IHÉS), Grothendieck attracted attention by an intense and highly productive activity of seminars (*de facto* working groups drafting into foundational work some of the ablest French and other mathematicians of the younger generation). Grothendieck himself practically ceased publication of papers through the conventional, learned journal route. He was, however, able to play a dominant role in mathematics for around a decade, gathering a strong school.

During this time he had officially as students Michel Demazure (who worked on SGA3, on group schemes), Luc Illusie (cotangent complex), Michel Raynaud, Jean-Louis Verdier (cofounder of the derived category theory) and Pierre Deligne. Collaborators on the SGA projects also included Mike Artin (étale cohomology) and Nick Katz (monodromy theory and Lefschetz pencils). Jean Giraud worked out torsor theory extensions of non-abelian cohomology. Many others were involved.

The 'Golden Age'

Alexander Grothendieck's work during the 'Golden Age' period at IHÉS established several unifying themes in algebraic geometry, number theory, topology, category theory and complex analysis. His first (pre-IHÉS) breakthrough in algebraic geometry was the Grothendieck–Hirzebruch–Riemann–Roch theorem, a far-reaching generalisation of the Hirzebruch–Riemann–Roch theorem proved algebraically; in this context he also introduced K-theory. Then, following the programme he outlined in his talk at the 1958 International Congress of Mathematicians, he introduced the theory of schemes, developing it in detail in his *Éléments de géométrie algébrique* (EGA) and providing the new more flexible and general foundations for algebraic geometry that has been adopted in the field since that time. He went on to introduce the étale cohomology theory of schemes, providing the key tools for proving the Weil conjectures, as well as crystalline cohomology and algebraic de Rham cohomology to complement it. Closely linked to these cohomology theories, he originated topos theory as a generalisation of topology (relevant also in categorical logic). He also provided an algebraic definition of fundamental groups of schemes and more generally the main structures of a categorical Galois theory. As a framework for his coherent duality theory he also introduced derived categories, which were further developed by Verdier.

The results of work on these and other topics were published in the EGA and in less polished form in the notes of the Séminaire de géométrie algébrique (SGA) that he directed at IHÉS.

Politics and retreat from scientific community

Grothendieck's political views were radical and pacifist. Thus he strongly opposed both United States aggression in Vietnam and Soviet military expansionism. He gave lectures on category theory in the forests surrounding Hanoi while the city was being bombed, to protest against the Vietnam War (*The Life and Work of Alexander Grothendieck*, *American Mathematical Monthly*, vol. 113, no. 9, footnote 6). He retired from scientific life around 1970, after having discovered the partly military funding of IHÉS (see pp. xii and xiii of SGA1, Springer Lecture Notes 224). He returned to academia a few years later as a professor at the University of Montpellier, where he stayed until his retirement in 1988. His criticisms of the scientific community, and especially of several mathematics circles, are also contained in a letter, written in 1988, in which he states the reasons for his refusal of the Crafoord Prize.^[11] He declined the prize on ethical grounds in an open letter to the media.^[12]

While the issue of military funding was perhaps the most obvious explanation for Grothendieck's departure from IHÉS, those who knew him say that the causes of the rupture ran deeper. Pierre Cartier, a *visiteur de longue durée* ("long-term guest") at the IHÉS, wrote a piece about Grothendieck for a special volume published on the occasion of the IHÉS's fortieth anniversary. The *Grothendieck Festschrift* was a three-volume collection of research papers to mark his sixtieth birthday (falling in 1988), and published in 1990.^[13]

In it Cartier notes that, as the son of an antimilitary anarchist and one who grew up among the disenfranchised, Grothendieck always had a deep compassion for the poor and the downtrodden. As Cartier puts it, Grothendieck came to find Bures-sur-Yvette "*une cage dorée*" ("a golden cage"). While Grothendieck was at the IHÉS, opposition to the Vietnam War was heating up, and Cartier suggests that this also reinforced Grothendieck's distaste at having become a mandarin of the scientific world. In addition, after several years at the IHÉS Grothendieck seemed to cast about for new intellectual interests. By the late 1960s he had started to become interested in scientific areas outside of mathematics. David Ruelle, a physicist who joined the IHÉS faculty in 1964, said that Grothendieck came to talk to him a few times about physics. (In the 1970s Ruelle and the Dutch mathematician Floris Takens produced a new model for turbulence, and it was Ruelle who invented the concept of a strange attractor in a dynamical system.) Biology interested Grothendieck much more than physics, and he organized some seminars on biological topics.^[14]

After leaving the IHÉS, Grothendieck became a temporary professor at Collège de France for two years. A permanent position became open at the end of his tenure, but the application Grothendieck submitted made it clear that he had no plans to continue his mathematical research. The position was given to Jacques Tits.

He then went to Université de Montpellier, where he became increasingly estranged from the mathematical community. Around this time, he founded a group called *Survivre*, which was dedicated to antimilitary and ecological issues. His mathematical career, for the most part, ended when he left the IHÉS. In 1984 he wrote a proposal to get a position through the Centre National de la Recherche Scientifique. The proposal, entitled *Esquisse d'un Programme* ("Program Sketch") describes new ideas for studying the moduli space of complex curves. Although Grothendieck himself never published his work in this area, the proposal became the inspiration for work by other mathematicians and the source of the theory of dessin d'enfants. *Esquisse d'un Programme* was published in the two-volume proceedings *Geometric Galois Actions* (Cambridge University Press, 1997).^[15]

Manuscripts written in the 1980s

While not publishing mathematical research in conventional ways during the 1980s, he produced several influential manuscripts with limited distribution, with both mathematical and biographical content. During that period he also released his work on Bertini type theorems contained in EGA 5, published by the Grothendieck Circle^[16] in 2004.

La Longue Marche à travers la théorie de Galois [*The Long March Through Galois Theory*] is an approximately 1600-page handwritten manuscript produced by Grothendieck during the years 1980–1981, containing many of the ideas leading to the *Esquisse d'un programme*^[17] (see below, and also a more detailed entry), and in particular studying the Teichmüller theory.

In 1983 he wrote a huge extended manuscript (about 600 pages) entitled *Pursuing Stacks*, stimulated by correspondence with Ronald Brown, (see also R. Brown^[18] and Tim Porter at University of Bangor in Wales), and starting with a letter addressed to Daniel Quillen. This letter and successive parts were distributed from Bangor (see External Links below): in an informal manner, as a kind of diary, Grothendieck explained and developed his ideas on the relationship between algebraic homotopy theory and algebraic geometry and prospects for a noncommutative theory of stacks. The manuscript, which is being edited for publication by G. Maltsiniotis, later led to another of his monumental works, *Les Dérivateurs*. Written in 1991, this latter opus of about 2000 pages further developed the homotopical ideas begun in *Pursuing Stacks*. Much of this work anticipated the subsequent development of the motivic homotopy theory of Fabien Morel and V. Voevodsky in the mid 1990s.

His *Esquisse d'un programme*^[17] (1984) is a proposal for a position at the Centre National de la Recherche Scientifique, which he held from 1984 to his retirement in 1988. Ideas from it have proved influential, and have been developed by others, in particular dessins d'enfants and a new field emerging as anabelian geometry. In *La Clef des Songes* he explains how the reality of dreams convinced him of God's existence.

The 1000-page autobiographical manuscript *Récoltes et semailles* (1986) is now available on the internet in the French original, and an English translation is underway (these parts of *Récoltes et semailles* have already been translated into Russian] and published in Moscow^[19]). Some parts of *Récoltes et semailles*^[20] ^[21] and the whole *La Clef des Songes*^[22] have been translated into Spanish.

Retirement into reclusion

Grothendieck was co-awarded (but declined) the Crafoord Prize with Pierre Deligne in 1988.

In 1991, Grothendieck moved to an address he did not provide to his previous contacts in the mathematical community. He is now said to live in southern France or Andorra and to be reclusive.

In January 2010, Grothendieck wrote a letter to Luc Illusie. In this "Déclaration d'intention de non-publication", he states that essentially all materials that have been published in his absence have been done without his permission. He asks that none of his work should be reproduced in whole or in part, and even further that libraries containing such copies of his work remove them.^[23]

Notes

- [1] (Dieudonné 1990)
- [2] See, for example, (Deligne 1998).
- [3] Jackson, Allyn (2004), "Comme Appelé du Néant — As If Summoned from the Void: The Life of Alexandre Grothendieck I" (<http://www.ams.org/notices/200409/fea-grothendieck-part1.pdf>) (PDF), *Notices of the American Mathematical Society* **51** (4): 1049,
- [4] McLarty, Colin. "The Rising Sea: Grothendieck on simplicity and generality I" (<http://people.math.jussieu.fr/~leila/grothendieckcircle/mclarty1.pdf>) (PDF). . Retrieved 2008-01-13.
- [5] Peck, Morgen, *Equality of Mathematicians* (http://scienceline.org/2007/01/31/math_controversy_peck/), , "Alexandre Grothendieck is arguably the most important mathematician of the 20th century..."
- [6] Leith, Sam (20 March 2004), "The Einstein of maths" (<http://www.lewrockwell.com/spectator/spec262.html>), *The Spectator*, , "[A] mathematician of staggering accomplishment ... a legendary figure in the mathematical world."
- [7] at p. 2. (<http://www.ams.org/notices/200309/rev-raynaud.pdf>.)
- [8] Society for Industrial and Applied Mathematics (<http://www.siam.org/news/news.php?id=1405>)
- [9] Allyn Jackson, The Life of Alexander Grothendieck, p. 1040 (<http://www.ams.org/notices/200409/fea-grothendieck-part1.pdf>)
- [10] See Jackson (2004:1).
- [11] Crafoord Prize letter (<http://web.archive.org/web/20060106062005/http://www.math.columbia.edu/~lipyan/CrafoordPrize.pdf>)
- [12] Matthews, Robert (20 August 2006). "Mathematics, where nothing is ever as simple as it seems" (<http://www.telegraph.co.uk/news/1526781/Mathematics-where-nothing-is-ever-as-simple-as-it-seems.html>). *Daily Telegraph*. . Retrieved 5 July 2009.
- [13] The editors were Pierre Cartier, Luc Illusie, Nick Katz, Gérard Laumon, Yuri Manin, and Ken Ribet. A second edition has been printed (2007) by Birkhauser.
- [14] The IHÉS at Forty (<http://www.ams.org/notices/199903/ihes-changes.pdf>) Allyn Jackson, March 1999, Noticed of the AMS pp. 329-337
- [15] Google book link (http://books.google.co.uk/books?id=H1f1j2XmXkcC&dq=Geometric+Galois+Actions&printsec=frontcover&source=bl&ots=q9iN4QMkYj&sig=fAsvVlPOhN9K9LjGFJC5zm_IA4&hl=en&ei=KW-3ScSkHYHIMuDukOEK&sa=X&oi=book_result&resnum=1&ct=result)
- [16] <http://www.math.jussieu.fr/~leila/grothendieckcircle/index.php>
- [17] ESQUISSE D'UN PROGRAMME par Alexandre Grothendieck (<http://matematicas.unex.es/~navarro/res/esquissefr.pdf>)
- [18] <http://www.bangor.ac.uk/r.brown>
- [19] In Russian (<http://www.mccme.ru/free-books/grothendieck/RS.html>)
- [20] COSECHAS Y SIEMBRAS: Reflexiones y testimonios sobre un pasado de matemático (<http://matematicas.unex.es/~navarro/res/preludio.pdf>) Preludio
- [21] COSECHAS Y SIEMBRAS: Reflexiones y testimonios sobre un pasado de matemático (<http://matematicas.unex.es/~navarro/res/carta.pdf>) Carta
- [22] La Clef des Songes (<http://matematicas.unex.es/~navarro/res/clef1-6.pdf>)
- [23] <http://sbseminar.wordpress.com/2010/02/09/grothendiecks-letter>

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External links

- O'Connor, John J.; Robertson, Edmund F., "Alexander Grothendieck" (<http://www-history.mcs.st-andrews.ac.uk/Biographies/Grothendieck.html>), *MacTutor History of Mathematics archive*, University of St Andrews.
- Alexander Grothendieck (<http://www.genealogy.ams.org/id.php?id=31245>) at the Mathematics Genealogy Project
- Grothendieck Circle (<http://www.grothendieckcircle.org/>), collection of mathematical and biographical information, photos, links to his writings
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Charles Ehresmann

Charles Ehresmann



Charles Ehresmann (right) at the topology conference 1949 in Oberwolfach, together with Paul Vincensini (middle) and Georges Reeb (left)

Born	19 April 1905Straßburg, Alsace-Lorraine, German Empire (today Strasbourg, Alsace, France)
Died	22 September 1979Amiens, Picardy, France
Fields	Mathematics
Alma mater	École Normale Supérieure
Doctoral advisor	Élie Cartan
Doctoral students	Georges Reeb Wu Wen-Tsün André Haefliger Valentin Poénaru Daniel Tanré
Known for	Ehresmann's theorem Ehresmann connection

Charles Ehresmann (1905-1979) was a French mathematician who worked on differential topology and category theory. He is known for work on the topology of Lie groups, the *jet* concept (see jet bundle), and his seminar on category theory.

He attended the École Normale Supérieure in Paris before performing one year of military service. He finished his PhD thesis *Sur la topologie de certains espaces homogènes* (French: On the topology of certain homogeneous spaces) in 1934 under the supervision of Élie Cartan.

In 1957 he founded the mathematical journal *Cahiers de Topologie et Géométrie Différentielle Categoricals*.

Jean Dieudonné describes Ehresmann's personality as "... distinguished by forthrightness, simplicity, and total absence of conceit or careerism. As a teacher he was outstanding, not so much for the brilliance of his lectures as for the inspiration and tireless guidance he generously gave to his research students ... "

He had 76 Ph.D. students, including Georges Reeb, Wu Wen-Tsün, André Haefliger, Valentin Poénaru, Daniel Tanré.

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[1] <http://www-history.mcs.st-andrews.ac.uk/Biographies/Ehresmann.html>

[2] <http://www.genealogy.ams.org/id.php?id=96080>

Samuel Eilenberg

Samuel Eilenberg	
 <p>Samuel Eilenberg (1970)</p>	
Born	September 30, 1913Warsaw, Russian Empire
Died	January 30, 1998 (aged 84)New York City, New York
Nationality	Polish American
Fields	Mathematics
Institutions	Columbia University
Alma mater	University of Warsaw
Doctoral advisor	Kazimierz Kuratowski Karol Borsuk
Doctoral students	David Buchsbaum Alex Heller Daniel Kan William Lawvere Ramaiyengar Sridharan
Known for	Eilenberg–Steenrod axioms Eilenberg swindle

Samuel Eilenberg (September 30, 1913 – January 30, 1998) was a Polish and American mathematician of Jewish descent. He was born in Warsaw, Russian Empire (now in Poland) and died in New York City, USA, where he had spent much of his career as a professor at Columbia University.

He earned his Ph.D. from University of Warsaw in 1936. His thesis advisor was Karol Borsuk. His main interest was algebraic topology. He worked on the axiomatic treatment of homology theory with Norman Steenrod (whose names the Eilenberg–Steenrod axioms bear), and on homological algebra with Saunders Mac Lane. In the process, Eilenberg and Mac Lane created category theory.

Eilenberg was a member of the Bourbaki and with Henri Cartan, wrote the 1956 book *Homological Algebra*, which became a classic.

Later in life he worked mainly in pure category theory, being one of the founders of the field. The Eilenberg swindle (or *telescope*) is a construction applying the telescoping cancellation idea to projective modules.

Eilenberg also wrote an important book on automata theory. The X-machine, a form of automaton, was introduced by Eilenberg in 1974.

Eilenberg was also a prominent collector of Asian art. His collection mainly consisted of small sculptures and other artifacts from India, Indonesia, Pakistan, Nepal, Thailand, Cambodia, Sri Lanka and Central Asia. In 1991-1992, the Metropolitan Museum of Art in New York staged an exhibition from more than 400 items that Eilenberg had

donated to the museum, entitled *The Lotus Transcendent: Indian and Southeast Asian Art From the Samuel Eilenberg Collection*".^[1]

Selected publications

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Footnotes

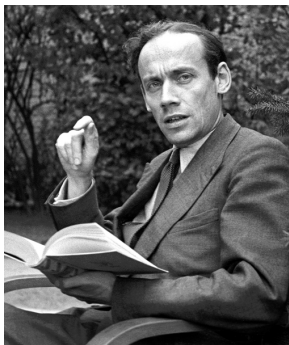
[1] New York Times obituary, February 3, 1998.

[2] <http://links.jstor.org/sici?sici=0003-486X%28195705%292%3A65%3A3%3C517%3AOTLCOA%3E2.0.CO%3B2-J>

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Emil Artin

Emil Artin	
	
Born	March 3, 1898Vienna, Austria
Died	December 20, 1962 (aged 64)Hamburg, Germany
Fields	Mathematics
Institutions	University of Hamburg University of Notre Dame Indiana University Princeton University
Alma mater	University of Vienna University of Leipzig
Doctoral advisor	Gustav Herglotz Otto Ludwig Hölder
Doctoral students	Bernard Dwork Serge Lang Kollagunta Ramanathan John Tate Hans Zassenhaus Max Zorn

Emil Artin (March 3, 1898, in Vienna – December 20, 1962, in Hamburg) was an Austrian-Armenian mathematician.

Biography

Parents

The mathematician Emil Artin was born on March 3, 1898 in Vienna to parents Emma Maria, née Laura (stage name Clarus), a soubrette on the operetta stages of Austria and Germany, and Emil Hadochadus Maria Artin, Austrian-born of Armenian descent. Several documents, including Emil's birth certificate, list the father's occupation as "opera singer" though others list it as "art dealer." It seems at least plausible that he and Emma had met as colleagues in the theater. They had been married in St. Stephen's Parish on July 24, 1895.

Early education

Emil entered school in September 1904, presumably in Vienna. By then, his father was already suffering symptoms of advanced syphilis, among them increasing mental instability, and was eventually institutionalized at the recently established (and imperially sponsored) insane asylum at Mauer Öhling, 125 kilometers west of Vienna. It is notable that neither wife nor child contracted this highly infectious disease. The senior Emil Artin died there July 20, 1906. Young Emil was eight.

On July 15, 1907, Emil's mother remarried—her second husband, Rudolf Hübner a prosperous manufacturer in the German-speaking city of Reichenberg, Czechoslovakia (now Liberec, in the Czech Republic). Documentary evidence suggests that Emma had already been resident in Reichenberg the previous year, and in deference to her new husband, she had abandoned her vocal career. Hübner deemed a life in the theater unseemly in the wife of a man of his position.

In September, 1907, Emil entered the Volksschule in Strobnitz, a small town in southern Czechoslovakia near the Austrian border. For that year, he lived away from home, boarding on a local farm. The following year, he returned to the home of his mother and stepfather, and entered the Realschule in Reichenberg, where he pursued his secondary education to June, 1916.

In Reichenberg, Emil formed a life-long friendship with a young neighbor, Arthur Baer, who became an astronomer, teaching for many years at Cambridge University. Astronomy was an interest the two boys shared already at this time. They each had telescopes. They also rigged a telegraph between their houses, over which once Baer excitedly reported to his friend an astronomical discovery he thought he had made—perhaps a supernova, he thought—and told Emil where in the sky to look. Emil tapped back the terse reply “A-N-D-R-O-M-E-D-A N-E-B-E-L.” (Andromeda nebula)

Emil's academic performance in the first years at the Realschule was spotty. Up to the end of the 1911–1912 school year, for instance, his grade in mathematics was merely “genügend,” (satisfactory). Of his mathematical inclinations at this early period he later wrote, “Meine eigene Vorliebe zur Mathematik zeigte sich erst im sechzehnten Lebensjahr, während vorher von irgendeiner Anlage dazu überhaupt nicht die Rede sein konnte.” (“My own predilection for mathematics manifested itself only in my sixteenth year, whereas earlier there was absolutely no question of any particular aptitude for it.”) His grade in French for 1912 was actually “nicht genügend” (unsatisfactory). He did rather better work in physics and chemistry. But from 1910 to 1912, his grade for “Comportment” was “nicht genügend.”

Emil spent the school year 1912–1913 away from home, in France, a period he spoke of later as one of the happiest of his life. He lived that year with the family of Edmond Fritz, in the vicinity of Paris, and attended a school there. When he returned from France to Reichenberg, his academic work markedly improved, and he began consistently receiving grades of “gut” or “sehr gut” (good or very good) in virtually all subjects—including French and “Comportment.” By the time he completed studies at the Realschule in June, 1916, he was awarded the Reifezeugnis (diploma—not to be confused with the Abitur) that affirmed him “reif mit Auszeichnung” (qualified with distinction) for graduation to a technical university.

University education

Now that it was time to move on to university studies, Emil was no doubt content to leave Reichenberg, for relations with his stepfather were clouded. According to him, Hübner reproached him “day and night” as a financial burden, and even when Emil became a university lecturer and then a professor, Hübner deprecated his academic career as self-indulgent and belittled its paltry remuneration.

In October, 1916, Emil matriculated at the University of Vienna, having focused by now on mathematics. He studied there with Phillip Furtwängler, and also took courses in astrophysics and Latin.

Studies at Vienna were interrupted when Emil was drafted in June, 1918 into the Austrian army (his Army photo ID is dated July 1, 1918). Assigned to the K.u. K. 44th Infantry Regiment, he was stationed northwest of Venice at Primolano, on the Italian front in the foothills of the Dolomites. To his great relief, Emil managed to avoid combat by volunteering for service as a translator—his ignorance of Italian notwithstanding. He did know French, of course, and some Latin, was generally a quick study, and was motivated by a highly rational fear in a theater of that war that had all too often proven a meat-grinder. In his scramble to learn at least some Italian, Emil had recourse to an encyclopedia, which he once consulted for help in controlling the cockroaches infesting the Austrian barracks. At some length, the article described a variety of elaborate methods, concluding finally with—Emil laughingly recalled in later years—“la caccia diretta” (“the direct hunt”). Indeed, “la caccia diretta” was the straight-forward method he and his fellow infantrymen adopted.

Emil survived both war and vermin on the Italian front, and returned late in 1918 to the University of Vienna, where he remained through Easter of the following year.

By June 1919, he had moved to Leipzig and matriculated at the University there as a “Class 2 Auditor” (“Hörer zweiter Ordnung”). Late the same year, Emil undertook the formality of standing for a qualifying examination by an academic board of the Oberrealschule in Leipzig, which he passed with the grade of “gut” (good), receiving for the second time the Reifezeugnis (diploma attesting the equivalence of satisfactory completion of 6 years at a Realschule.) How this Leipzig Reifezeugnis differed technically from the one he had been granted at Reichenberg is unclear from the document, but it apparently qualified him to matriculate as a regular student at the University, which normally required the Abitur.

From 1919 to June 1921, Emil pursued mostly mathematical studies at Leipzig. His principal teacher and dissertation advisor was Gustav Herglotz. Additionally, Emil took courses in chemistry and various fields of physics, including mechanics, atomic theory, quantum theory, Maxwellian theory, radioactivity, and astrophysics. In June, 1921 he was awarded the Doctor of Philosophy degree, based on his “excellent” dissertation, “Quadratische Körper im Gebiete der höheren Kongruenzen” (“On the Arithmetic of Quadratic Function Fields over Finite Fields”), and the oral examination which—his diploma affirms—he had passed three days earlier “with extraordinary success.”

In the fall of 1921, Emil moved to Göttingen, considered the Mecca of mathematics at the time, where he pursued one year of post-doctoral studies in mathematics and mathematical physics with Richard Courant and David Hilbert. While at Göttingen, he worked closely with Emmy Noether and Helmut Hasse.

Aside from consistently good school grades in singing, the first documentary evidence of Emil’s deep and life-long engagement with music comes from the year in Göttingen, where he was regularly invited to join in the chamber music sessions hosted by Richard Courant. He played all the keyboard instruments, and was an especially accomplished flautist, although it is not known exactly by what instruction he had achieved proficiency on these instruments. He became especially devoted to the music of J. S. Bach.

Professorship at Hamburg

Courant arranged for Emil to receive a stipend for the summer of 1922 in Göttingen, which occasioned his declining a position offered him at the University of Kiel. The following October, however, he accepted an equivalent position at Hamburg, where in 1923, he completed the Habilitation thesis (required of aspirants to a professorship in Germany), and on July 24 advanced to the rank of Privatdozent.

On April 1, 1925, Emil was promoted to Associate Professor (außerordentlicher Professor). In this year also, Emil applied for and was granted German citizenship. And on October 15, 1926, he was promoted to full Professor (ordentlicher Professor).

Early in the summer of 1925, Emil attended the Congress of the Wandervogel youth movement at Wilhelmshausen near Kassel with the intention of gathering a congenial group to undertake a trek through Iceland later that summer. Iceland (before the transforming presence of American and British forces stationed there during WWII) was still a primitive country in 1925, with a thinly scattered population and little transportation infrastructure. Emil succeeded in finding six young men to join him in this adventure. In the second half of August, 1925, the group set out by steamer from Hamburg, first to Norway, where they boarded a second steamer that took them to Iceland, stopping in several of the east fjords before arriving at their destination, Husavik in the north of the island. Here the Wandervogel group disembarked, their initial goal trekking down the Laxá River to Lake Myvatn. They made a circuit of the large, irregular lake, staying in farm houses, barns, and occasionally a tent as they went. When they slept in barns, it was often on piles of wet straw or hay. On those lucky occasions when they slept in beds, it could be nearly as damp on account of the rain trickling through the sod roofs. The tent leaked as well.

Emil kept a meticulous journal of this trip, making daily entries in a neat, minuscule hand. He and several of the young men had brought cameras, so that the trek is documented also by nearly 200 small photographs. Emil's journal attests to his over-arching interest in the geology of this mid-Atlantic island, situated over the boundary of two tectonic plates whose shifting relation makes it geologically hyperactive.

In keeping with the Wandervogel ethos, Emil and his companions carried music with them wherever they visited. The young men had packed guitars and violins, and Emil played the harmoniums common in the isolated farmsteads where they found lodging. The group regularly entertained their Icelandic hosts, not in full exchange for board and lodging, to be sure, but for goodwill certainly, and occasioning sometimes even a little extra on their plates and a modestly discounted tariff.

From Lake Myvatn, Emil and his companions headed west towards Akureyri, passing the large waterfall Goðafoss on the way. From Akureyri, they trekked west down the Öxnadalur (Ox Valley) intending to rent pack horses and cross the high and barren interior by foot to Reykjavik. By the time they reached the lower end of Skagafjörður, however, they were persuaded by a local farmer from whom they had hoped to rent the horses that a cross-country trek was by then impracticable since, with the approach of winter, highland routes were already snow-bound and impassable. Instead of turning south, then, they turned north to Siglufjörður, where they boarded another steamer that took them around the western peninsula and down the coast to Reykjavik. From Reykjavik, they returned via Norway to Hamburg. By Emil's calculation the distance they had covered on foot through Iceland totaled 450 kilometers.

Early in 1926, the University of Münster offered Emil a professorial position; however, Hamburg matched the offer financially, and (as noted above) promoted him to full professor, making him (along with his young colleague Helmut Hasse) one of the two youngest professors of mathematics in Germany.

It was in this period that he acquired his lifelong nickname, "Ma," short for mathematics, which he came to prefer to his given name, and by which virtually everyone who knew him well called him. Although the nickname might seem to imply a narrow intellectual focus, quite the reverse was true of Emil. Even his teaching at the University of Hamburg went beyond the strict boundaries of mathematics to include mechanics and relativity theory. He kept up on a serious level with advances in astronomy, chemistry and biology (he owned and used a fine microscope), and the circle of his friends in Hamburg attests to the catholicity of his interests. It included the painter Heinrich

Stegemann, and the author and organ-builder Hans Henny Jahn. Stegemann was a particularly close friend, and made portraits of Emil, Natascha and the two children born in Hamburg. Music continued to play a central role in his life; he acquired a Neupert double manual harpsichord, and a clavichord made by the Hamburg builder Walter Ebeloe, as well as a silver flute made in Hamburg by G. Urban. Chamber music gatherings became a regular event at the Artin apartment as they had been at the Courants in Göttingen.

On August 15, 1929, Emil married Natalia Naumovna Jasny (Natascha), a young Russian émigré who had been a student in several of his classes. One of their shared interests was photography, and when Emil bought a Leica for their joint use (a Leica A, the first commercial model of this legendary camera), Natascha began chronicling the life of the family, as well as the city of Hamburg. For the next decade, she made a series of artful and expressive portraits of Emil that remain by far the best images of him taken at any age. Emil, in turn, took many fine and evocative portraits of Natascha. Lacking access to a professional darkroom, their films and prints had to be developed in a makeshift darkroom set up each time (and then dismantled again) in the small bathroom of whatever apartment they were occupying. The makeshift darkroom notwithstanding, the high artistic level of the resulting photographic prints is attested to by the exhibit of Natascha's photographs mounted in 2001 by the Museum für Kunst und Gewerbe Hamburg, and its accompanying catalogue, "Hamburg—Wie Ich Es Sah."

In 1930, Emil was offered a professorship at ETH (Eidgenössische Technische Hochschule) in Zürich, to replace Hermann Weyl, who had moved to Göttingen. He chose to remain at Hamburg, however. Two years later, in 1932, for contributions leading to the advancement of mathematics, Emil was honored—jointly with Emmy Noether—with the award of the Alfred Ackermann-Teubner Memorial Prize, which carried a grant of 500 marks.

Nazi period

In January 1933—a tragically fateful month in German history—Natascha gave birth to their first child, Karin. A year and a half later, in the summer of 1934, son Michael was born. The political climate at Hamburg was not so poisonous as that at Göttingen, where by 1935 the mathematics department had been purged of Jewish and dissident professors. Still, Emil's situation became increasingly precarious, not only because Natascha was half Jewish, but also because Emil made no secret of his distaste for the Hitler regime. At one point Blaschke, by then a Nazi Party member, but nonetheless solicitous of the Artins' well-being, warned Emil discreetly to close his classroom door so his frankly anti-Nazi comments couldn't be heard by passersby in the hallway.

Natascha recalled going down to the newsstand on the corner one day and being warned in hushed tones by the man from whom she and Emil bought their paper that a man had daily been watching their apartment from across the street. Once tipped off, she and Emil became very aware of the watcher (Natascha liked to refer to him as their "spy"), and even rather enjoyed the idea of his being forced to follow them on the long walks they loved taking in the afternoons to a café far out in the countryside.

Toying with their watcher on a fine Autumn afternoon was one thing, but the atmosphere was in fact growing inexorably serious. Natascha's Jewish father and her sister, seeing the handwriting on the wall, had already left for the U.S. in the summer of 1933. As half-Jewish, Natascha's status was, if not ultimately quite hopeless, certainly not good. Hasse, like Blaschke a nationalistic supporter of the regime, had applied for Party membership, but was nonetheless no anti-Semite. Besides he was a long-time friend and colleague of Emil's. He suggested that the two Artin children—only one quarter Jewish, or in Nazi terminology, "Mischlinge zweiten Grades"—might, if a few strategic strings could be pulled, be officially "aryanized." Hasse offered to exert his influence with the Ministry of Education (Kultur- und Schulbehörde, Hochschulwesen), and Emil—not daring to leave any stone unturned, especially with respect to the safety of his children—went along with this effort. He asked his father-in-law, by then resident in Washington D.C., to draft and have notarized an affidavit attesting to the Christian lineage of his late wife, Natascha's mother. Emil submitted this affidavit to the Ministry of Education, but to no avail.

By this time, to be precise, on July 15, 1937, because of Natascha's status as "Mischling ersten Grades," Emil had lost his post at the University—technically, compelled into early retirement—on the grounds of paragraph 6 of the

Act to Restore the Professional Civil Service (Gesetz zur Wiederherstellung des Berufsbeamtentums) of April 7, 1933. Ironically, he had applied only some months earlier, on February 8, 1937, for a leave of absence from the University in order to accept a position offered him at Stanford. On March 15, 1937, the response had come back denying his application for leave on the grounds that his services to the University were indispensable (“Da die Tätigkeit des Professors Dr. Artin an der Universität Hamburg nicht entbehrt werden kann. . .”)

By July, when he was summarily “retired,” (“in Ruhestand versetzt”) the position at Stanford had been filled. However, through the efforts of Richard Courant (by then in New York), and Solomon Lefschetz at Princeton, a position was found for him at Notre Dame University in South Bend, Indiana.

Emigration to the U.S.

The family must have worked feverishly to prepare for emigration to the United States, for this entailed among other things packing their entire household for shipment. Since German law forbade emigrants taking more than a token sum of money out of the country, the Artins sank all the funds at their disposal into shipping their entire household, from beds, tables, chairs and double-manual harpsichord down to the last kitchen knife, cucumber slicer, and potato masher to their new home. This is why each of their residences in the United States bore such a striking resemblance to the rooms photographed so beautifully by Natascha in their Hamburg apartment (see Natascha A. Brunswick, “Hamburg: Wie Ich Es Sah,” *Dokumente der Photographie* 6, Museum für Kunst und Gewerbe Hamburg, 2001, pp. 48–53).

On the morning they were to board the Hamburg-Amerika line ship in Bremerhaven, October 21, 1937, daughter Karin woke with a high temperature. Terrified that should this opportunity be missed, the window of escape from Nazi Germany might close forever, Emil and Natascha chose to risk somehow getting Karin past emigration and customs officials without their noticing her condition. Anxiously, they managed to conceal Karin’s feverish state, and without incident boarded the ship, as many left behind were tragically never able to do. When they landed a week later at Hoboken, Richard Courant and Natascha’s father, the Russian agronomist Naum Jasny (then working for the U.S. Department of Agriculture) were on the dock to welcome the family to the United States.

Bloomington years

It was early November, 1937 by the time they arrived in South Bend, where Emil joined the faculty at Notre Dame, and taught for the rest of that academic year. He was offered a permanent position the following year 170 miles to the south at Indiana University, in Bloomington. Shortly after the family resettled there, a second son, Thomas, was born on November 12, 1938.

After moving to Bloomington, Emil quickly acquired a piano, and soon after that a Hammond Organ, a recently invented electronic instrument that simulated the sound of a pipe organ. He wanted this instrument in order primarily to play the works of J. S. Bach, and because the pedal set that came with the production model had a range of only two octaves (not quite wide enough for all the Bach pieces), he set about extending its range. Music was a constant presence in the Artin household. Karin played the cello, and then the piano as well, and Michael played the violin. As in Hamburg, the Artin living room was regularly the venue for amateur chamber music performances.

The circle of the Artins’ University friends reflected Emil’s wide cultural and intellectual interests. Notable among them were Alfred Kinsey and his wife of the Psychology Department, as well as prominent members of the Fine Arts, Art History, Anthropology, German Literature, and Music Departments. For several summer semesters, Emil accepted teaching positions at other universities, viz., Stanford in 1939 and 1940, The University of Michigan at Ann Arbor in 1941 and 1951, and The University of Colorado, in Boulder, in 1953. On each of these occasions, the family accompanied him.

Emil insisted that only German be spoken in the house. Even Tom, born in the U.S., spoke German as his first language, acquiring English only from his siblings and his playmates in the neighborhood; for the first four or five years of his life, he spoke English with a pronounced German accent. Consistent with his program of maintaining the

family's German cultural heritage, Emil gave high priority to regularly reading German literature aloud to the children. The text was frequently from Goethe's autobiographical "Dichtung und Wahrheit," or his poems, "Erkönig," for instance. Occasionally, he would read from an English text. Favorites were Mark Twain's "Tom Sawyer," Charles Dickens's "A Christmas Carol," and Oscar Wilde's "The Canterville Ghost." For the Artin children, these readings replaced radio entertainment, which was strictly banned from the house. There was a radio, but (with the notable exception of Sunday morning broadcasts by E. Power Biggs from the organ at the Busch-Reisinger Museum in Cambridge, to which Emil and Natascha listened still lounging in bed) it was switched on only to hear news of the war. Similarly, the Artin household would never in years to come harbor a television set. Once the war had ended, the radio was retired to the rear of a dark closet.

As German citizens, Emil and Natascha were technically classified as enemy aliens for the duration of the war. On April 12, 1945, with the end of the war in Europe only weeks away, they applied for naturalization as American citizens. American citizenship was granted them on February 7, 1946.

On the orders of a Hamburg doctor whom he had consulted about a chronic cough, Emil had given up smoking years before. He had vowed not to smoke so long as Hitler remained in power. On May 8, 1945, at the news of Germany's surrender and the fall of the Third Reich, Natascha made the mistake of reminding him of this vow, and in lieu of a champagne toast, he indulged in what was to be the smoking of a single cigarette. Unfortunately, the single cigarette led to a second, and another after that. Emil returned to heavy smoking for the rest of his life, a habit that unquestionably contributed to his premature death.

Princeton years

If Göttingen had been the "Mecca" of mathematics in the 1920s and early '30's, Princeton, following the decimation of German mathematics under the Nazis, had become the center of the mathematical world in the 1940's. In April, 1946, Emil was appointed Professor at Princeton, at a yearly salary of \$8,000. The family moved there in the fall of 1946.

Notable among his graduate students at Princeton are Serge Lang, John Tate, and Timothy O'Meara. But Emil chose also to teach the honors section of Freshman calculus each year. He was renowned for the elegance of his teaching. Frei and Roquette write that Artin's "main medium of communication was teaching and conversation: in groups, seminars and in smaller circles. We have many statements of people near to him describing his unpretentious way of communicating with everybody, demanding quick grasp of the essentials but never tired of explaining the necessary. He was open to all kinds of suggestions, and distributed joyfully what he knew. He liked to teach, also to young students, and his excellent lectures, always well prepared but without written notes, were hailed for their clarity and beauty." (Emil Artin and Helmut Hasse: *Their Correspondence 1923–1934*, Introduction.)

Whenever he was asked whether mathematics was a science, Emil would reply unhesitatingly, "No. An art." His elegant elaboration of this idea is often cited, and worth repeating here: "We all believe that mathematics is an art. The author of a book, the lecturer in a classroom tries to convey the structural beauty of mathematics to his readers, to his listeners. In this attempt, he must always fail. Mathematics is logical to be sure, each conclusion is drawn from previously derived statements. Yet the whole of it, the real piece of art, is not linear; worse than that, its perception should be instantaneous. We have all experienced on some rare occasion the feeling of elation in realizing that we have enabled our listeners to see at a glance the whole architecture and all its ramifications."

It has even been said—only half in jest—that his lectures could be too perfect, lulling a hearer into believing he had understood and assimilated an idea or a proof which, on waking the following day might seem as remote and chimerical as ever.

During the Princeton years, Emil built a 6 inch reflecting telescope to plans he found in "Sky and Telescope" magazine, to which he subscribed. He spent weeks in the basement attempting to grind the mirror to specifications, without success, and his continued failure to get it right led to increasing frustration. Then, in California to give a talk, he made a side trip to the Mt. Wilson Observatory, where he discussed his project with the astronomers.

Whether it was their technical advice, or Natascha's intuitive suggestion that it might be too cold in the basement, and that he should try the procedure upstairs in the warmth of his study (which he did), he completed the grinding of the mirror in a matter of days. With this telescope, he surveyed the night skies over Princeton.

In September 1955, Emil accepted an invitation to visit Japan. From his letters, it is clear he was treated like royalty by the Japanese mathematical community, and was charmed by the country. A confirmed atheist most of his life, he was nonetheless interested in learning about the diverse threads of Buddhism, and visiting its holy sites. In a letter home he describes his visit to the temples at Nara. "Then we were driven to a place nearby, Horiuji [Horyu-ji] where a very beautiful Buddhist temple is. We were received by the abbot, and a priest translated into English. We obtained the first sensible explanation about modern Buddhism. The difficulty of obtaining such an explanation is enormous. To begin with most Japanese do not know and do not understand our questions. All this is made more complicated by the fact that there are numerous sects and each one has another theory. Since you get your information only piece wise, you cannot put it together. This results in an absurd picture. I am talking of the present day, not of its original form."

His letter goes on to outline at length the general eschatological framework of Buddhist belief. Then he adds, "By the way, a problem given by the Zens for meditation is the following: If you clap your hands, does the sound come from the left hand or from the right?"

Return to Hamburg

The following year, Emil took a leave of absence to return to Germany for the first time since emigration, nearly twenty years earlier. He spent the fall semester at Göttingen, and the next at Hamburg. For the Christmas holidays, he travelled to his birthplace, Vienna, to visit his mother, a city he had not seen in decades. In a letter home he described the experience of his return in a single, oddly laconic sentence: "It is kind of amusing to walk through Vienna again." In 1957, an honorary doctorate was conferred on Emil by the University of Freiburg. That fall, he returned to Princeton for what would be his final academic year at that institution.

Emil's marriage to Natascha had by this time seriously frayed. Though nominally still husband and wife, resident in the same house, they were for all intents and purposes living separate lives. Emil was offered a professorship at Hamburg, and at the conclusion of Princeton's spring semester, 1958, he moved permanently to Germany. His decision to leave Princeton University and the United States was complicated, based on multiple factors, prominent among them Princeton's (then operative) mandatory retirement age of 65. Emil had no wish to retire from teaching and direct involvement with students. Hamburg's offer was open-ended.

Emil and Natascha were divorced in 1959. In Hamburg, Emil had taken an apartment, but soon gave it over to his mother whom he had brought from Vienna to live near him in Hamburg. He in turn moved in with Hel Braun into an apartment in the same neighborhood. On January 4, 1961, he was granted German citizenship. In June, 1962, on the occasion of the 300th anniversary of the death Blaise Pascal, the University of Clermont-Ferrand conferred an honorary doctorate on him. On December 20 of the same year, Emil Artin died at home in Hamburg, aged 64, of a heart attack.

The University of Hamburg honored his memory on April 26, 2005 by naming one of its newly renovated lecture halls The Emil Artin Lecture Hall.^[1]

Influence and work

He was one of the leading algebraists of the century, with an influence larger than might be guessed from the one volume of his *Collected Papers* edited by Serge Lang and John Tate. He worked in algebraic number theory, contributing largely to class field theory and a new construction of L-functions. He also contributed to the pure theories of rings, groups and fields. He developed the theory of braids as a branch of algebraic topology.

He was also an important expositor of Galois theory, and of the group cohomology approach to class ring theory (with John Tate), to mention two theories where his formulations became standard. The influential treatment of abstract algebra by van der Waerden is said to derive in part from Artin's ideas, as well as those of Emmy Noether. He wrote a book on geometric algebra that gave rise to the contemporary use of the term, reviving it from the work of W. K. Clifford.

Conjectures

He left two conjectures, both known as **Artin's conjecture**. The first concerns Artin L-functions for a linear representation of a Galois group; and the second the frequency with which a given integer a is a primitive root modulo primes p , when a is fixed and p varies. These are unproven; Hooley proved a result for the second conditional on the first.

Supervision of research

Artin advised over thirty doctoral students, including Bernard Dwork, Serge Lang, K. G. Ramanathan, John Tate, Hans Zassenhaus and Max Zorn. A more complete list of his students can be found at the Mathematics Genealogy Project website (see "External Links," below).

Family

In 1932 he married Natascha Jasny, born in Russia to mixed parentage (her mother was Christian, her father, Jewish).^[2] Artin was not himself Jewish, but, on account of his wife's racial status in Nazi Germany, was dismissed from his university position in 1937. They had three children, one of whom is Michael Artin, an American algebraist currently at MIT.

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- (http://www.princeton.edu/~mudd/finding_aids/mathoral/pmcxrota.htm) "Fine Hall in its golden age: Remembrances of Princeton in the early fifties", by Gian-Carlo Rota. Contains a section on Artin at Princeton.

Ronald Brown

Ronald Brown	
Born	4 January 1935London
Nationality	 United Kingdom
Fields	Mathematics
Alma mater	University of Oxford
Doctoral advisor	J. H. C. Whitehead

Ronald Brown is an English mathematician. Emeritus Professor in the School of Computer Science at Bangor University,^[1] he has authored many books and journal articles.

Education and career

Born on 4 January 1935 in London, Brown attended Oxford University, obtaining a B.A. in 1956 and a Ph.D. in 1962.^[2] Brown began his teaching career during his Ph.D. work, serving as an assistant lecturer at Liverpool University before assuming the position as lecturer. In 1964, he took a position at Hull University, serving first as a senior lecturer and then as a reader before becoming a professor of pure mathematics at Bangor University, then a part of the University of Wales, in 1970.

Brown served as professor of pure mathematics for 29 years, also serving briefly during the 1983-1984 term as an associate professor at Louis Pasteur University in Strasbourg.^[2] In 1999, Brown took a half-time research professorship until he became Professor Emeritus in 2001.

Editing and writing

Brown has served as an editor or on the editorial board for a number of print and electronic journals. He began in 1968 with the *Chapman & Hall Mathematics Series*, contributing through 1986.^[2] In 1975, he joined the editorial advisory board of the London Mathematical Society, remaining through 1994. Two years later, he joined the editorial board of *Applied Categorical Structures*,^[3] continuing through 2007. From 1995 and 1999, respectively, he has been active with the electronic journals [*Theory and Applications of Categories* <http://www.tac.mta.ca/tac/>]^[4] and [*Homology, Homotopy and Applications* <http://www.intlpress.com/>],^[5] which he helped found. Since 2006, he has been involved with *Journal of Homotopy and Related Structures*.^[6] ^[7] His mathematical research interests range from algebraic topology and groupoids, to homology theory, category theory, mathematical biology, mathematical physics and higher dimensional algebra.^[8] ^[9] ^[10] ^[11] ^[12]

Brown has authored or edited a number of books and over 150 academic papers published in academic journals or collections. His first published paper was "Ten topologies for $X\times Y$ ", which was published in the *Quarterly Journal of Math* in 1963^[13] Since then, his publications have appeared in many journals, including but not limited to the *Journal of Algebra*, *Proceedings of the American Mathematical Society*, *Mathematische Zeitschrift*, *College Mathematics Journal*, and *American Mathematical Monthly*. He is also known for several recent co-authored papers on Categorical ontology.^[14]

Among his several books and standard topology and algebraic topology textbooks are: *Elements of Modern Topology* (1968), *Low-Dimensional Topology* (1979, co-edited with T.L. Thickstun), *Topology: a geometric account of general topology, homotopy types, and the fundamental groupoid* (1998),^[15] ^[16] *Topology and Groupoids* (2006)^[17] and *Nonabelian Algebraic Topology: Filtered Spaces, Crossed Complexes, Cubical Homotopy Groupoids* (EMS, 2010).^[18] ^[19] ^[20] ^[17] ^[21] ^[22] ^[23] ^[24] ^[25] ^[26]

His recent fundamental results that extend the classical Van Kampen theorem to higher homotopy in higher dimensions (HHSvKT) are of substantial interest for solving several problems in algebraic topology, both old and new.^[27] Moreover, developments in algebraic topology have often had wider implications, as for example in algebraic geometry and also in algebraic number theory. Such higher dimensional (HHSvKT) theorems are about homotopy invariants of structured spaces, and especially those for filtered spaces or n -cubes of spaces. A nice example is the fact that the relative Hurewicz theorem is a consequence of HHSvKT, and this then suggested a triadic Hurewicz theorem.


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External links

- Ronald Brown's Biography and publications (<http://planetphysics.org/encyclopedia/RonaldBrown.html>)
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Henri Poincaré

Henri Poincaré	
<div></div> <p>Jules Henri Poincaré (1854–1912). Photograph from the frontispiece of the 1913 edition of <i>Last Thoughts</i>.</p>	
Born	29 April 1854Nancy, Meurthe-et-Moselle
Died	17 July 1912 (aged 58)Paris
Residence	France
Nationality	French
Fields	Mathematician and physicist
Institutions	Corps des Mines Caen University La Sorbonne Bureau des Longitudes
Alma mater	Lycée Nancy École Polytechnique École des Mines
Doctoral advisor	Charles Hermite
Doctoral students	Louis Bachelier Dimitrie Pompeiu Mihailo Petrović
Other notable students	Tobias Dantzig
Known for	Poincaré conjecture Three-body problem Topology Special relativity Poincaré–Hopf theorem Poincaré duality Poincaré–Birkhoff–Witt theorem Poincaré inequality Hilbert–Poincaré series Poincaré metric Rotation number Coining term 'Betti number' Chaos theory Sphere-world Poincaré–Bendixson theorem Poincaré–Lindstedt method Poincaré recurrence theorem
Influences	Lazarus Fuchs

Influenced	Louis Rougier George David Birkhoff
Notable awards	RAS Gold Medal (1900) Sylvester Medal (1901) Matteucci Medal (1905) Bolyai Prize (1905) Bruce Medal (1911)
Signature 	
Notes	He was a cousin of Pierre Boutroux.

Jules Henri Poincaré (29 April 1854 – 17 July 1912) (French pronunciation: [ˈʒyl ɑ̃ʁi pwɛ̃kaˈʁe])^[1] was a French mathematician, theoretical physicist, engineer, and a philosopher of science. He is often described as a polymath, and in mathematics as *The Last Universalist*, since he excelled in all fields of the discipline as it existed during his lifetime.

As a mathematician and physicist, he made many original fundamental contributions to pure and applied mathematics, mathematical physics, and celestial mechanics. He was responsible for formulating the Poincaré conjecture, one of the most famous problems in mathematics. In his research on the three-body problem, Poincaré became the first person to discover a chaotic deterministic system which laid the foundations of modern chaos theory. He is also considered to be one of the founders of the field of topology.

Poincaré introduced the modern principle of relativity and was the first to present the Lorentz transformations in their modern symmetrical form. Poincaré discovered the remaining relativistic velocity transformations and recorded them in a letter to Dutch physicist Hendrik Lorentz (1853–1928) in 1905. Thus he obtained perfect invariance of all of Maxwell's equations, an important step in the formulation of the theory of special relativity.

The Poincaré group used in physics and mathematics was named after him.

Life

Poincaré was born on 29 April 1854 in Cité Ducale neighborhood, Nancy, Meurthe-et-Moselle into an influential family (Belliver, 1956). His father Leon Poincaré (1828–1892) was a professor of medicine at the University of Nancy (Sagaret, 1911). His adored younger sister Aline married the spiritual philosopher Emile Boutroux. Another notable member of Jules' family was his cousin, Raymond Poincaré, who would become the President of France, 1913 to 1920, and a fellow member of the Académie française.^[2]

Education

During his childhood he was seriously ill for a time with diphtheria and received special instruction from his mother, Eugénie Launois (1830–1897).

In 1862, Henri entered the Lycée in Nancy (now renamed the Lycée Henri Poincaré in his honour, along with the University of Nancy). He spent eleven years at the Lycée and during this time he proved to be one of the top students in every topic he studied. He excelled in written composition. His mathematics teacher described him as a "monster of mathematics" and he won first prizes in the concours général, a competition between the top pupils from all the Lycées across France. His poorest subjects were music and physical education, where he was described as "average at best" (O'Connor et al., 2002). However, poor eyesight and a tendency towards absentmindedness may explain these difficulties (Carl, 1968). He graduated from the Lycée in 1871 with a Bachelor's degree in letters and sciences.

During the Franco-Prussian War of 1870 he served alongside his father in the Ambulance Corps.

Poincaré entered the École Polytechnique in 1873. There he studied mathematics as a student of Charles Hermite, continuing to excel and publishing his first paper (*Démonstration nouvelle des propriétés de l'indicatrice d'une surface*) in 1874. He graduated in 1875 or 1876. He went on to study at the École des Mines, continuing to study mathematics in addition to the mining engineering syllabus and received the degree of ordinary engineer in March 1879.

As a graduate of the École des Mines he joined the Corps des Mines as an inspector for the Vesoul region in northeast France. He was on the scene of a mining disaster at Magny in August 1879 in which 18 miners died. He carried out the official investigation into the accident in a characteristically thorough and humane way.

At the same time, Poincaré was preparing for his doctorate in sciences in mathematics under the supervision of Charles Hermite. His doctoral thesis was in the field of differential equations. It was named *Sur les propriétés des fonctions définies par les équations différences*. Poincaré devised a new way of studying the properties of these equations. He not only faced the question of determining the integral of such equations, but also was the first person to study their general geometric properties. He realised that they could be used to model the behaviour of multiple bodies in free motion within the solar system. Poincaré graduated from the University of Paris in 1879.



The young Henri Poincaré

Career

Soon after, he was offered a post as junior lecturer in mathematics at Caen University, but he never fully abandoned his mining career to mathematics. He worked at the Ministry of Public Services as an engineer in charge of northern railway development from 1881 to 1885. He eventually became chief engineer of the Corps des Mines in 1893 and inspector general in 1910.

Beginning in 1881 and for the rest of his career, he taught at the University of Paris (the Sorbonne). He was initially appointed as the *maître de conférences d'analyse* (associate professor of analysis) (Sageret, 1911). Eventually, he held the chairs of Physical and Experimental Mechanics, Mathematical Physics and Theory of Probability, and Celestial Mechanics and Astronomy.

Also in that same year, Poincaré married Miss Poulain d'Andecy. Together they had four children: Jeanne (born 1887), Yvonne (born 1889), Henriette (born 1891), and Léon (born 1893).

In 1887, at the young age of 32, Poincaré was elected to the French Academy of Sciences. He became its president in 1906,

and was elected to the Académie française in 1909.

In 1887 he won Oscar II, King of Sweden's mathematical competition for a resolution of the three-body problem concerning the free motion of multiple orbiting bodies. (See #The three-body problem section below)

In 1893, Poincaré joined the French Bureau des Longitudes, which engaged him in the synchronisation of time around the world. In 1897 Poincaré backed an unsuccessful proposal for the decimalisation of circular measure, and hence time and longitude (see Galison 2003). It was this post which led him to consider the question of establishing international time zones and the synchronisation of time between bodies in relative motion. (See #Work on relativity section below)

In 1899, and again more successfully in 1904, he intervened in the trials of Alfred Dreyfus. He attacked the spurious scientific claims of some of the evidence brought against Dreyfus, who was a Jewish officer in the French army charged with treason by anti-Semitic colleagues.

In 1912, Poincaré underwent surgery for a prostate problem and subsequently died from an embolism on 17 July 1912, in Paris. He was 58 years of age. He is buried in the Poincaré family vault in the Cemetery of Montparnasse, Paris.

A former French Minister of Education, Claude Allègre, has recently (2004) proposed that Poincaré be reburied in the Panthéon in Paris, which is reserved for French citizens only of the highest honour.^[3]



The Poincaré family grave at the Cimetière du Montparnasse

Students

Poincaré had two notable doctoral students at the University of Paris, Louis Bachelier (1900) and Dimitrie Pompeiu (1905).^[4]

Work

Summary

Poincaré made many contributions to different fields of pure and applied mathematics such as: celestial mechanics, fluid mechanics, optics, electricity, telegraphy, capillarity, elasticity, thermodynamics, potential theory, quantum theory, theory of relativity and physical cosmology.

He was also a populariser of mathematics and physics and wrote several books for the lay public.

Among the specific topics he contributed to are the following:

- algebraic topology
- the theory of analytic functions of several complex variables
- the theory of abelian functions
- algebraic geometry
- Poincaré was responsible for formulating one of the most famous problems in mathematics, the Poincaré conjecture.
- Poincaré recurrence theorem
- hyperbolic geometry
- number theory
- the three-body problem
- the theory of diophantine equations
- the theory of electromagnetism
- the special theory of relativity
- In an 1894 paper, he introduced the concept of the fundamental group.

- In the field of differential equations Poincaré has given many results that are critical for the qualitative theory of differential equations, for example the Poincaré sphere and the Poincaré map.
- Poincaré on "everybody's belief" in the *Normal Law of Errors* (see normal distribution for an account of that "law")
- Published an influential paper providing a novel mathematical argument in support of quantum mechanics.^[5] ^[6]

The three-body problem

The problem of finding the general solution to the motion of more than two orbiting bodies in the solar system had eluded mathematicians since Newton's time. This was known originally as the three-body problem and later the n -body problem, where n is any number of more than two orbiting bodies. The n -body solution was considered very important and challenging at the close of the nineteenth century. Indeed in 1887, in honour of his 60th birthday, Oscar II, King of Sweden, advised by Gösta Mittag-Leffler, established a prize for anyone who could find the solution to the problem. The announcement was quite specific:

“Given a system of arbitrarily many mass points that attract each according to Newton's law, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that is some known function of time and for all of whose values the series converges uniformly.”

In case the problem could not be solved, any other important contribution to classical mechanics would then be considered to be prizeworthy. The prize was finally awarded to Poincaré, even though he did not solve the original problem. One of the judges, the distinguished Karl Weierstrass, said, *"This work cannot indeed be considered as furnishing the complete solution of the question proposed, but that it is nevertheless of such importance that its publication will inaugurate a new era in the history of celestial mechanics."* (The first version of his contribution even contained a serious error; for details see the article by Diacu^[7]). The version finally printed contained many important ideas which lead to the theory of chaos. The problem as stated originally was finally solved by Karl F. Sundman for $n = 3$ in 1912 and was generalised to the case of $n > 3$ bodies by Qiudong Wang in the 1990s.

Work on relativity



Marie Curie and Poincaré talk at the 1911 Solvay Conference.

Local time

Poincaré's work at the Bureau des Longitudes on establishing international time zones led him to consider how clocks at rest on the Earth, which would be moving at different speeds relative to absolute space (or the "luminiferous aether"), could be synchronised. At the same time Dutch theorist Hendrik Lorentz was developing Maxwell's theory into a theory of the motion of charged particles ("electrons" or "ions"), and their interaction with radiation. In 1895 Lorentz had introduced an auxiliary quantity (without physical interpretation) called "local time" $t' = t - vx/c^2$ ^[8] and introduced the hypothesis of length contraction to explain the failure of optical and electrical experiments to detect motion relative to the aether (see Michelson–Morley experiment).^[9] Poincaré was a

constant interpreter (and sometimes friendly critic) of Lorentz's theory. Poincaré as a philosopher was interested in the "deeper meaning". Thus he interpreted Lorentz's theory and in so doing he came up with many insights that are now associated with special relativity. In *The Measure of Time* (1898), Poincaré said, "A little reflection is sufficient to understand that all these affirmations have by themselves no meaning. They can have one only as the result of a convention." He also argued that scientists have to set the constancy of the speed of light as a postulate to give physical theories the simplest form.^[10] Based on these assumptions he discussed in 1900 Lorentz's "wonderful invention" of local time and remarked that it arose when moving clocks are synchronised by exchanging light signals assumed to travel with the same speed in both directions in a moving frame.^[11]

Principle of relativity and Lorentz transformations

He discussed the "principle of relative motion" in two papers in 1900^[11] ^[12] and named it the principle of relativity in 1904, according to which no physical experiment can discriminate between a state of uniform motion and a state of rest.^[13] In 1905 Poincaré wrote to Lorentz about Lorentz's paper of 1904, which Poincaré described as a "paper of supreme importance." In this letter he pointed out an error Lorentz had made when he had applied his transformation to one of Maxwell's equations, that for charge-occupied space, and also questioned the time dilation factor given by Lorentz.^[14] In a second letter to Lorentz, Poincaré gave his own reason why Lorentz's time dilation factor was indeed correct after all: it was necessary to make the Lorentz transformation form a group and gave what is now known as the relativistic velocity-addition law.^[15] Poincaré later delivered a paper at the meeting of the Academy of Sciences in Paris on 5 June 1905 in which these issues were addressed. In the published version of that he wrote:^[16]

“The essential point, established by Lorentz, is that the equations of the electromagnetic field are not altered by a certain transformation (which I will call by the name of Lorentz) of the form:

$$x' = k\ell(x + \varepsilon t), \quad t' = k\ell(t + \varepsilon x), \quad y' = \ell y, \quad z' = \ell z, \quad k = 1/\sqrt{1 - \varepsilon^2}.$$

and showed that the arbitrary function $\ell(\varepsilon)$ must be unity for all ε (Lorentz had set $\ell = 1$ by a different argument) to make the transformations form a group. In an enlarged version of the paper that appeared in 1906 Poincaré pointed out that the combination $x^2 + y^2 + z^2 - c^2 t^2$ is invariant. He noted that a Lorentz transformation is merely a rotation in four-dimensional space about the origin by introducing $ct\sqrt{-1}$ as a fourth imaginary coordinate, and he used an early form of four-vectors.^[17] Poincaré's attempt at a four-dimensional reformulation of the new mechanics was rejected by himself in 1907, because in his opinion the translation of physics into the language of four-dimensional geometry would entail too much effort for limited profit.^[18] So it was Hermann Minkowski who worked out the consequences of this notion in 1907.

Mass–energy relation

Like others before, Poincaré (1900) discovered a relation between mass and electromagnetic energy. While studying the conflict between the action/reaction principle and Lorentz ether theory, he tried to determine whether the center of gravity still moves with a uniform velocity when electromagnetic fields are included.^[11] He noticed that the action/reaction principle does not hold for matter alone, but that the electromagnetic field has its own momentum. Poincaré concluded that the electromagnetic field energy of an electromagnetic wave behaves like a fictitious fluid ("fluide fictif") with a mass density of E/c^2 . If the center of mass frame is defined by both the mass of matter *and* the mass of the fictitious fluid, and if the fictitious fluid is indestructible — it's neither created or destroyed — then the motion of the center of mass frame remains uniform. But electromagnetic energy can be converted into other forms of energy. So Poincaré assumed that there exists a non-electric energy fluid at each point of space, into which electromagnetic energy can be transformed and which also carries a mass proportional to the energy. In this way, the motion of the center of mass remains uniform. Poincaré said that one should not be too surprised by these assumptions, since they are only mathematical fictions.

However, Poincaré's resolution led to a paradox when changing frames: if a Hertzian oscillator radiates in a certain direction, it will suffer a recoil from the inertia of the fictitious fluid. Poincaré performed a Lorentz boost (to order

v/c) to the frame of the moving source. He noted that energy conservation holds in both frames, but that the law of conservation of momentum is violated. This would allow perpetual motion, a notion which he abhorred. The laws of nature would have to be different in the frames of reference, and the relativity principle would not hold. Therefore he argued that also in this case there has to be another compensating mechanism in the ether.

Poincaré himself came back to this topic in his St. Louis lecture (1904).^[13] This time (and later also in 1908) he rejected^[19] the possibility that energy carries mass and criticized the ether solution to compensate the above mentioned problems:

“The apparatus will recoil as if it were a cannon and the projected energy a ball, and that contradicts the principle of Newton, since our present projectile has no mass; it is not matter, it is energy. [...] Shall we say that the space which separates the oscillator from the receiver and which the disturbance must traverse in passing from one to the other, is not empty, but is filled not only with ether, but with air, or even in inter-planetary space with some subtle, yet ponderable fluid; that this matter receives the shock, as does the receiver, at the moment the energy reaches it, and recoils, when the disturbance leaves it? That would save Newton's principle, but it is not true. If the energy during its propagation remained always attached to some material substratum, this matter would carry the light along with it and Fizeau has shown, at least for the air, that there is nothing of the kind. Michelson and Morley have since confirmed this. We might also suppose that the motions of matter proper were exactly compensated by those of the ether; but that would lead us to the same considerations as those made a moment ago. The principle, if thus interpreted, could explain anything, since whatever the visible motions we could imagine hypothetical motions to compensate them. But if it can explain anything, it will allow us to foretell nothing; it will not allow us to choose between the various possible hypotheses, since it explains everything in advance. It therefore becomes useless.”

He also discussed two other unexplained effects: (1) non-conservation of mass implied by Lorentz's variable mass γm , Abraham's theory of variable mass and Kaufmann's experiments on the mass of fast moving electrons and (2) the non-conservation of energy in the radium experiments of Madame Curie.

It was Albert Einstein's concept of mass–energy equivalence (1905) that a body losing energy as radiation or heat was losing mass of amount $m = E/c^2$ that resolved^[20] Poincaré's paradox, without using any compensating mechanism within the ether.^[21] The Hertzian oscillator loses mass in the emission process, and momentum is conserved in any frame. However, concerning Poincaré's solution of the Center of Gravity problem, Einstein noted that Poincaré's formulation and his own from 1906 were mathematically equivalent.^[22]

Poincaré and Einstein

Einstein's first paper on relativity was published three months after Poincaré's short paper,^[16] but before Poincaré's longer version.^[17] It relied on the principle of relativity to derive the Lorentz transformations and used a similar clock synchronisation procedure (Einstein synchronisation) that Poincaré (1900) had described, but was remarkable in that it contained no references at all. Poincaré never acknowledged Einstein's work on special relativity. Einstein acknowledged Poincaré in the text of a lecture in 1921 called *Geometrie und Erfahrung* in connection with non-Euclidean geometry, but not in connection with special relativity. A few years before his death Einstein commented on Poincaré as being one of the pioneers of relativity, saying "Lorentz had already recognised that the transformation named after him is essential for the analysis of Maxwell's equations, and Poincaré deepened this insight still further"^[23]

Assessments

Poincaré's work in the development of special relativity is well recognised,^[20] though most historians stress that despite many similarities with Einstein's work, the two had very different research agendas and interpretations of the work.^[24] Poincaré developed a similar physical interpretation of local time and noticed the connection to signal velocity, but contrary to Einstein he continued to use the ether-concept in his papers and argued that clocks in the ether show the "true" time, and moving clocks show the local time. So Poincaré tried to keep the relativity principle in accordance with classical concepts, while Einstein developed a mathematically equivalent kinematics based on the new physical concepts of the relativity of space and time.^{[25] [26] [27] [28] [29]} While this is the view of most historians, a minority go much further, such as E.T. Whittaker, who held that Poincaré and Lorentz were the true discoverers of Relativity.^[30]

Character

Poincaré's work habits have been compared to a bee flying from flower to flower. Poincaré was interested in the way his mind worked; he studied his habits and gave a talk about his observations in 1908 at the Institute of General Psychology in Paris. He linked his way of thinking to how he made several discoveries.

The mathematician Darboux claimed he was *un intuitif* (intuitive), arguing that this is demonstrated by the fact that he worked so often by visual representation. He did not care about being rigorous and disliked logic. He believed that logic was not a way to invent but a way to structure ideas and that logic limits ideas.

Toulouse' characterisation

Poincaré's mental organisation was not only interesting to Poincaré himself but also to Toulouse, a psychologist of the Psychology Laboratory of the School of Higher Studies in Paris. Toulouse wrote a book entitled *Henri Poincaré* (1910).^[31] In it, he discussed Poincaré's regular schedule:

- He worked during the same times each day in short periods of time. He undertook mathematical research for four hours a day, between 10 a.m. and noon then again from 5 p.m. to 7 p.m.. He would read articles in journals later in the evening.
- His normal work habit was to solve a problem completely in his head, then commit the completed problem to paper.
- He was ambidextrous and nearsighted.
- His ability to visualise what he heard proved particularly useful when he attended lectures, since his eyesight was so poor that he could not see properly what the lecturer wrote on the blackboard.

These abilities were offset to some extent by his shortcomings:

- He was physically clumsy and artistically inept.
- He was always in a rush and disliked going back for changes or corrections.
- He never spent a long time on a problem since he believed that the subconscious would continue working on the problem while he consciously worked on another problem.

In addition, Toulouse stated that most mathematicians worked from principles already established while Poincaré started from basic principles each time (O'Connor et al., 2002).

His method of thinking is well summarised as:

"Habitué à négliger les détails et à ne regarder que les cimes, il passait de l'une à l'autre avec une promptitude surprenante et les faits qu'il découvrait se groupant d'eux-mêmes autour de leur centre étaient instantanément et automatiquement classés dans sa mémoire." ("Accustomed to neglecting details and to looking only at mountain tops, he went from one peak to another with surprising rapidity, and the facts he discovered, clustering around their center, were instantly and automatically pigeonholed in his memory.") Belliver (1956)



Crédit Henri Manuel

Henri Poincaré

Photographic portrait of H. Poincaré by Henri Manuel

Attitude towards transfinite numbers

Poincaré was dismayed by Georg Cantor's theory of transfinite numbers, and referred to it as a "disease" from which mathematics would eventually be cured.^[32] Poincaré said, "There is no actual infinite; the Cantorians have forgotten this, and that is why they have fallen into contradiction."^[33]

View on economics

Poincaré saw mathematical work in economics and finance as peripheral. In 1900 Poincaré commented on Louis Bachelier's thesis "The Theory of Speculation", saying: "M. Bachelier has evidenced an original and precise mind [but] the subject is somewhat remote from those our other candidates are in the habit of treating." (Bernstein, 1996, pp. 199–200) Bachelier's work explained what was then the French government's pricing options on French Bonds and anticipated many of the pricing theories in financial markets used today.^[34]

Honours

Awards

- Oscar II, King of Sweden's mathematical competition (1887)
- American Philosophical Society 1899
- Gold Medal of the Royal Astronomical Society of London (1900)
- Bolyai Prize in 1905
- Matteucci Medal 1905
- French Academy of Sciences 1906
- Académie Française 1909
- Bruce Medal (1911)

Named after him

- Poincaré Prize (Mathematical Physics International Prize)
- Annales Henri Poincaré (Scientific Journal)
- Poincaré Seminar (nicknamed "Bourbaphy")
- The crater Poincaré on the Moon
- Asteroid 2021 Poincaré

Philosophy

Poincaré had philosophical views opposite to those of Bertrand Russell and Gottlob Frege, who believed that mathematics was a branch of logic. Poincaré strongly disagreed, claiming that intuition was the life of mathematics. Poincaré gives an interesting point of view in his book *Science and Hypothesis*:

For a superficial observer, scientific truth is beyond the possibility of doubt; the logic of science is infallible, and if the scientists are sometimes mistaken, this is only from their mistaking its rule.

Poincaré believed that arithmetic is a synthetic science. He argued that Peano's axioms cannot be proven non-circularly with the principle of induction (Murzi, 1998), therefore concluding that arithmetic is *a priori* synthetic and not analytic. Poincaré then went on to say that mathematics cannot be deduced from logic since it is not analytic. His views were similar to those of Immanuel Kant (Kolak, 2001, Folina 1992). He strongly opposed Cantorian set theory, objecting to its use of impredicative definitions.

However, Poincaré did not share Kantian views in all branches of philosophy and mathematics. For example, in geometry, Poincaré believed that the structure of non-Euclidean space can be known analytically. Poincaré held that convention plays an important role in physics. His view (and some later, more extreme versions of it) came to be known as "conventionalism". Poincaré believed that Newton's first law was not empirical but is a conventional

framework assumption for mechanics. He also believed that the geometry of physical space is conventional. He considered examples in which either the geometry of the physical fields or gradients of temperature can be changed, either describing a space as non-Euclidean measured by rigid rulers, or as a Euclidean space where the rulers are expanded or shrunk by a variable heat distribution. However, Poincaré thought that we were so accustomed to Euclidean geometry that we would prefer to change the physical laws to save Euclidean geometry rather than shift to a non-Euclidean physical geometry.^[35]

Free will

Poincaré's famous lectures before the Société de Psychologie in Paris (published as *Science and Hypothesis*, *The Value of Science*, and *Science and Method*) were cited by Jacques Hadamard as the source for the idea that creativity and invention consist of two mental stages, first random combinations of possible solutions to a problem, followed by a critical evaluation.^[36]

Although he most often spoke of a deterministic universe, Poincaré said that the subconscious generation of new possibilities involves chance.

"It is certain that the combinations which present themselves to the mind in a kind of sudden illumination after a somewhat prolonged period of unconscious work are generally useful and fruitful combinations... all the combinations are formed as a result of the automatic action of the subliminal ego, but those only which are interesting find their way into the field of consciousness... A few only are harmonious, and consequently at once useful and beautiful, and they will be capable of affecting the geometrician's special sensibility I have been speaking of; which, once aroused, will direct our attention upon them, and will thus give them the opportunity of becoming conscious... In the subliminal ego, on the contrary, there reigns what I would call liberty, if one could give this name to the mere absence of discipline and to disorder born of chance."^[37]

Poincaré's two stages – random combinations followed by selection – became the basis for Daniel Dennett's two-stage model of free will.^[38]

Poincaré Model of the Subconscious Mind in Mathematics

Poincaré proposes a model of the Subconscious Mind stresses the Subconscious or Unconscious mind is capable of evaluating and processing even complex mathematical or scientific ideas, and evaluate and elevate them on the basis of their elegance and beauty.^[39]

Poincaré thus shares with Freud a belief that mental functions are at work in creating thinking which are not present to our conscious mind. That the discontinuity in time between thinking of a problem and suddenly relieving a solution proves that some mental function outside the space of conscious awareness must be at work^[40]

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Henri Cartan

Henri Cartan	
	
Born	July 8, 1904Nancy, France
Died	August 13, 2008 (aged 104)Paris, France
Nationality	 France
Fields	Mathematics
Institutions	University of Paris
Alma mater	École Normale Supérieure
Doctoral advisor	Paul Montel
Doctoral students	Jean-Paul Benzécri Jean-Paul Brasselet Pierre Cartier Jacques Deny Adrien Douady Roger Godement Max Karoubi Jean-Louis Koszul Jean-Pierre Serre René Thom
Known for	Cartan's theorems A and B
Notable awards	Wolf Prize (1980)

Henri Paul Cartan (July 8, 1904 – August 13, 2008) was a French mathematician with substantial contributions in algebraic topology. He was the son of the French mathematician Élie Cartan.^[1]

Life

Cartan studied at the Lycée Hoche in Versailles, then at the ENS, receiving his doctorate in mathematics. He taught at the University of Strasbourg from November 1931 until the outbreak of the Second World War, after which he held academic positions at a number of other French universities, spending the bulk of his working life in Paris.

Cartan was known for work in algebraic topology, in particular on cohomology operations, the method of "killing homotopy groups", and group cohomology. His seminar in Paris in the years after 1945 covered ground on several complex variables, sheaf theory, spectral sequences and homological algebra, in a way that deeply influenced Jean-Pierre Serre, Armand Borel, Alexander Grothendieck and Frank Adams, amongst others of the leading lights of the younger generation. The number of his official students was small, but includes Adrien Douady, Roger Godement, Max Karoubi, Jean-Pierre Serre and René Thom.

Cartan also was a founding member of the Bourbaki group and one of its most active participants. His book with Samuel Eilenberg *Homological Algebra* (1956)^[2] was an important text, treating the subject with a moderate level of abstraction and category theory.

Cartan used his influence to help obtain the release of some dissident mathematicians, including Leonid Plyushch and Jose Luis Massera. For his humanitarian efforts he received the Pagels Award from the New York Academy of Sciences.^[3]

Cartan died on 13 August 2008 at the age of 104. His funeral took place the following Wednesday on 20 August in Die, Drome.^[1]

Honours and awards

Cartan received numerous honours and awards including the Wolf Prize in 1980. From 1974 until his death he had been a member of the French Academy of Sciences. He was a foreign member of the Finnish Academy of Science and Letters, Royal Danish Academy of Sciences and Letters, Royal Society of London, Russian Academy of Sciences, Royal Swedish Academy of Sciences, United States National Academy of Sciences, Polish Academy of Sciences and other academies and societies.

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- [3] Notices of the AMS, Vol. 46(7), page 788 (<http://www.ams.org/notices/199907/fea-cartan.pdf>)

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Jacques Hadamard

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<div><div><div><div></div></div><div><div></div></div></div><div><div><div></div></div><div><div></div></div></div><div><div><div></div></div><div><div></div></div></div></div> <div>J. Hadamard</div> <div>Jacques Salomon Hadamard</div>	
Born	December 8, 1865Versailles, France
Died	October 17, 1963 (aged 97)Paris, France
Residence	France
Nationality	French
Fields	Mathematician
Institutions	University of Bordeaux Sorbonne Collège de France École Polytechnique École Centrale
Alma mater	École Normale Supérieure
Doctoral advisor	C. Émile Picard Jules Tannery
Doctoral students	Maurice René Fréchet Paul Lévy Szolem Mandelbrojt André Weil Xinmou Wu
Known for	Hadamard product Proof of prime number theorem Hadamard matrices
Notable awards	Grand Prix des Sciences Mathématiques (1892) Prix Poncelet (1898) CNRS Gold medal (1956)

Jacques Salomon Hadamard (December 8, 1865 – October 17, 1963) was a French mathematician who made major contributions in number theory, complex function theory, differential geometry and partial differential equations.

Biography

The son of a teacher, Amédée Hadamard, of Jewish descent, and Claire Marie Jeanne Picard, Hadamard attended the Lycée Charlemagne and Lycée Louis-le-Grand, where his father taught. In 1884 Hadamard entered the École Normale Supérieure, having been placed first in the entrance examinations both there and at the École Polytechnique. His teachers included Tannery, Hermite, Darboux, Appell, Goursat and Picard. He obtained his doctorate in 1892 and in the same year was awarded the *Grand Prix des Sciences Mathématiques* for his prize essay on the Riemann zeta function.

In 1892 Hadamard married Louise-Anna Trénel, also of Jewish descent, with whom he had three sons and two daughters. The following year he took up a lectureship in the University of Bordeaux, where he proved his celebrated inequality on determinants, which led to the discovery of Hadamard matrices when equality holds. In 1896 he made two important contributions: he proved the prime number theorem, using complex function theory (also proved independently by de la Vallée Poussin); and he was awarded the Bordin Prize of the French Academy of Sciences for his work on geodesics in the differential geometry of surfaces and dynamical systems. In the same year he was appointed Professor of Astronomy and Rational Mechanics in Bordeaux. His foundational work on geometry and symbolic dynamics continued in 1898 with the study of geodesics on surfaces of negative curvature. For his cumulative work, he was awarded the Prix Poncelet in 1898.

After the Dreyfus affair, which involved him personally because his wife was related to Dreyfus, Hadamard became politically active and a staunch supporter of Jewish causes^[1] though he professed to be an atheist in his religion.^[2]

In 1897 he moved back to Paris, holding positions in the Sorbonne and the Collège de France, where he was appointed Professor of Mechanics in 1909. In addition to this post, he was appointed to chairs of analysis at the École Polytechnique in 1912 and at the École Centrale in 1920, succeeding Jordan and Appell. In Paris Hadamard concentrated his interests on the problems of mathematical physics, in particular partial differential equations, the calculus of variations and the foundations of functional analysis. He introduced the idea of *well-posed problem* and the *method of descent* in the theory of partial differential equations, culminating in his seminal book on the subject, based on lectures given at Yale University in 1922. He was elected to the French Academy of Sciences in 1916, in succession to Poincaré, whose complete works he helped edit. Later in his life he wrote on probability theory and mathematical education. He was awarded the CNRS Gold medal for his lifetime achievements in 1956.

Hadamard's students included Maurice Fréchet, Paul Lévy, Szolem Mandelbrojt and André Weil.

On creativity

In his book *Psychology of Invention in the Mathematical Field*, Hadamard uses introspection to describe mathematical thought processes. In sharp contrast to authors who identify language and cognition, he describes his own mathematical thinking as largely wordless, often accompanied by mental images that represent the entire solution to a problem. He surveyed 100 of the leading physicists of the day (approximately 1900), asking them how they did their work.

Hadamard described the experiences of the mathematicians/theoretical physicists Carl Friedrich Gauss, Hermann von Helmholtz, Henri Poincaré and others as viewing entire solutions with “sudden spontaneousness.”^[3]

Hadamard described the process as having four steps of the five-step Graham Wallas creative process model, with the first three also having been put forth by Helmholtz:^[4] Preparation, Incubation, Illumination, and Verification.

Notes

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Niels Bohr

Niels Bohr	
	
Born	Niels Henrik David Bohr7 October 1885Copenhagen, Denmark
Died	18 November 1962 (aged 77)Copenhagen, Denmark
Nationality	Danish
Fields	Physics
Institutions	University of Copenhagen University of Cambridge University of Manchester
Alma mater	University of Copenhagen
Doctoral advisor	Christian Christiansen
Other academic advisors	J. J. Thomson Ernest Rutherford
Doctoral students	Hendrik Anthony Kramers
Known for	Copenhagen interpretation Complementarity Bohr model Sommerfeld–Bohr theory BKS theory Bohr-Einstein debates Bohr magneton
Influences	Ernest Rutherford
Influenced	Werner Heisenberg Wolfgang Pauli Paul Dirac Lise Meitner Max Delbrück and many others
Notable awards	Nobel Prize in Physics (1922) Franklin Medal (1929)
<div>Signature</div> 	
Notes	Harald Bohr is his younger brother, and Aage Bohr is his son.

Niels Henrik David Bohr (Danish pronunciation: [nɛls 'bo̥ʁ]; 7 October 1885 – 18 November 1962) was a Danish physicist who made fundamental contributions to understanding atomic structure and quantum mechanics, for which he received the Nobel Prize in Physics in 1922. Bohr mentored and collaborated with many of the top physicists of the century at his institute in Copenhagen. He was part of a team of physicists working on the Manhattan Project. Bohr married Margrethe Nørlund in 1912, and one of their sons, Aage Bohr, grew up to be an important physicist who in 1975 also received the Nobel prize. Bohr has been described as one of the most influential scientists of the 20th century.^[1]

Biography

Early years

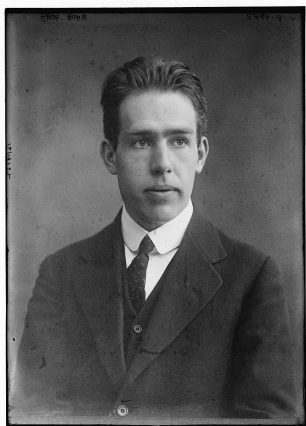
Bohr was born in Copenhagen, Denmark, in 1885. His father, Christian Bohr, was professor of physiology at the University of Copenhagen (it is his name which is given to the Bohr shift or Bohr effect), while his mother, Ellen Adler Bohr, came from a wealthy Jewish family prominent in Danish banking and parliamentary circles. His brother was Harald Bohr, a mathematician and Olympic footballer who played on the Danish national team. Niels Bohr was a passionate footballer as well, and the two brothers played a number of matches for the Copenhagen-based Akademisk Boldklub, with Niels in goal. There is, however, no truth in the oft-repeated claim that Niels Bohr emulated his brother Harald by playing for the Danish national team.^[2]

In 1903 Bohr enrolled as an undergraduate at Copenhagen University, initially studying philosophy and mathematics. In 1905, prompted by a gold medal competition sponsored by the Royal Danish Academy of Sciences and Letters, he conducted a series of experiments to examine the properties of surface tension, using his father's laboratory in the university, familiar to him from assisting there since childhood. His essay won the prize, and it was this success that decided Bohr to abandon philosophy and adopt physics.^[3] As a student under Christian Christiansen he received his doctorate in 1911. As a post-doctoral student, Bohr first conducted experiments under J. J. Thomson at Trinity College, Cambridge. In 1912 he joined Ernest Rutherford at Manchester University and he adapted Rutherford's nuclear structure to Max Planck's quantum theory and so obtained a theory of atomic structure which, with later improvements, mainly as a result of Heisenberg's concepts, remains valid to this day. On the basis of Rutherford's theories, Bohr published his model of atomic structure in 1913, introducing the theory of electrons traveling in orbits around the atom's nucleus, the chemical properties of the element being largely determined by the number of electrons in the outer orbits. Bohr introduced the idea that an electron could drop from a higher-energy orbit to a lower one, emitting a photon (light quantum) of discrete energy. This became a basis for quantum theory. After four productive years with Ernest Rutherford in Manchester, Bohr returned to Denmark becoming in 1918 director of the newly created Institute of Theoretical Physics.

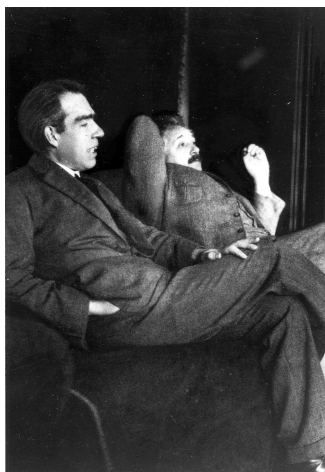
Bohr and his wife Margrethe Nørlund Bohr had six sons. Their oldest died in a tragic boating accident and another died from childhood meningitis. The others went on to lead successful lives, including Aage Bohr, who became a very successful physicist and, like his father, won a Nobel Prize in physics, in 1975.

Physics

In 1916, Bohr became a professor at the University of Copenhagen. With the assistance of the Danish government and the Carlsberg Foundation, he succeeded in founding the Institute of Theoretical Physics in 1921, of which he became director.^[4] In 1922, Bohr was awarded the Nobel Prize in physics "for his services in the investigation of the structure of atoms and of the radiation emanating from them." Bohr's institute served as a focal point for theoretical physicists in the 1920s and '30s, and most of the world's best known theoretical physicists of that period spent some time there.



Niels Bohr as a young man. Exact date of photo unknown.



Niels Bohr and Albert Einstein debating quantum theory at Paul Ehrenfest's home in Leiden (December 1925).

Bohr also conceived the principle of complementarity: that items could be separately analyzed as having several contradictory properties. For example, physicists currently conclude that light behaves either as a wave or a stream of particles depending on the experimental framework — two apparently mutually exclusive properties — on the basis of this principle. Bohr found philosophical applications for this daringly original principle. Albert Einstein much preferred the determinism of classical physics over the probabilistic new quantum physics (to which Max Planck and Einstein himself had contributed). He and Bohr had good-natured arguments over the truth of this principle throughout their lives (see Bohr–Einstein debates).

Werner Heisenberg worked as an assistant to Bohr and university lecturer in Copenhagen from 1926 to 1927. It was in Copenhagen, in 1927, that Heisenberg developed his uncertainty principle, while working on the mathematical foundations of quantum mechanics. Heisenberg later became head of the German atomic bomb project. In 1941, during the German occupation of Denmark in World War II, Bohr was visited by Heisenberg in Copenhagen (see section below). In 1943, shortly before he was to be arrested by the German police, Bohr escaped to Sweden, and then traveled to London.

Atomic research

Bohr worked at the top-secret Los Alamos laboratory in New Mexico, U.S., on the Manhattan Project, where he was known by the assumed name of *Nicholas Baker* for security reasons.^[5] His role in the project was important as he was a knowledgeable consultant or "father confessor" on the project. He was concerned about a nuclear arms race, and is quoted as saying, "That is why I went to America. They didn't need my help in making the atom bomb."^[6]

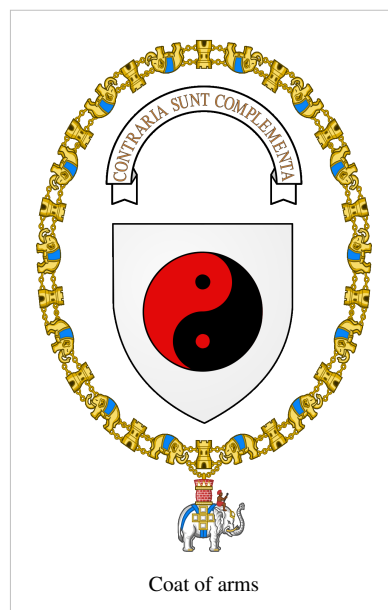
Bohr believed that atomic secrets should be shared by the international scientific community. After meeting with Bohr, J. Robert Oppenheimer suggested that Bohr visit President Franklin D. Roosevelt to convince him that the Manhattan Project should be shared with the Russians in the hope of speeding up its results.

Roosevelt suggested that Bohr return to the United Kingdom to try to win British approval. Winston Churchill disagreed with the idea of openness towards the Russians to the point that he wrote in a letter: "It seems to me Bohr ought to be confined or at any rate made to see that he is very near the edge of mortal crimes."^[7]

After the war Bohr returned to Copenhagen, advocating the peaceful use of nuclear energy. When awarded the Order of the Elephant by the Danish government, he designed his own coat of arms which featured a *taijitu* (symbol of yin and yang) and the Latin motto *contraria sunt complementa*: opposites are complementary.^[8] He died in Copenhagen in 1962 of heart failure.^[9] He is buried in the Assistens Kirkegård in the Nørrebro section of Copenhagen.

Political activity

As regards the Occupation of Denmark during World War II, and especially the events surrounding the Danish policy of cooperation with Nazi Germany and the treatment of Danish Jews, most Danish archives remained sealed until 1998, many remain sealed to this day, and opinions of Danish historical scholars and politicians on these dark topics remain deeply divided.^[10] So too, scholarly opinions remain divided on the importance of the political activities of Niels Bohr during this period. All sources agree that almost as soon as Hitler had taken power in Germany Bohr played an active role in rescuing Jewish physicists out of Germany, typically offering them haven in Copenhagen before they could take up permanent residence elsewhere.^[11] As for Sweden during World War II and especially in the autumn of 1943, it was far from certain that they would accept Danish Jews attempting to escape Hitler's deportation order. As related by Bohr's friend Stefan Rozental and the historian Richard Rhodes (sources cited in note 11), Bohr was immediately smuggled out of Denmark in order to secure his services for the Manhattan project. But rather than proceeding promptly to the United States, as had been planned for him, on 30 September 1943 Bohr persuaded King Gustav of Sweden to make public Sweden's willingness to provide asylum, on 2 October 1943 Swedish radio broadcast that Sweden was ready to offer asylum, and there followed quickly thereafter the mass Rescue of the Danish Jews by their countrymen. Historians are divided not on Bohr's political actions in Sweden, but rather on the implications and impacts of those actions. Some argue that Bohr was among those rescued and therefore could have played no role in facilitating the mass rescue, whereas Richard Rhodes and others (note 11) interpret Bohr's political action in Sweden as being a decisive event without which that mass rescue could not have occurred. Whether or not the mass Rescue of the Danish Jews could have happened without Bohr's political activity in Sweden, there is no doubt that he did all that he could for his countrymen.



Contributions to Physics and Chemistry

- The Bohr model of the atom, the theory that electrons travel in discrete orbits around the atom's nucleus.
- The shell model of the atom, where the chemical properties of an element are determined by the electrons in the outermost orbit.
- The correspondence principle, the basic tool of Old quantum theory.
- The liquid drop model of the atomic nucleus.
- Identified the isotope of uranium that was responsible for slow-neutron fission - ^{235}U .^[12]
- Much work on the Copenhagen interpretation of quantum mechanics.
- The principle of complementarity: that items could be separately analyzed as having several contradictory properties.

Kierkegaard's influence on Bohr

It is generally accepted that Bohr read the 19th century Danish Christian existentialist philosopher, Søren Kierkegaard. Richard Rhodes argues in *The Making of the Atomic Bomb* that Bohr was influenced by Kierkegaard via the philosopher Harald Høffding, who was strongly influenced by Kierkegaard and who was an old friend of Bohr's father. In 1909, Bohr sent his brother Kierkegaard's *Stages on Life's Way* as a birthday gift. In the enclosed letter, Bohr wrote, "It is the only thing I have to send home; but I do not believe that it would be very easy to find anything better.... I even think it is one of the most delightful things I have ever read." Bohr enjoyed Kierkegaard's language and literary style, but mentioned that he had some "disagreement with [Kierkegaard's ideas]."^[13]

Given this, there has been some dispute over whether Kierkegaard influenced Bohr's philosophy and science. David Favrholdt^[14] argues that Kierkegaard had minimal influence over Bohr's work, taking Bohr's statement about disagreeing with Kierkegaard at face value; while Jan Faye^[15] endorses the opposing point of view by arguing that one can disagree with the content of a theory while accepting its general premises and structure.^[16]

Relationship with Heisenberg

Bohr and Werner Heisenberg enjoyed a strong mentor/protégé relationship up to the onset of World War II. Bohr became aware of Heisenberg's talent during a lecture Heisenberg gave in Göttingen in 1922. During the mid-1920s, Heisenberg worked with Bohr at the institute in Copenhagen. Heisenberg, like most of Bohr's assistants, learned Danish. Heisenberg's uncertainty principle was developed during this period, as was Bohr's complementarity principle.

By the time of World War II, the relationship became strained; this was in part because Bohr, with his partially Jewish heritage, remained in occupied Denmark, while Heisenberg remained in Germany and became head of the German nuclear effort. Heisenberg made a famous visit to Bohr in September 1941 and during a private moment it seems that he began to address nuclear energy and morality as well as the war. Neither Bohr nor Heisenberg spoke about it in any detail or left written records of this part of the meeting and they were alone and outside.^[17] Bohr seems to have reacted by terminating that conversation abruptly while not giving Heisenberg hints in any direction.

While some suggest that the relationship became strained at this meeting, other evidence shows that the level of contact had been reduced considerably for some time already. Heisenberg suggested that the fracture occurred later. In correspondence to his wife, Heisenberg described the final visit of the trip: "Today I was once more, with Weizsäcker, at Bohr's. In many ways this was especially nice, the conversation revolved for a large part of the evening around purely human concerns, Bohr was reading aloud, I played a Mozart Sonata (A-Major)."^[18] Ivan Supek, one of Heisenberg's students and friends, claimed that the main figure of the meeting was actually Weizsäcker who tried to persuade Bohr to mediate peace between Great Britain and Germany.^[19]

Tube Alloys

"Tube Alloys" was the code-name for the British nuclear weapon program. British intelligence inquired about Bohr's availability for work or insights of particular value. Bohr's reply made it clear that he could not help. This reply, like his reaction to Heisenberg, made sure that if Gestapo intercepted anything attributed to Bohr it would point to no knowledge regarding nuclear energy as it stood in 1941. This does not exclude the possibility that Bohr privately made calculations going further than his work in 1939 with Wheeler.

After leaving Denmark in the dramatic day and night (October 1943) when most Jews were able to escape to Sweden due to exceptional circumstances (see Rescue of the Danish Jews), Bohr was quickly asked again to join the British effort and he was flown to the UK. He was evacuated from Stockholm in 1943 in an unarmed De Havilland Mosquito operated by British Overseas Airways Corporation (BOAC). Passengers on BOAC's Mosquitos were carried in an improvised cabin in the bomb bay. The flight almost ended in tragedy as Bohr did not don his oxygen equipment as instructed and passed out at high altitude. He would have died had not the pilot surmising from Bohr's

lack of response to intercom communication that he had lost consciousness, descended to a lower altitude for the remainder of the flight. Bohr's comment was that he had slept like a baby for the entire flight.

As part of the UK team on "Tube Alloys" Bohr went to Los Alamos. Oppenheimer credited Bohr warmly for his guiding help during certain discussions among scientists there. Discreetly, he met President Franklin D. Roosevelt and later Winston Churchill to warn against the perilous perspectives that would follow from separate development of nuclear weapons by several powers rather than some form of controlled sharing of the knowledge, which would spread quickly in any case. Only in the 1950s, after the Soviet Union's first nuclear weapon test, was it possible to create the International Atomic Energy Agency along the lines of Bohr's suggestion.

Speculation

In 1957, while the author Robert Jungk was working on the book *Brighter Than a Thousand Suns*, Heisenberg wrote to Jungk explaining that he had visited Copenhagen to communicate to Bohr his view that scientists on either side should help prevent development of the atomic bomb, that the German attempts were entirely focused on energy production and that Heisenberg's circle of colleagues tried to keep it that way.^[20] Heisenberg acknowledged that his cryptic approach of the subject had so alarmed Bohr that the discussion failed. Heisenberg nuanced his claims and avoided the implication that he and his colleagues had sabotaged the bomb effort; this nuance was lost in Jungk's original publication of the book, which implied that the German atomic bomb project was obstructed by Heisenberg.

When Bohr saw Jungk's erroneous depiction in the Danish translation of the book, he disagreed. He drafted (but never sent) a letter to Heisenberg, stating that while Heisenberg had indeed discussed the subject of nuclear weapons in Copenhagen, Heisenberg had never alluded to the fact that he might be resisting efforts to build such weapons. Bohr dismissed the idea of any pact as hindsight.^[21]

Michael Frayn's play *Copenhagen*, which was performed in London (for five years), Copenhagen, Gothenburg, Rome, Athens, Geneva and on Broadway in New York, explores what might have happened at the 1941 meeting between Heisenberg and Bohr. Frayn points in particular to the onus of being one of the few to understand what it would mean to create a nuclear weapon.

Open World

Bohr advocated informing the Soviet authorities that the atomic bomb would soon be in use. In 1944 he obtained an audience with Winston Churchill, who became worried about whether Bohr was a security risk.^[22] In 1950 he addressed an 'Open Letter' to the United Nations.^[23]

Legacy

- He was one of the founding fathers of CERN in 1954.^[24]
 - Received the first ever Atoms for Peace Award in 1957.
 - In 1965, three years after Bohr's death, the Institute of Physics at the University of Copenhagen changed its name to the Niels Bohr Institute.
 - The Bohr models semicentennial was commemorated in Denmark on 21 November 1963 with a postage stamp depicting Bohr, the hydrogen atom and the formula for the difference of any two hydrogen energy levels:
$$h\nu = \epsilon_2 - \epsilon_1.$$
 - Bohrium (a chemical element, atomic number 107) is named in honour of Bohr.
 - Hafnium, another chemical element, whose properties were predicted by Bohr, was named by him after Hafnia, Copenhagen's Latin name.
 - Asteroid 3948 Bohr is named after him.
 - The Centennial of Bohr's birth was commemorated in Denmark on 3 October 1985 with a postage stamp depicting Bohr with his wife Margrethe.
-

- In 1997 the Danish National Bank started circulating the 500-krone banknote with the portrait of Bohr smoking a pipe.^[25] ^[26]
- Bohr has been a common name in Europe since the Middle Ages.^[27] It remains fairly common in Europe and spread to the U.S. with pilgrims named Bohr settling there. There was an notable increase in the middle name Bohr throughout Europe and America following Bohr's death.
- Bohr was referenced in The Simpsons thirteenth season episode entitled I Am Furious Yellow. In the episode, Homer Simpson is looking forward to watching a television show called *When Dinosaurs Get Drunk*, when it is suddenly announced that it will be replaced with another called *The Boring World of Niels Bohr*. The opening image is of the scientist and an expanded version of the Bohr model of the atom.

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- *Copenhagen* Michael Frayn ISBN 0413724905
- *Faust in Copenhagen: A Struggle for the Soul of Physics* by Gino Segre; ISBN 0-670-03858-X.

External links

- Niels Bohr Collected Works. (<http://elsevierdirect.com/nielsbohr>)
- Niels Bohr on the 500 Danish Kroner banknote. (<http://www-personal.umich.edu/~jbourj/money1.htm>)
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- Annotated bibliography for Niels Bohr from the Alsos Digital Library for Nuclear Issues (<http://alsos.wlu.edu/qsearch.aspx?browse=people/Bohr,+Niels>)
- Quantum Chemistry I Lecture - Bohr's Model of the Atom (<http://cinarz.zdo.com/moodle/mod/resource/view.php?id=15>)
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- Oral History interview transcript with Niels Bohr 31 October 1962, American Institute of Physics, Niels Bohr Library and Archives (http://www.aip.org/history/ohilist/4517_1.html)
 - Short video documentary about Niels Bohr (<http://www.science.tv/watch/3bcea3b7513ccef5857a/THE-CORE>)
 - The gunfighter's dilemma (http://news.bbc.co.uk/today/hi/today/newsid_8493000/8493203.stm) BBC News story about Bohr's researches on reaction times.
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Werner Heisenberg

Werner Heisenberg	
	
Born	Werner Karl Heisenberg5 December 1901Würzburg, Germany
Died	1 February 1976 (aged 74)Munich, Germany
Nationality	German
Fields	Physics
Institutions	University of Göttingen University of Copenhagen University of Leipzig University of Berlin University of Munich
Alma mater	University of Munich
Doctoral advisor	Arnold Sommerfeld
Other academic advisors	Niels Bohr Max Born
Doctoral students	Felix Bloch Edward Teller Rudolph E. Peierls Reinhard Oehme Friedwardt Winterberg Peter Mittelstaedt Șerban Țițeica Ivan Supek Erich Bagge Hermann Arthur Jahn Raziuddin Siddiqui Heimo Dolch Hans Heinrich Euler Edwin Gora Bernhard Kockel Arnold Siegert Wang Foh-san
Other notable students	William Vermillion Houston Guido Beck Ugo Fano

Known for	Uncertainty Principle Heisenberg's microscope Matrix mechanics Kramers-Heisenberg formula Heisenberg group Isospin
Influenced	Robert Döpel Carl Friedrich von Weizsäcker
Notable awards	Nobel Prize in Physics (1932) Max Planck Medal (1933)
Notes	He was the father of the neurobiologist Martin Heisenberg and the son of August Heisenberg

Werner Heisenberg (5 December 1901 – 1 February 1976) was a German theoretical physicist who made foundational contributions to quantum mechanics and is best known for asserting the uncertainty principle of quantum theory. In addition, he made important contributions to nuclear physics, quantum field theory, and particle physics.

Heisenberg, along with Max Born and Pascual Jordan, set forth the matrix formulation of quantum mechanics in 1925. Heisenberg was awarded the 1932 Nobel Prize in Physics for the creation of quantum mechanics, and its application especially to the discovery of the allotropic forms of hydrogen.^[1]

Following World War II, he was appointed director of the Kaiser Wilhelm Institute for Physics, which was soon thereafter renamed the Max Planck Institute for Physics. He was director of the institute until it was moved to Munich in 1958, when it was expanded and renamed the Max Planck Institute for Physics and Astrophysics.

Heisenberg was also president of the German Research Council, chairman of the Commission for Atomic Physics, chairman of the Nuclear Physics Working Group, and president of the Alexander von Humboldt Foundation.

Biography

Early years

Heisenberg was born in Würzburg, Germany to Kaspar Earnesta August Heisenberg, a secondary school teacher of classical languages who became Germany's only *ordentlicher Professor* (ordinarius professor) of medieval and modern Greek studies in the university system, and his wife Annie Wecklein.^[2]

He studied physics and mathematics from 1920 to 1923 at the *Ludwig-Maximilians-Universität München* and the *Georg-August-Universität Göttingen*. At Munich, he studied under Arnold Sommerfeld and Wilhelm Wien. At Göttingen, he studied physics with Max Born and James Franck, and he studied mathematics with David Hilbert. He received his doctorate in 1923, at Munich under Sommerfeld. He completed his Habilitation in 1924, at Göttingen under Born.^{[3] [4]}

Because Sommerfeld had a sincere interest in his students and knew of Heisenberg's interest in Niels Bohr's theories on atomic physics, Sommerfeld took Heisenberg to Göttingen to the *Bohr-Festspiele* (Bohr Festival) in June 1922. At the event, Bohr was a guest lecturer and gave a series of comprehensive lectures on quantum atomic physics. There, Heisenberg met Bohr for the first time, and it had a significant and continuing effect on him.^{[5] [6] [7]}

Heisenberg's doctoral thesis, the topic of which was suggested by Sommerfeld, was on turbulence;^[8] the thesis discussed both the stability of laminar flow and the nature of turbulent flow. The problem of stability was investigated by the use of the Orr–Sommerfeld equation, a fourth order linear differential equation for small disturbances from laminar flow. He briefly returned to this topic after World War II.^[9]

Heisenberg's paper on the anomalous Zeeman effect^[10] was accepted as his *Habilitationsschrift* under Max Born at Göttingen.^[11]

In his youth he was a member and Scoutleader of the *Neupfadfinder*, a German Scout association and part of the German Youth Movement.^{[12] [13] [14]} In August 1923 Robert Honsell and Heisenberg organized a trip (*Großfahrt*) to Finland with a Scout group of this association from Munich.^[15]

Career

Göttingen, Copenhagen, and Leipzig

From 1924 to 1927, Heisenberg was a Privatdozent at Göttingen. From 17 September 1924 to 1 May 1925, under an International Education Board Rockefeller Foundation fellowship, Heisenberg went to do research with Niels Bohr, director of the Institute of Theoretical Physics at the University of Copenhagen. He returned to Göttingen and with Max Born and Pascual Jordan, over a period of about six months, developed the matrix mechanics formulation of quantum mechanics. On 1 May 1926, Heisenberg began his appointment as a university lecturer and assistant to Bohr in Copenhagen. It was in Copenhagen, in 1927, that Heisenberg developed his uncertainty principle, while working on the mathematical foundations of quantum mechanics. On 23 February, Heisenberg wrote a letter to fellow physicist Wolfgang Pauli, in which he first described his new principle.^[16] In his paper^[17] on the uncertainty principle, Heisenberg used the word "*Ungenauigkeit*" (imprecision).^{[3] [18] [19]}

In 1927, Heisenberg was appointed *ordentlicher Professor* (ordinarius professor) of theoretical physics and head of the department of physics at the Universität Leipzig; he gave his inaugural lecture on 1 February 1928. In his first paper published from Leipzig,^[20] Heisenberg used the Pauli exclusion principle to solve the mystery of ferromagnetism.^{[3] [4] [18] [21]}

In Heisenberg's tenure at Leipzig, the quality of doctoral students, post-graduate and research associates who studied and worked with Heisenberg there is attested to by the acclaim later earned by these people; at various times, they included: Erich Bagge, Felix Bloch, Ugo Fano, Siegfried Flügge, William Vermillion Houston, Friedrich Hund, Robert S. Mulliken, Rudolf Peierls, George Placzek, Isidor Isaac Rabi, Fritz Sauter, John C. Slater, Edward Teller, John Hasbrouck van Vleck, Victor Frederick Weisskopf, Carl Friedrich von Weizsäcker, Gregor Wentzel and Clarence Zener.^[22]

In early 1929, Heisenberg and Pauli submitted the first of two papers^{[23] [24]} laying the foundation for relativistic quantum field theory. Also in 1929, Heisenberg went on a lecture tour in the United States, Japan, China, and India.^{[18] [22]}

Shortly after the discovery of the neutron by James Chadwick in 1932, Heisenberg submitted the first of three papers^{[25] [26] [27]} on his neutron-proton model of the nucleus. He was awarded the 1932 Nobel Prize in Physics.^{[18] [28]}

In 1928, the British mathematical physicist P. A. M. Dirac had derived the relativistic wave equation of quantum mechanics, which implied the existence of positive electrons, later to be named positrons. In 1932, from a cloud chamber photograph of cosmic rays, the American physicist Carl David Anderson identified a track as having been made by a positron. In mid-1933, Heisenberg presented his theory of the positron. His thinking on Dirac's theory and further development of the theory were set forth in two papers. The first, *Bemerkungen zur Diracschen Theorie des Positrons* (*Remarks on Dirac's theory of the positron*) was published in 1934,^[29] and the second, *Folgerungen aus der Diracschen Theorie des Positrons* (*Consequences of Dirac's Theory of the Positron*), was published in 1936.^{[18] [30] [31]} In these papers Heisenberg was the first to reinterpret the Dirac equation as a "classical" field equation for any point particle of spin $\hbar/2$, itself subject to quantization conditions involving anti-commutators. Thus reinterpreting it as a (quantum) field equation accurately describing electrons, Heisenberg put matter on the same footing as electromagnetism: as being described by relativistic quantum field equations which allowed the possibility of particle creation and destruction.

In the early 1930s in Germany, the *deutsche Physik* movement was anti-Semitic and anti-theoretical physics, especially including quantum mechanics and the theory of relativity. As applied in the university environment,

political factors took priority over the historically applied concept of scholarly ability,^[32] even though its two most prominent supporters were the Nobel Laureates in Physics Philipp Lenard^[33] and Johannes Stark.^[34]

After Adolf Hitler came to power in 1933, Heisenberg was attacked in the press as a "White Jew"^[35] by elements of the *deutsche Physik* (German Physics) movement for his insistence on teaching about the roles of Jewish scientists. As a result, he came under investigation by the SS. This was over an attempt to appoint Heisenberg as successor to Arnold Sommerfeld at the University of Munich. The issue was resolved in 1938 by Heinrich Himmler, head of the SS. While Heisenberg was not chosen as Sommerfeld's successor, he was rehabilitated to the physics community during the Third Reich. Nevertheless, supporters of *deutsche Physik* launched vicious attacks against leading theoretical physicists, including Arnold Sommerfeld and Heisenberg. On 29 June 1936, a National Socialist Party newspaper published a column attacking Heisenberg. On 15 July 1937, he was attacked in a journal of the SS. This was the beginning of what is called the Heisenberg Affair.^[18]

In mid-1936, Heisenberg presented his theory of cosmic-ray showers in two papers.^{[36] [37]} Four more papers^{[38] [39] [40] [41]} appeared in the next two years.^{[18] [42]}

In June 1939, Heisenberg bought a summer home for his family in Urfeld, in southern Germany. He also traveled to the United States in June and July, visiting Samuel Abraham Goudsmit, at the University of Michigan in Ann Arbor. However, Heisenberg refused an invitation to emigrate to the United States. He did not see Goudsmit again until six years later, when Goudsmit was the chief scientific advisor to the American Operation Alsos at the close of World War II. Ironically, Heisenberg was arrested under Operation Alsos and detained in England under Operation Epsilon.^{[18] [43] [44]}

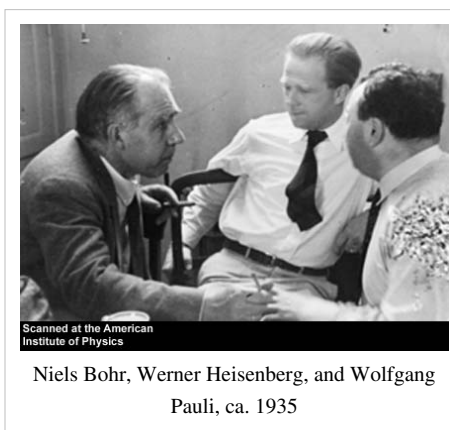
Matrix Mechanics and the Nobel Prize

Heisenberg's paper establishing quantum mechanics^[45] has puzzled physicists and historians. His methods assume that the reader is familiar with Kramers-Heisenberg transition probability calculations. The main new idea, noncommuting matrices, is justified only by a rejection of unobservable quantities. It introduces the non-commutative multiplication of matrices by physical reasoning, based on the correspondence principle, despite the fact that Heisenberg was not then familiar with the mathematical theory of matrices. The path leading to these results has been reconstructed in MacKinnon, 1977,^[46] and the detailed calculations are worked out in Aitchison et al.^[47]

In Copenhagen, Heisenberg and H. Kramers collaborated on a paper on dispersion, or the scattering from atoms of radiation whose wavelength is larger than the atoms. They showed that the successful formula Kramers had developed earlier could not be based on Bohr orbits, because the transition frequencies are based on level spacings which are not constant. The frequencies which occur in the Fourier transform of sharp classical orbits, by contrast, are equally spaced. But these results could be explained by a semi-classical Virtual State model: the incoming radiation excites the valence, or outer, electron to a virtual state from which it decays. In a subsequent paper Heisenberg showed that this virtual oscillator model could also explain the polarization of fluorescent radiation.

These two successes, and the continuing failure of the Bohr-Sommerfeld model to explain the outstanding problem of the anomalous Zeeman effect, led Heisenberg to use the virtual oscillator model to try to calculate spectral frequencies. The method proved too difficult to immediately apply to realistic problems, so Heisenberg turned to a simpler example, the anharmonic oscillator.

The dipole oscillator consists of a simple harmonic oscillator, which is thought of as a charged particle on a spring, perturbed by an external force, like an external charge. The motion of the oscillating charge can be expressed as a



Niels Bohr, Werner Heisenberg, and Wolfgang Pauli, ca. 1935

Fourier series in the frequency of the oscillator. Heisenberg solved for the quantum behavior by two different methods. First, he treated the system with the virtual oscillator method, calculating the transitions between the levels that would be produced by the external source.

He then solved the same problem by treating the anharmonic potential term as a perturbation to the harmonic oscillator and using the perturbation methods that he and Born had developed. Both methods led to the same results for the first and the very complicated second order correction terms. This suggested that behind the very complicated calculations lay a consistent scheme.

So Heisenberg set out to formulate these results without any explicit dependence on the virtual oscillator model. To do this, he replaced the Fourier expansions for the spatial coordinates by matrices, matrices which corresponded to the transition coefficients in the virtual oscillator method. He justified this replacement by an appeal to Bohr's correspondence principle and the Pauli doctrine that quantum mechanics must be limited to observables.

On 9 July, Heisenberg gave Born this paper to review and submit for publication. When Born read the paper, he recognized the formulation as one which could be transcribed and extended to the systematic language of matrices,^[48] which he had learned from his study under Jakob Rosanes^[49] at Breslau University. Born, with the help of his assistant and former student Pascual Jordan, began immediately to make the transcription and extension, and they submitted their results for publication; the paper was received for publication just 60 days after Heisenberg's paper.^[50] A follow-on paper was submitted for publication before the end of the year by all three authors.^[51] (A brief review of Born's role in the development of the matrix mechanics formulation of quantum mechanics along with a discussion of the key formula involving the non-commutivity of the probability amplitudes can be found in an article by Jeremy Bernstein, *Max Born and the Quantum Theory*.^[52] A detailed historical and technical account can be found in Mehra and Rechenberg's book *The Historical Development of Quantum Theory. Volume 3. The Formulation of Matrix Mechanics and Its Modifications 1925–1926*.^[53])

Up until this time, matrices were seldom used by physicists; they were considered to belong to the realm of pure mathematics. Gustav Mie had used them in a paper on electrodynamics in 1912 and Born had used them in his work on the lattices theory of crystals in 1921. While matrices were used in these cases, the algebra of matrices with their multiplication did not enter the picture as they did in the matrix formulation of quantum mechanics.^[54]

Born had learned matrix algebra from Rosanes, as already noted, but Born had also learned Hilbert's theory of integral equations and quadratic forms for an infinite number of variables as was apparent from a citation by Born of Hilbert's work *Grundzüge einer allgemeinen Theorie der Linearen Integralgleichungen* published in 1912.^[55] [56] Jordan, too was well equipped for the task. For a number of years, he had been an assistant to Richard Courant at Göttingen in the preparation of Courant and David Hilbert's book *Methoden der mathematischen Physik I*, which was published in 1924.^[57] This book, fortuitously, contained a great many of the mathematical tools necessary for the continued development of quantum mechanics. In 1926, John von Neumann became assistant to David Hilbert, and he coined the term Hilbert space to describe the algebra and analysis which were used in the development of quantum mechanics.^[58] [59]

In 1928, Albert Einstein nominated Heisenberg, Born, and Jordan for the Nobel Prize in Physics,^[60] but it was not to be. The announcement of the Nobel Prize in Physics for 1932 was delayed until November 1933.^[61] It was at that time that it was announced Heisenberg had won the Prize for 1932 "for the creation of quantum mechanics, the application of which has, inter alia, led to the discovery of the allotropic forms of hydrogen"^[62] [63] and Erwin Schrödinger and Paul Adrien Maurice Dirac shared the 1933 Prize "for the discovery of new productive forms of atomic theory".^[63] One can rightly ask why Born was not awarded the Prize in 1932 along with Heisenberg – Bernstein gives some speculations on this matter. One of them is related to Jordan joining the Nazi Party on 1 May 1933 and becoming a Storm Trooper.^[64] Hence, Jordan's Party affiliations and Jordan's links to Born may have affected Born's chance at the Prize at that time. Bernstein also notes that when Born won the Prize in 1954, Jordan was still alive, and the Prize was awarded for the statistical interpretation of quantum mechanics, attributable alone to Born.^[65]

Heisenberg's reaction to Born for Heisenberg receiving the Prize for 1932 and to Born for Born receiving the Prize in 1954 are also instructive in evaluating whether Born should have shared the Prize with Heisenberg. On 25 November 1933, Born received a letter from Heisenberg in which he said he had been delayed in writing due to a "bad conscience" that he alone had received the Prize "for work done in Göttingen in collaboration – you, Jordan and I." Heisenberg went on to say that Born and Jordan's contribution to quantum mechanics cannot be changed by "a wrong decision from the outside."^[66] In 1954, Heisenberg wrote an article honoring Max Planck for his insight in 1900. In the article, Heisenberg credited Born and Jordan for the final mathematical formulation of matrix mechanics and Heisenberg went on to stress how great their contributions were to quantum mechanics, which were not "adequately acknowledged in the public eye."^[67]

The *deutsche Physik* movement

On 1 April 1935, the eminent theoretical physicist Arnold Sommerfeld, Heisenberg's doctoral advisor at the University of Munich, achieved emeritus status. However, Sommerfeld stayed in his chair during the selection process for his successor, which took until 1 December 1939. The process was lengthy due to academic and political differences between the Munich Faculty's selection and that of the Reichserziehungsministerium (REM, Reich Education Ministry.) and the supporters of *Deutsche Physik*, which was anti-Semitic and had a bias against theoretical physics, especially quantum mechanics and the theory of relativity. In 1935, the Munich Faculty drew up a list of candidates to replace Sommerfeld as ordinarius professor of theoretical physics and head of the Institute for Theoretical Physics at the University of Munich. There were three names on the list: Werner Heisenberg, who received the Nobel Prize in Physics for 1932, Peter Debye, who received the Nobel Prize in Chemistry in 1936, and Richard Becker - all former students of Sommerfeld. The Munich Faculty was firmly behind these candidates, with Heisenberg as their first choice. However, supporters of *Deutsche Physik* and elements in the REM had their own list of candidates and the battle dragged on for over four years. During this time, Heisenberg came under vicious attack by the *Deutsche Physik* supporters. One attack was published in *Das Schwarze Korps*, the newspaper of the Schutzstaffel (SS), headed by Heinrich Himmler. In this, Heisenberg was called a "White Jew" (i.e. an Aryan who acts like a Jew) who should be made to "disappear".^[68] These attacks were taken seriously, as Jews were violently attacked and incarcerated. Heisenberg fought back with an editorial and a letter to Himmler, in an attempt to resolve this matter and regain his honour. At one point, Heisenberg's mother visited Himmler's mother. The two women knew each other as Heisenberg's maternal grandfather and Himmler's father were rectors and members of a Bavarian hiking club. Eventually, Himmler settled the Heisenberg affair by sending two letters, one to SS Gruppenführer Reinhard Heydrich and one to Heisenberg, both on 21 July 1938. In the letter to Heydrich, Himmler said Germany could not afford to lose or silence Heisenberg as he would be useful for teaching a generation of scientists. To Heisenberg, Himmler said the letter came on recommendation of his family and he cautioned Heisenberg to make a distinction between professional physics research results and the personal and political attitudes of the involved scientists. The letter to Heisenberg was signed under the closing "Mit freundlichem Gruss und, Heil Hitler!" (With friendly greetings, Heil Hitler!)^[69] Overall, the Heisenberg affair was a victory for academic standards and professionalism. However, the appointment of Wilhelm Müller to replace Sommerfeld was a political victory over academic standards. Müller was not a theoretical physicist, had not published in a physics journal, and was not a member of the Deutsche Physikalische Gesellschaft; his appointment was considered a travesty and detrimental to educating theoretical physicists.^{[69] [70] [71] [72] [73]}

During the SS investigation of Heisenberg, the three investigators had training in physics. Heisenberg had participated in the doctoral examination of one of them at the *Universität Leipzig*. The most influential of the three, however, was Johannes Juilfs. During their investigation, they had become supporters of Heisenberg as well as his position against the ideological policies of the *deutsche Physik* movement in theoretical physics and academia.^[74]

World War II

In 1939, shortly after the discovery of nuclear fission, the German nuclear energy project, also known as the *Uranverein* (Uranium Club), was begun. Heisenberg was one of the principal scientists leading research and development in the project.

From 15 to 22 September 1941, Heisenberg traveled to German-occupied Copenhagen to lecture and discuss nuclear research and theoretical physics with Niels Bohr. The meeting, and specifically what it might reveal about Heisenberg's intentions concerning developing nuclear weapons for the Nazi regime, is the subject of the award winning Michael Frayn play titled *Copenhagen*. Documents relating to the Bohr-Heisenberg meeting were released in 2002 by the Niels Bohr Archive and by the Heisenberg family.^{[75] [76]}

On 26 February 1942, Heisenberg presented a lecture to Reich officials on energy acquisition from nuclear fission, after the Army withdrew most of its funding.^[77] The Uranium Club was transferred to the Reich Research Council (RFR) in July 1942. On 4 June 1942, Heisenberg was summoned to report to Albert Speer, Germany's Minister of Armaments, on the prospects for converting the Uranium Club's research toward developing nuclear weapons. During the meeting, Heisenberg told Speer that a bomb could not be built before 1945, and would require significant monetary and manpower resources.^[78] Five days later, on 9 June 1942, Adolf Hitler issued a decree for the reorganization of the RFR as a separate legal entity under the Reich Ministry for Armament and Ammunition; the decree appointed Reich Marshall Göring as the president.^[79]

In September 1942, Heisenberg submitted his first paper of a three-part series on the scattering matrix, or S-matrix, in elementary particle physics. The first two papers were published in 1943^{[80] [81]} and the third in 1944.^[82] The S-matrix described only observables, i.e., the states of incident particles in a collision process, the states of those emerging from the collision, and stable bound states; there would be no reference to the intervening states. This was the same precedent as he followed in 1925 in what turned out to be the foundation of the matrix formulation of quantum mechanics through only the use of observables.^{[18] [42]}

In February 1943, Heisenberg was appointed to the Chair for Theoretical Physics at the *Friedrich-Wilhelms-Universität* (today, the Humboldt-Universität zu Berlin). In April, his election to the *Preußische Akademie der Wissenschaften* (Prussian Academy of Sciences) was approved. That same month, he moved his family to their retreat in Urfeld as Allied bombing increased in Berlin. In the summer, he dispatched the first of his staff at the *Kaiser-Wilhelm Institut für Physik* to Hechingen and its neighboring town of Haigerloch, on the edge of the Black Forest, for the same reasons. From 18–26 October, he traveled to German-occupied Netherlands. In December 1943, Heisenberg visited German-occupied Poland.^{[18] [83]}

From 24 January to 4 February 1944, Heisenberg traveled to occupied Copenhagen, after the German Army confiscated Bohr's Institute of Theoretical Physics. He made a short return trip in April. In December, Heisenberg lectured in neutral Switzerland.^[18]

In January 1945, Heisenberg vacated the *Kaiser-Wilhelm Institut für Physik* with about all of his staff for the facilities in the Black Forest.^[18]

Uranium Club

In December 1938, the German chemists Otto Hahn and Fritz Strassmann sent a manuscript to *Naturwissenschaften* reporting they had detected the element barium after bombarding uranium with neutrons,^[84] simultaneously, they communicated these results to Lise Meitner, who had in July of that year fled to the Netherlands and then went to Sweden.^[85] Meitner, and her nephew Otto Robert Frisch, correctly interpreted these results as being nuclear fission.^[86] Frisch confirmed this experimentally on 13 January 1939.^{[87] [88]}

Paul Harteck was director of the physical chemistry department at the University of Hamburg and an advisor to the *Heereswaffenamt* (HWA, Army Ordnance Office). On 24 April 1939, along with his teaching assistant Wilhelm Groth, Harteck made contact with the *Reichskriegsministerium* (RKM, Reich Ministry of War) to alert them to the potential of military applications of nuclear chain reactions. Two days earlier, on 22 April 1939, after hearing a

colloquium paper by Wilhelm Hanle on the use of uranium fission in a *Uranmaschine* (uranium machine, i.e., nuclear reactor), Georg Joos, along with Hanle, notified Wilhelm Dames, at the *Reichserziehungsministerium* (REM, Reich Ministry of Education), of potential military applications of nuclear energy. The communication was given to Abraham Esau, head of the physics section of the *Reichsforschungsrat* (RFR, Reich Research Council) at the REM. On 29 April, a group, organized by Esau, met at the REM to discuss the potential of a sustained nuclear chain reaction. The group included the physicists Walther Bothe, Robert Döpel, Hans Geiger, Wolfgang Gentner (probably sent by Walther Bothe), Wilhelm Hanle, Gerhard Hoffmann and Georg Joos; Peter Debye was invited, but he did not attend. After this, informal work began at the Georg-August University of Göttingen by Joos, Hanle and their colleague Reinhold Mannfopff; the group of physicists was known informally as the first *Uranverein* (Uranium Club) and formally as *Arbeitsgemeinschaft für Kernphysik*. The group's work was discontinued in August 1939, when the three were called to military training.^{[89] [90] [91] [92]}

The second *Uranverein* began after the *Heereswaffenamt* (HWA, Army Ordnance Office) squeezed the *Reichsforschungsrat* (RFR, Reich Research Council) out of the *Reichserziehungsministerium* (REM, Reich Ministry of Education) and started the formal German nuclear energy project under military auspices. The second *Uranverein* was formed on 1 September 1939, the day World War II began, and it had its first meeting on 16 September 1939. The meeting was organized by Kurt Diebner, advisor to the HWA, and held in Berlin. The invitees included Walther Bothe, Siegfried Flügge, Hans Geiger, Otto Hahn, Paul Harteck, Gerhard Hoffmann, Josef Mattauch and Georg Stetter. A second meeting was held soon thereafter and included Klaus Clusius, Robert Döpel, Werner Heisenberg and Carl Friedrich von Weizsäcker. Also at this time, the *Kaiser-Wilhelm Institut für Physik* (KWIP, Kaiser Wilhelm Institute for Physics, after World War II the Max Planck Institute for Physics), in Berlin-Dahlem, was placed under HWA authority, with Diebner as the administrative director, and the military control of the nuclear research commenced.^{[91] [92] [93]}

When it was apparent that the nuclear energy project would not make a decisive contribution to ending the war effort in the near term, control of the KWIP was returned in January 1942 to its umbrella organization, the *Kaiser-Wilhelm Gesellschaft* (KWG, Kaiser Wilhelm Society, after World War II the Max-Planck Gesellschaft), and HWA control of the project was relinquished to the RFR in July 1942. The nuclear energy project thereafter maintained its *kriegswichtig* (important for the war) designation and funding continued from the military. However, the German nuclear power project was then broken down into the following main areas: uranium and heavy water production, uranium isotope separation and the *Uranmaschine* (uranium machine, i.e., nuclear reactor). Also, the project was then essentially split up between a number of institutes, where the directors dominated the research and set their own research agendas.^{[91] [94] [95]} The dominant personnel and facilities were the following:^{[96] [97] [98]}

- *Institut für Physik* (Walther Bothe) of the *Kaiser-Wilhelm Institut für medizinische Forschung* (KWImF, Kaiser Wilhelm Institute for Medical Research),
- Institute for Physical Chemistry (Klaus Clusius) at the Ludwig Maximilian University of Munich,
- HWA *Versuchsstelle* (testing station) in Gottow (Kurt Diebner),
- *Kaiser-Wilhelm-Institut für Chemie* (Otto Hahn),
- Physical Chemistry Department (Paul Harteck) of the University of Hamburg,
- *Kaiser-Wilhelm-Institut für Physik* (Werner Heisenberg),
- Second Experimental Physics Institute (Hans Kopfermann) at the Georg-August University of Göttingen,
- Auergesellschaft (Nikolaus Riehl), and
- *II. Physikalisches Institut* (Georg Stetter) at the University of Vienna.

Heisenberg was appointed director-in-residence of the KWIP on 1 July 1942, as Peter Debye was still officially the director and on leave in the United States; Debye had gone on leave as he was a citizen of The Netherlands and had refused to become a German citizen when the HWA took administrative control of the KWIP. Heisenberg still also had his department of physics at the University of Leipzig where work was done for the *Uranverein* by Robert Döpel and his wife Klara Döpel. During the period Kurt Diebner administered the KWIP under the HWA program,

considerable personal and professional animosity developed between Diebner and the Heisenberg inner circle – Heisenberg, Karl Wirtz, and Carl Friedrich von Weizsäcker.^{[18] [99]}

The point in 1942, when the army relinquished its control of the German nuclear energy project, was the zenith of the project relative to the number of personnel devoting time to the effort. There were only about 70 scientists working on the project, with about 40 devoting more than half their time to nuclear fission research. After this, the number of scientists working on applied nuclear fission diminished dramatically. Many of the scientists not working with the main institutes stopped working on nuclear fission and devoted their efforts to more pressing war related work.^[100]

Over time, the HWA and then the RFR controlled the German nuclear energy project. The most influential people in the project were Kurt Diebner, Abraham Esau, Walther Gerlach and Erich Schumann. Schumann was one of the most powerful and influential physicists in Germany. Schumann was director of the Physics Department II at the Frederick William University (later, University of Berlin), which was commissioned and funded by the *Oberkommando des Heeres* (OKW, Army High Command) to conduct physics research projects. He was also head of the research department of the HWA, assistant secretary of the Science Department of the OKW and *Bevollmächtigter* (plenipotentiary) for high explosives. Diebner, throughout the life of the nuclear energy project, had more control over nuclear fission research than did Walther Bothe, Klaus Clusius, Otto Hahn, Paul Harteck or Werner Heisenberg.^{[101] [102]}

1945: Operation Alsos and Operation Epsilon

Operation Alsos was an Allied effort commanded by the Russian-American Colonel Boris T. Pash. He reported directly to General Leslie Groves, commander of the Manhattan Engineer District, which was developing atomic weapons for the United States. The chief scientific advisor to Operation Alsos was the physicist Samuel Abraham Goudsmit. Goudsmit was selected for this task because of his knowledge of physics, he spoke German, and he personally knew a number of the German scientists working on the German nuclear energy project. He also knew little of the Manhattan Project, so, if he were captured, he would have little intelligence value to the Germans. The objectives of Operation Alsos were to determine if the Germans had an atomic bomb program and to exploit German atomic related facilities, intellectual materials, materiel resources, and scientific personnel for the benefit of the United States. Personnel on this operation generally swept into areas which had just come under control of the Allied military forces, but sometimes they operated in areas still under control by German forces.^{[103] [104] [105]}

Berlin had been a location of many German scientific research facilities. To limit casualties and loss of equipment, many of these facilities were dispersed to other locations in the latter years of the war. The *Kaiser-Wilhelm-Institut für Physik* (KWIP, Kaiser Wilhelm Institute for Physics) had mostly been moved in 1943 and 1944 to Hechingen and its neighboring town of Haigerloch, on the edge of the Black Forest, which eventually became the French occupation zone. This move and a little luck allowed the Americans to take into custody a large number of German scientists associated with nuclear research. The only section of the institute which remained in Berlin was the low-temperature physics section, headed by Ludwig Bewilogua (1906–83), who was in charge of the exponential uranium pile.^{[106] [107]}

Nine of the prominent German scientists who published reports in *Kernphysikalische Forschungsberichte* as members of the *Uranverein*^[108] were picked up by Operation Alsos and incarcerated in England under Operation Epsilon: Erich Bagge, Kurt Diebner, Walther Gerlach, Otto Hahn, Paul Harteck, Werner Heisenberg, Horst Korsching, Carl Friedrich von Weizsäcker and Karl Wirtz. Also, incarcerated was Max von Laue, although he had nothing to do with the nuclear energy project. Goudsmit, the chief scientific advisor to Operation Alsos, thought von Laue might be beneficial to the postwar rebuilding of Germany and would benefit from the high level contacts he would have in England.^[109]

Heisenberg had been captured and arrested by Colonel Pash at Heisenberg's retreat in Urfeld, on 3 May 1945, in what was a true alpine-type operation in territory still under control by German forces. He was taken to Heidelberg,

where, on 5 May, he met Goudsmit for the first time since the Ann Arbor visit in 1939. Germany surrendered just two days later. Heisenberg did not see his family again for eight months. Heisenberg was moved across France and Belgium and flown to England on 3 July 1945.^{[110] [111] [112]}

The 10 German scientists were held at Farm Hall in England. The facility had been a safe house of the British foreign intelligence MI6. During their detention, their conversations were recorded. Conversations thought to be of intelligence value were transcribed and translated into English. The transcripts were released in 1992. Bernstein has published an annotated version of the transcripts in his book *Hitler's Uranium Club: The Secret Recordings at Farm Hall*, along with an introduction to put them in perspective. A complete, unedited publication of the British version of the reports appeared as *Operation Epsilon: The Farm Hall Transcripts*, which was published in 1993 by the Institute of Physics in Bristol and by the University of California Press in the United States.^{[113] [114] [115]}

Post 1945

On 3 January 1946, the 10 Operation Epsilon detainees were transported to Alswede, Germany, which was in the British occupation zone. Heisenberg settled in Göttingen, also in the British zone. In July, he was named director of the *Kaiser-Wilhelm Institut für Physik* (KWIP, Kaiser Wilhelm Institute for Physics), then located in Göttingen. Shortly thereafter, it was renamed the *Max-Planck Institut für Physik*, in honor of Max Planck and to assuage political objections to the continuation of the institute. Heisenberg was its director until 1958. In 1958, the institute was moved to Munich, expanded, and renamed *Max-Planck-Institut für Physik und Astrophysik* (MPIFA). Heisenberg was its director from 1960 to 1970; in the interim, Heisenberg and the astrophysicist Ludwig Biermann were co-directors. Heisenberg resigned his directorship of the MPIFA on 31 December 1970. Upon the move to Munich, Heisenberg also became an *ordentlicher Professor* (ordinarius professor) at the University of Munich.^{[4] [18]}

Just as the Americans did with Operation Alsos, the Russians inserted special search teams into Germany and Austria in the wake of their troops. Their objective, under the Russian Alsos, was also the exploitation of German atomic related facilities, intellectual materials, materiel resources and scientific personnel for the benefit of the Soviet Union. One of the German scientists recruited under this Russian operation was the nuclear physicist Heinz Pose, who was made head of Laboratory V in Obninsk. When he returned to Germany on a recruiting trip for his laboratory, Pose wrote a letter to the Werner Heisenberg inviting him to work in Russia. The letter lauded the working conditions in Russia and the available resources, as well as the favorable attitude of the Russians towards German scientists. A courier hand delivered the recruitment letter, dated 18 July 1946, to Heisenberg; Heisenberg politely declined in a return letter to Pose.^{[116] [117]}

In 1947, Heisenberg presented lectures in Cambridge, Edinburgh and Bristol. Heisenberg also contributed to the understanding of the phenomenon of superconductivity with a paper in 1947^[118] and two papers in 1948,^{[119] [120]} one of them with Max von Laue.^{[18] [121]}

In the period shortly after World War II, Heisenberg briefly returned to the subject of his doctoral thesis, turbulence. Three papers were published in 1948^{[122] [123] [124]} and one in 1950.^{[9] [125]}

In the post-war period, Heisenberg continued his interests in cosmic-ray showers with considerations on multiple production of mesons. He published three papers^{[126] [127] [128]} in 1949, two^{[129] [130]} in 1952, and one^[131] in 1955.^[132]

On 9 March 1949, the *Deutsche Forschungsrat* (German Research Council) was established by the *Max-Planck Gesellschaft* (MPG, Max Planck Society, successor organization to the *Kaiser-Wilhelm Gesellschaft*). Heisenberg was appointed president of the *Deutsche Forschungsrat*. In 1951, the organization was fused with the *Notgemeinschaft der Deutschen Wissenschaft* (NG, Emergency Association of German Science) and that same year renamed the *Deutsche Forschungsgemeinschaft* (DFG, German Research Foundation). With the merger, Heisenberg was appointed to the presidium.^{[18] [133] [134]}

In 1952, Heisenberg served as the chairman of the Commission for Atomic Physics of the DFG. Also that year, he headed the German delegation to the European Council for Nuclear Research.^{[3] [18]}

In 1953, Heisenberg was appointed president of the *Alexander von Humboldt-Stiftung* by Konrad Adenauer. Heisenberg served until 1975. Also, from 1953, Heisenberg's theoretical work concentrated on the unified field theory of elementary particles.^{[3] [4] [18]}

In late 1955 to early 1956, Heisenberg gave the Gifford Lectures at St Andrews University, in Scotland, on the intellectual history of physics. The lectures were later published as *Physics and Philosophy: The Revolution in Modern Science*.^[135]

During 1956 and 1957, Heisenberg was the chairman of the *Arbeitskreis Kernphysik* (Nuclear Physics Working Group) of the *Fachkommission II "Forschung und Nachwuchs"* (Commission II "Research and Growth") of the *Deutschen Atomkommission* (DAtK, German Atomic Energy Commission). Other members of the Nuclear Physics Working Group in both 1956 and 1957 were: Walther Bothe, Hans Kopfermann (vice-chairman), Fritz Bopp, Wolfgang Gentner, Otto Haxel, Willibald Jentschke, Heinz Maier-Liebnitz, Josef Mattauch, Wolfgang Riezler, Wilhelm Walcher and Carl Friedrich von Weizsäcker. Wolfgang Paul was also a member of the group during 1957.^[136]

In 1957, Heisenberg was a signatory of the manifesto of the *Göttinger Achtzehn* (Göttingen Eighteen).^[137]

From 1957, Heisenberg was interested in plasma physics and the process of nuclear fusion. He also collaborated with the International Institute of Atomic Physics in Geneva. He was a member of the Institute's Scientific Policy Committee, and for several years was the Committee's chairman.^[3]

In 1973, Heisenberg gave a lecture at Harvard University on the historical development of the concepts of quantum theory.^[138]

On 24 March 1973, Heisenberg gave a speech before the Catholic Academy of Bavaria, accepting the Romano Guardini Prize. An English translation of its title is "Scientific and Religious Truth." And its stated goal was "In what follows, then, we shall first of all deal with the unassailability and value of scientific truth, and then with the much wider field of religion, of which - so far as the Christian religion is concerned - Guardini himself has so persuasively written; finally - and this will be the hardest part to formulate - we shall speak of the relationship of the two truths."^[139] A more detail insight in Planck and Heisenberg on religion has been discussed by Wilfried Schröder in "Natural science and religion" (Bremen 1999, Science edition) and Wilfried Schröder "Naturerkenntnis und Religion" Bremen, science edition 2008).

Personal life

In January 1937 Heisenberg met Elisabeth Schumacher at a private music recital. Elisabeth was the daughter of a well-known Berlin economics professor. They were married on 29 April. The fraternal twins, Maria and Wolfgang, were born to them in January 1938, whereupon, Wolfgang Pauli congratulated Heisenberg on his "pair creation" – a word play on a process from elementary particle physics, pair production. They had five more children over the next 12 years: Barbara, Christine, Jochen, Martin and Verena. Jochen became a physics professor at the University of New Hampshire.^{[140] [141]}

Heisenberg enjoyed classical music and was an accomplished pianist.^[3] He was a Lutheran Christian.^[142]

Heisenberg died of cancer of the kidneys and gall bladder at his home, on 1 February 1976.^[143] The next evening, his colleagues and friends walked in remembrance from the Institute of Physics to his home and each put a candle near the front door.^[144]

Honors and awards

Heisenberg was awarded a number of honors:^[3]

- Honorary doctorates from the University of Bruxelles, the Technological University of Karlsruhe, and the University of Budapest.
- Order of Merit of Bavaria
- Romano Guardini Prize^[139]
- Grand Cross for Federal Service with Star
- Knight of the Order of Merit (Peace Class)
- Fellow of the Royal Society of London
- Member of the Academies of Sciences of Göttingen, Bavaria, Saxony, Prussia, Sweden, Rumania, Norway, Spain, The Netherlands, Rome (Pontifical), the *Deutsche Akademie der Naturforscher Leopoldina* (Halle), the Accademia dei Lincei (Rome), and the American Academy of Sciences.
- 1932–Nobel Prize in Physics "for the creation of quantum mechanics, the application of which has, inter alia, led to the discovery of the allotropic forms of hydrogen".^[62]
- 1933–*Max-Planck-Medaille* of the *Deutsche Physikalische Gesellschaft*

Internal reports

The following reports were published in *Kernphysikalische Forschungsberichte* (*Research Reports in Nuclear Physics*), an internal publication of the German *Uranverein*. The reports were classified Top Secret, they had very limited distribution, and the authors were not allowed to keep copies. The reports were confiscated under the Allied Operation Alsos and sent to the United States Atomic Energy Commission for evaluation. In 1971, the reports were declassified and returned to Germany. The reports are available at the Karlsruhe Nuclear Research Center and the American Institute of Physics.^{[145] [146]}

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- Robert Döpel, K. Döpel, and Werner Heisenberg *Bestimmung der Diffusionslänge thermischer Neutronen in schwerem Wasser* G-23 (7 August 1940)
- Werner Heisenberg *Die Möglichkeit der technischer Energiegewinnung aus der Uranspaltung* G-39 (6 December 1939)
- Werner Heisenberg *Bericht über die Möglichkeit technischer Energiegewinnung aus der Uranspaltung (II)* G-40 (29 February 1940)
- Robert Döpel, K. Döpel, and Werner Heisenberg *Versuche mit Schichtenanordnungen von D_2O und 38* G-75 (28 October 1941)
- Werner Heisenberg *Über die Möglichkeit der Energieerzeugung mit Hilfe des Isotops 238* G-92 (1941)
- Werner Heisenberg *Bericht über Versuche mit Schichtenanordnungen von Präparat 38 und Paraffin am Kaiser Wilhelm Institut für Physik in Berlin-Dahlem* G-93 (May 1941)
- Fritz Bopp, Erich Fischer, Werner Heisenberg, Carl-Friedrich von Weizsäcker, and Karl Wirtz *Untersuchungen mit neuen Schichtenanordnungen aus U-metall und Paraffin* G-127 (March 1942)
- Robert Döpel *Bericht über Unfälle beim Umgang mit Uranmetall* G-135 (9 July 1942)
- Werner Heisenberg *Bemerkungen zu dem geplanten halbtechnischen Versuch mit 1,5 to D_2O und 3 to 38-Metall* G-161 (31 July 1942)
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- Werner Heisenberg *Die Energiegewinnung aus der Atomkernspaltung* G-217 (6 May 1943)

- Fritz Bopp, Walther Bothe, Erich Fischer, Erwin Fünfer, Werner Heisenberg, O. Ritter, and Karl Wirtz *Bericht über einen Versuch mit D_2O und U und 40 cm Kohlerückstreumantel (B7)* G-300 (3 January 1945)
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- A. Sommerfeld and W. Heisenberg *Eine Bemerkung über relativistische Röntgendoublets und Linienschärfe*, *Z. Phys.* Volume 10, 393-398 (1922)
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Albert Einstein

Albert Einstein	
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Died	18 April 1955 (aged 76)Princeton, New Jersey, United States
Residence	Germany, Italy, Switzerland, United States
Citizenship	<div><ul style="list-style-type: none">Württemberg/Germany (until 1896)Stateless (1896–1901)Switzerland (from 1901)Austria (1911–12)Germany (1914–33)United States (from 1940)^[1]</div>
<i>Alma mater</i>	<div><ul style="list-style-type: none">ETH ZurichUniversity of Zurich</div>
Known for	<div><ul style="list-style-type: none">General relativity and special relativityPhotoelectric effectMass-energy equivalenceQuantification of the Brownian motionEinstein field equationsBose–Einstein statisticsUnified Field Theory</div>
Spouse	<div><ul style="list-style-type: none">Mileva Marić (1903–1919)Elsa Löwenthal, née Einstein, (1919–1936)</div>
Awards	<div><ul style="list-style-type: none">Nobel Prize in Physics (1921)Copley Medal (1925)Max Planck Medal (1929)<i>Time</i> Person of the Century</div>
Signature	
<div><div><div></div><div></div></div></div>	

Albert Einstein (ⓘ[ⓘ] /ˈælbərt ˈaɪnʃtaɪn/; German: [ˈalbɐt ˈaɪnʃtaɪn] (ⓘ[ⓘ] listen); 14 March 1879 – 18 April 1955) was a German-born theoretical physicist who discovered the theory of general relativity, effecting a revolution in physics. For this achievement, Einstein is often regarded as the father of modern physics.^[2] He received the 1921 Nobel Prize in Physics "for his services to theoretical physics, and especially for his discovery of the law of the photoelectric

effect".^[3]

Near the beginning of his career, Einstein thought that Newtonian mechanics was no longer enough to reconcile the laws of classical mechanics with the laws of the electromagnetic field. This led to the development of his special theory of relativity. He realized, however, that the principle of relativity could also be extended to gravitational fields, and with his subsequent theory of gravitation in 1916, he published a paper on the general theory of relativity. He continued to deal with problems of statistical mechanics and quantum theory, which led to his explanations of particle theory and the motion of molecules. He also investigated the thermal properties of light which laid the foundation of the photon theory of light. In 1917, Einstein applied the general theory of relativity to model the structure of the universe as a whole.^[4]

He was visiting the United States when Hitler came to power in 1933, and did not go back to Germany, where he had been a professor at the Berlin Academy of Sciences. He settled in the U.S., becoming a citizen in 1940. On the eve of World War II, he helped alert President Franklin D. Roosevelt that Germany might be developing an atomic weapon, and recommended that the U.S. begin similar research. Later, together with Bertrand Russell, Einstein signed the Russell–Einstein Manifesto, which highlighted the danger of nuclear weapons. Einstein taught physics at the Institute for Advanced Study at Princeton, New Jersey, until his death in 1955.

Einstein published more than 300 scientific papers along with over 150 non-scientific works.^{[4] [5]} His great intelligence and originality have made the word "Einstein" synonymous with genius.^[6]

Biography

Early life and education

Albert Einstein was born in Ulm, in the Kingdom of Württemberg in the German Empire on 14 March 1879.^[7] His father was Hermann Einstein, a salesman and engineer. His mother was Pauline Einstein (née Koch). In 1880, the family moved to Munich, where his father and his uncle founded *Elektrotechnische Fabrik J. Einstein & Cie*, a company that manufactured electrical equipment based on direct current.^[7]

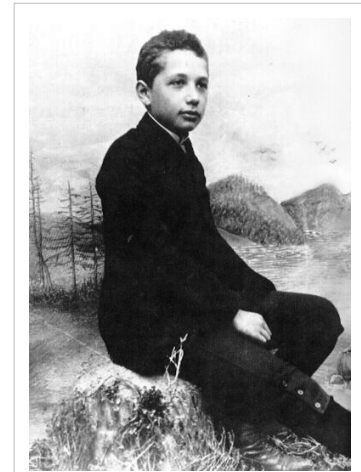
The Einsteins were non-observant Jews. Albert attended a Catholic elementary school from the age of five for three years. Later, at the age of eight, Einstein was transferred to the Luitpold Gymnasium where he received advanced primary and secondary school education till he left Germany seven years later.^[8] Although it has been thought that Einstein had early speech difficulties, this is disputed by the Albert Einstein Archives, and he excelled at the first school that he attended.^[9]

His father once showed him a pocket compass; Einstein realized that there must be something causing the needle to move, despite the apparent "empty space".^[10] As he grew, Einstein built models and mechanical devices for fun and began to show a talent for mathematics.^[7] In 1889, Max Talmud (later changed to Max Talmey) introduced the ten-year old Einstein to key texts in science, mathematics and philosophy, including Immanuel Kant's *Critique of Pure Reason* and *Euclid's Elements* (which Einstein called the "holy little geometry book").^[11] Talmud was a poor Jewish medical student from Poland. The Jewish community arranged for Talmud to take meals with the Einsteins each week on Thursdays for six years. During this time Talmud wholeheartedly guided Einstein through many secular educational interests.^{[12] [13]}



Einstein at the age of 4

In 1894, his father's company failed: direct current (DC) lost the War of Currents to alternating current (AC). In search of business, the Einstein family moved to Italy, first to Milan and then, a few months later, to Pavia. When the family moved to Pavia, Einstein stayed in Munich to finish his studies at the Luitpold Gymnasium. His father intended for him to pursue electrical engineering, but Einstein clashed with authorities and resented the school's regimen and teaching method. He later wrote that the spirit of learning and creative thought were lost in strict rote learning. In the spring of 1895, he withdrew to join his family in Pavia, convincing the school to let him go by using a doctor's note.^[7] During this time, Einstein wrote his first scientific work, "The Investigation of the State of Aether in Magnetic Fields".^[14]



Albert Einstein in 1893 (age 14)

Einstein applied directly to the Eidgenössische Polytechnische Schule (ETH) in Zurich, Switzerland. Lacking the requisite Matura certificate, he took an entrance examination, which he failed, although he got exceptional marks in mathematics and physics.^[15] The Einsteins sent Albert to Aarau, in northern Switzerland to finish secondary school.^[7] While lodging with the family of Professor Jost Winteler, he fell in love with Winteler's daughter, Marie. (His sister Maja later married the Winteler's son, Paul.)^[16] In Aarau, Einstein studied Maxwell's electromagnetic theory. At age 17, he graduated, and, with his father's approval, renounced his citizenship in the German Kingdom of Württemberg to avoid military service, and in 1896 he enrolled in the four year mathematics and physics teaching diploma program at the Polytechnic in Zurich. Marie Winteler moved to Olsberg, Switzerland for a teaching post.

Einstein's future wife, Mileva Marić, also enrolled at the Polytechnic that same year, the only woman among the six students in the mathematics and physics section of the teaching diploma course. Over the next few years, Einstein and Marić's friendship developed into romance, and they read books together on extra-curricular physics in which Einstein was taking an increasing interest. In 1900 Einstein was awarded the Zurich Polytechnic teaching diploma, but Marić failed the examination with a poor grade in the mathematics component, theory of functions.^[17] There have been claims that Marić collaborated with Einstein on his celebrated 1905 papers,^{[18] [19]} but historians of physics who have studied the issue find no evidence that she made any substantive contributions.^{[20] [21] [22] [23]}

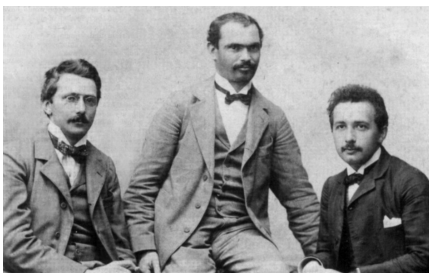
Marriages and children

In early 1902, Einstein and Mileva Marić had a daughter they named Lieserl in their correspondence, who was born in Novi Sad where Marić's parents lived.^[24] Her full name is not known, and her fate is uncertain after 1903.^[25]

Einstein and Marić married in January 1903. In May 1904, the couple's first son, Hans Albert Einstein, was born in Bern, Switzerland. Their second son, Eduard, was born in Zurich in July 1910. In 1914, Einstein moved to Berlin, while his wife remained in Zurich with their sons. Marić and Einstein divorced on 14 February 1919, having lived apart for five years.

Einstein married Elsa Löwenthal (née Einstein) on 2 June 1919, after having had a relationship with her since 1912. She was his first cousin maternally and his second cousin paternally. In 1933, they emigrated permanently to the United States. In 1935, Elsa Einstein was diagnosed with heart and kidney problems and died in December 1936.^[26]

Patent office



Left to right: Conrad Habicht, Maurice Solovine and Einstein, who founded the Olympia Academy



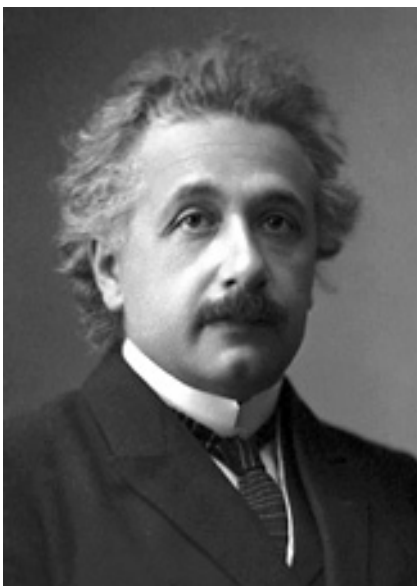
Einstein's home in Bern

After graduating, Einstein spent almost two frustrating years searching for a teaching post, but a former classmate's father helped him secure a job in Bern, at the Federal Office for Intellectual Property, the patent office, as an assistant examiner.^[27] He evaluated patent applications for electromagnetic devices. In 1903, Einstein's position at the Swiss Patent Office became permanent, although he was passed over for promotion until he "fully mastered machine technology".^[28]

Much of his work at the patent office related to questions about transmission of electric signals and electrical-mechanical synchronization of time, two technical problems that show up conspicuously in the thought experiments that eventually led Einstein to his radical conclusions about the nature of light and the fundamental connection between space and time.^[29]

With a few friends he met in Bern, Einstein started a small discussion group, self-mockingly named "The Olympia Academy", which met regularly to discuss science and philosophy. Their readings included the works of Henri Poincaré, Ernst Mach, and David Hume, which influenced his scientific and philosophical outlook.

Academic career



Einstein's official 1921 portrait after receiving the Nobel Prize in Physics.

In 1901, Einstein had a paper on the capillary forces of a straw published in the prestigious *Annalen der Physik*.^[30] On 30 April 1905, he completed his thesis, with Alfred Kleiner, Professor of Experimental Physics, serving as pro-forma advisor. Einstein was awarded a PhD by the University of Zurich. His dissertation was entitled "A New Determination of Molecular Dimensions".^[31] That same year, which has been called Einstein's *annus mirabilis* or "miracle year", he published four groundbreaking papers, on the photoelectric effect, Brownian motion, special relativity, and the equivalence of matter and energy, which were to bring him to the notice of the academic world.

By 1908, he was recognized as a leading scientist, and he was appointed lecturer at the University of Bern. The following year, he quit the patent office and the lectureship to take the position of physics docent^[32] at the University of Zurich. He became a full professor at Karl-Ferdinand University in Prague in 1911. In 1914, he returned to Germany after being appointed director of the Kaiser Wilhelm Institute for Physics (1914–1932)^[33] and a professor at the Humboldt University of Berlin,

although with a special clause in his contract that freed him from most teaching obligations. He became a member of the Prussian Academy of Sciences. In 1916, Einstein was appointed president of the German Physical Society (1916–1918).^[34] ^[35]

In 1911, he had calculated that, based on his new theory of general relativity, light from another star would be bent by the Sun's gravity. That prediction was claimed confirmed by observations made by a British expedition led by Sir Arthur Eddington during the solar eclipse of May 29, 1919. International media reports of this made Einstein world famous. On 7 November 1919, the leading British newspaper *The Times* printed a banner headline that read: "Revolution in Science – New Theory of the Universe – Newtonian Ideas Overthrown".^[36] (Much later, questions were raised whether the measurements were accurate enough to support Einstein's theory.)

In 1921, Einstein was awarded the Nobel Prize in Physics. Because relativity was still considered somewhat controversial, it was officially bestowed for his explanation of the photoelectric effect. He also received the Copley Medal from the Royal Society in 1925.

Travels abroad

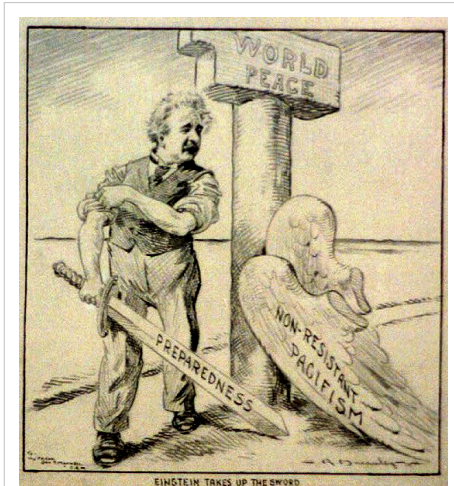
Einstein visited New York City for the first time on 2 April 1921, where he received an official welcome by the Mayor, followed by three weeks of lectures and receptions. He went on to deliver several lectures at Columbia University and Princeton University, and in Washington he accompanied representatives of the National Academy of Science on a visit to the White House. On his return to Europe he was the guest of the British statesman and philosopher Viscount Haldane in London, where he met several renowned scientific, intellectual and political figures, and delivered a lecture at Kings College.^[37]

In 1922, he traveled throughout Asia and later to Palestine, as part of a six-month excursion and speaking tour. His travels included Singapore, Ceylon, and Japan, where he gave a series of lectures to thousands of Japanese. His first lecture in Tokyo lasted four hours, after which he met the emperor and empress at the Imperial Palace where thousands came to watch. Einstein later gave his impressions of the Japanese in a letter to his sons:^[38] ^{:307} "Of all the people I have met, I like the Japanese most, as they are modest, intelligent, considerate, and have a feel for art."^[38] ^{:308}

On his return voyage, he also visited Palestine for 12 days in what would become his only visit to that region. "He was greeted with great British pomp, as if he were a head of state rather than a theoretical physicist", writes Isaacson. This included a cannon salute upon his arrival at the residence of the British high commissioner, Sir Herbert Samuel. During one reception given to him, the building was "stormed by throngs who wanted to hear him". In Einstein's talk to the audience, he expressed his happiness over the event:

I consider this the greatest day of my life. Before, I have always found something to regret in the Jewish soul, and that is the forgetfulness of its own people. Today, I have been made happy by the sight of the Jewish people learning to recognize themselves and to make themselves recognized as a force in the world.^[39] ^{:308}

Emigration from Germany



Cartoon of Einstein, who has shed his "Pacifism" wings, standing next to a pillar labeled "World Peace." He is rolling up his sleeves and holding a sword labeled "Preparedness." (circa 1933)

In 1933, Einstein decided to emigrate to the United States due to the rise to power of the Nazis under Germany's new chancellor, Adolf Hitler.^[40] While visiting American universities in April, 1933, he learned that the new German government had passed a law barring Jews from holding any official positions, including teaching at universities. A month later, the Nazi book burnings occurred, with Einstein's works being among those burnt, and Nazi propaganda minister Joseph Goebbels proclaimed, "Jewish intellectualism is dead."^[39] Einstein also learned that his name was on a list of assassination targets, with a "\$5,000 bounty on his head". One German magazine included him in a list of enemies of the German regime with the phrase, "not yet hanged".^[39]

Einstein was undertaking his third two-month visiting professorship at the California Institute of Technology when Hitler came to power in Germany. On his return to Europe in March 1933 he resided in Belgium for some months, before temporarily moving to England.^[41]

He took up a position at the Institute for Advanced Study at Princeton, New Jersey,^[42] an affiliation that lasted until his death in 1955. There, he tried to develop a unified field theory and to refute the accepted interpretation of quantum physics, both unsuccessfully. He and Kurt Gödel, another Institute member, became close friends. They would take long walks together discussing their work. His last assistant was Bruria Kaufman, who later became a renowned physicist.

Other scientists also fled to America. Among them were Nobel laureates and professors of theoretical physics. With so many other Jewish scientists now forced by circumstances to live in America, often working side by side, Einstein wrote to a friend, "For me the most beautiful thing is to be in contact with a few fine Jews—a few millennia of a civilized past do mean something after all." In another letter he writes, "In my whole life I have never felt so Jewish as now."^[39]

World War II and the Manhattan Project

In 1939, a group of Hungarian scientists that included Hungarian emigre physicist Leo Szilard attempted to alert Washington of ongoing Nazi atomic bomb research. The group's warnings were discounted.^[43]

In the summer of 1939, a few months before the beginning of World War II in Europe, Einstein was persuaded to lend his prestige by writing a letter, with Leo Szilard, to President Franklin D. Roosevelt, in order to alert him of the possibility that Nazi Germany might be developing an atomic bomb. At the same time, the letter recommended that the U.S. government should pay attention to and become directly involved with uranium research, and associated chain reaction research. Einstein and Szilard, along with other refugees such as Edward Teller and Eugene Wigner, "regarded it as their responsibility to alert Americans to the possibility that German scientists might win the race to build an atomic bomb, and to warn that Hitler would be more than willing to resort to such a weapon."^[38] :630 ^[44]

The letter is believed to be "arguably the key stimulus for the U.S. adoption of serious investigations into nuclear weapons on the eve of the U.S. entry into World War II".^[45] President Roosevelt could not take the risk of allowing Hitler to possess atomic bombs first. As a result of Einstein's letter and his meetings with Roosevelt, the U.S. entered the "race" to develop the bomb, drawing on its "immense material, financial, and scientific resources" to initiate the Manhattan Project. It became the only country to develop an atomic bomb during World War II.

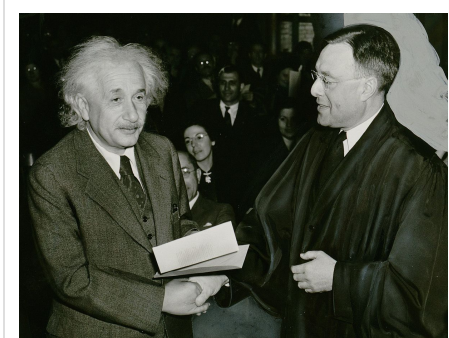
For Einstein, "war was a disease . . . [and] he called for resistance to war." But in 1933, after Hitler assumed full power in Germany, "he renounced pacifism altogether . . . In fact, he urged the Western powers to prepare

themselves against another German onslaught."^[46] :110 In 1954, a year before his death, Einstein said to his old friend, Linus Pauling, "I made one great mistake in my life — when I signed the letter to President Roosevelt recommending that atom bombs be made; but there was some justification — the danger that the Germans would make them..."^[47]

U.S. citizenship

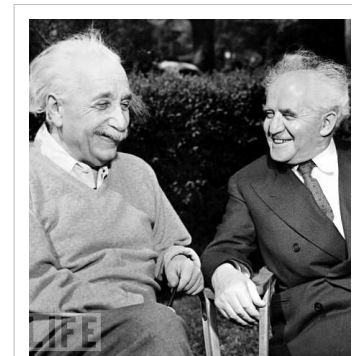
Einstein became an American citizen in 1940. Not long after settling into his career at Princeton, he expressed his appreciation of the "meritocracy" in American culture when compared to Europe. According to Isaacson, he recognized the "right of individuals to say and think what they pleased", without social barriers, and as result, the individual was "encouraged" to be more creative, a trait he valued from his own early education. Einstein writes:

What makes the new arrival devoted to this country is the democratic trait among the people. No one humbles himself before another person or class. . . American youth has the good fortune not to have its outlook troubled by outworn traditions.^[39] :432



Accepting U.S. citizenship, 1940

As a member of the National Association for the Advancement of Colored People NAACP at Princeton who campaigned for the civil rights of African Americans, Einstein corresponded with civil rights activist W. E. B. Du Bois, and in 1946 Einstein called racism America's "worst disease".^[48] He later stated, "Race prejudice has unfortunately become an American tradition which is uncritically handed down from one generation to the next. The only remedies are enlightenment and education".^[49]



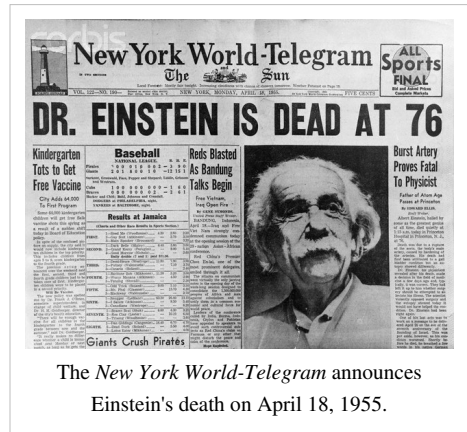
Einstein with David Ben-Gurion,
1951

After the death of Israel's first president, Chaim Weizmann, in November 1952, Prime Minister David Ben-Gurion offered Einstein the position of President of Israel, a mostly ceremonial post.^[50] The offer was presented by Israel's ambassador in Washington, Abba Eban, who explained that the offer "embodies the deepest respect which the Jewish people can repose in any of its sons".^[38] :522 However, Einstein declined, and wrote in his response that he was "deeply moved", and "at once saddened and ashamed" that he could not accept it:

All my life I have dealt with objective matters, hence I lack both the natural aptitude and the experience to deal properly with people and to exercise official function. I am the more distressed over these circumstances because my relationship with the Jewish people became my strongest human tie once I achieved complete clarity about our precarious position among the nations of the world.^[38] :522 [50] [51]

Death

On April 17, 1955, Albert Einstein experienced internal bleeding caused by the rupture of an abdominal aortic aneurysm, which had previously been reinforced surgically by Dr. Rudolph Nissen in 1948.^[52] He took the draft of a speech he was preparing for a television appearance commemorating the State of Israel's seventh anniversary with him to the hospital, but he did not live long enough to complete it.^[53] Einstein refused surgery, saying: "I want to go when I want. It is tasteless to prolong life artificially. I have done my share, it is time to go. I will do it elegantly."^[54] He died in Princeton Hospital early the next morning at the age of 76, having continued to work until near the end.



Einstein's remains were cremated and his ashes were scattered at an undisclosed location.^[55] ^[56] During the autopsy, the pathologist of Princeton Hospital, Thomas Stoltz Harvey, removed Einstein's brain for preservation, without the permission of his family, in hope that the neuroscience of the future would be able to discover what made Einstein so intelligent.^[57] In his lecture at Einstein's memorial, nuclear physicist Robert Oppenheimer summarized his impression of him as a person:^[46]

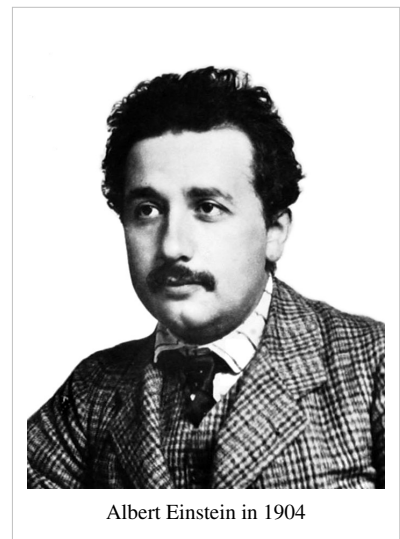
"He was almost wholly without sophistication and wholly without worldliness . . . There was always with him a wonderful purity at once childlike and profoundly stubborn."

Scientific career

Throughout his life, Einstein published hundreds of books and articles.^[5] ^[7] In addition to the work he did by himself he also collaborated with other scientists on additional projects including the Bose–Einstein statistics, the Einstein refrigerator and others.^[58]

Annus Mirabilis papers

The *Annus Mirabilis* papers are four articles pertaining to the photoelectric effect, Brownian motion, the special theory of relativity, and $E = mc^2$ that Albert Einstein published in the *Annalen der Physik* scientific journal in 1905. These four works contributed substantially to the foundation of modern physics and changed views on space, time, and matter.

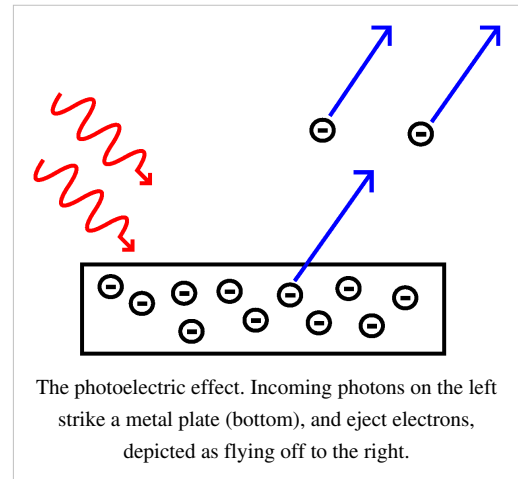


Thermodynamic fluctuations and statistical physics

Albert Einstein's first paper^[59] submitted in 1900 to *Annalen der Physik* was on capillary attraction. It was published in 1901 titled *Folgerungen aus den Capillaritätserscheinungen*, which was translated as "Conclusions from the capillarity phenomena". Two papers he published in 1902-1903 (thermodynamics) attempted to interpret phenomena from a statistical atomic point of view. These

papers were the foundation for the 1905 paper on Brownian motion. These published calculations (1905) showed that Brownian movement can be construed as firm evidence that molecules exist.

His research in 1903 and 1904 was mainly concerned with the effect of finite atomic size on diffusion phenomena.^[59] Einstein's theory of Brownian motion was the first paper in the field of statistical physics. It is the most frequently cited, of the *Annus Mirabilis* papers.



General principles postulated by Einstein

He articulated the principle of relativity. This was understood by Hermann Minkowski to be a generalization of rotational invariance from space to space-time. Other principles postulated by Einstein and later vindicated are the principle of equivalence and the principle of adiabatic invariance of the quantum number. Another of Einstein's general principles, Mach's principle, is fiercely debated, and whether it holds in our world or not is still not definitively established.

Theory of relativity and $E = mc^2$

Einstein's "Zur Elektrodynamik bewegter Körper" ("On the Electrodynamics of Moving Bodies") was received on June 30, 1905 and published September 26 of that same year. It reconciles Maxwell's equations for electricity and magnetism with the laws of mechanics, by introducing major changes to mechanics close to the speed of light. This later became known as Einstein's special theory of relativity.

Consequences of this include the time-space frame of a moving body slowing down and contracting (in the direction of motion) relative to the frame of the observer. This paper also argued that the idea of a luminiferous aether – one of the leading theoretical entities in physics at the time – was superfluous.^[60]

In his paper on *mass–energy equivalence*, which had previously been considered to be distinct concepts, Einstein deduced from his equations of special relativity what has been called the 20th century's best-known equation: $E = mc^2$.^[61] ^[62] This equation suggests that tiny amounts of mass could be converted into huge amounts of energy and presaged the development of nuclear power.^[63] Einstein's 1905 work on relativity remained controversial for many years, but was accepted by leading physicists, starting with Max Planck.^[64] ^[65]

Photons

In a 1905 paper,^[66] Einstein postulated that light itself consists of localized particles (*quanta*). Einstein's light quanta were nearly universally rejected by all physicists, including Max Planck and Niels Bohr. This idea only became universally accepted in 1919, with Robert Millikan's detailed experiments on the photoelectric effect, and with the measurement of Compton scattering.

Einstein concluded that each wave of frequency f is associated with a collection of photons with energy hf each, where h is Planck's constant. He does not say much more, because he is not sure how the particles are related to the wave. But he does suggest that this idea would explain certain experimental results, notably the photoelectric effect.^[67]

Quantized atomic vibrations

In 1907 Einstein proposed a model of matter where each atom in a lattice structure is an independent harmonic oscillator. In the Einstein model, each atom oscillates independently - a series of equally spaced quantized states for each oscillator. Einstein was aware that getting the frequency of the actual oscillations would be different, but he nevertheless proposed this theory because it was a particularly clear demonstration that quantum mechanics could solve the specific heat problem in classical mechanics. Peter Debye refined this model.^[68]

This work was the foundation of condensed matter physics.

Adiabatic principle and action-angle variables

Throughout the 1910s, quantum mechanics expanded in scope to cover many different systems. After Ernest Rutherford discovered the nucleus and proposed that electrons orbit like planets, Niels Bohr was able to show that the same quantum mechanical postulates introduced by Planck and developed by Einstein would explain the discrete motion of electrons in atoms, and the periodic table of the elements.

Einstein contributed to these developments by linking them with the 1898 arguments Wilhelm Wien had made. Wien had shown that the hypothesis of adiabatic invariance of a thermal equilibrium state allows all the blackbody curves at different temperature to be derived from one another by a simple shifting process. Einstein noted in 1911 that the same adiabatic principle shows that the quantity which is quantized in any mechanical motion must be an adiabatic invariant. Arnold Sommerfeld identified this adiabatic invariant as the action variable of classical mechanics. The law that the action variable is quantized was a basic principle of the quantum theory as it was known between 1900 and 1925.

Wave-particle duality

Although the patent office promoted Einstein to Technical Examiner Second Class in 1906, he had not given up on academia. In 1908, he became a *privatdozent* at the University of Bern.^[69] In "über die Entwicklung unserer Anschauungen über das Wesen und die Konstitution der Strahlung" ("The Development of Our Views on the Composition and Essence of Radiation"), on the quantization of light, and in an earlier 1909 paper, Einstein showed that Max Planck's energy quanta must have well-defined momenta and act in some respects as independent, point-like particles. This paper introduced the *photon* concept (although the name *photon* was introduced later by Gilbert N. Lewis in 1926) and inspired the notion of wave-particle duality in quantum mechanics.

Theory of critical opalescence

Einstein returned to the problem of thermodynamic fluctuations, giving a treatment of the density variations in a fluid at its critical point. Ordinarily the density fluctuations are controlled by the second derivative of the free energy with respect to the density. At the critical point, this derivative is zero, leading to large fluctuations. The effect of density fluctuations is that light of all wavelengths is scattered, making the fluid look milky white. Einstein relates this to Raleigh scattering, which is what happens when the fluctuation size is much smaller than the wavelength, and which explains why the sky is blue.^[70] Einstein quantitatively derived critical opalescence from a treatment of density fluctuations, and demonstrated how both the effect and Rayleigh scattering originate from the atomistic constitution of matter.

Zero-point energy

Einstein's physical intuition led him to note that Planck's oscillator energies had an incorrect zero point. He modified Planck's hypothesis by stating that the lowest energy state of an oscillator is equal to $\frac{1}{2}hf$, to half the energy spacing between levels. This argument, which was made in 1913 in collaboration with Otto Stern, was based on the thermodynamics of a diatomic molecule which can split apart into two free atoms.

General relativity and the Equivalence Principle

General relativity (GR) is a theory of gravitation that was developed by Albert Einstein between 1907 and 1915. According to general relativity, the observed gravitational attraction between masses results from the warping of space and time by those masses. General relativity has developed into an essential tool in modern astrophysics. It provides the foundation for the current understanding of black holes, regions of space where gravitational attraction is so strong that not even light can escape.

As Albert Einstein later said, the reason for the development of general relativity was that the preference of inertial motions within special relativity was unsatisfactory, while a theory which from the outset prefers no state of motion (even accelerated ones) should appear more satisfactory.^[71] So in 1908 he published an article on acceleration under special relativity. In that article, he argued that free fall is really inertial motion, and that for a freefalling observer the rules of special relativity must apply. This argument is called the Equivalence principle. In the same article, Einstein also predicted the phenomenon of gravitational time dilation. In 1911, Einstein published another article expanding on the 1907 article, in which additional effects such as the deflection of light by massive bodies were predicted.

Hole argument and Entwurf theory

While developing general relativity, Einstein became confused about the gauge invariance in the theory. He formulated an argument that led him to conclude that a general relativistic field theory is impossible. He gave up looking for fully generally covariant tensor equations, and searched for equations that would be invariant under general linear transformations only.

In June, 1913 the Entwurf ("draft") theory was the result of these investigations. As its name suggests, it was a sketch of a theory, with the equations of motion supplemented by additional gauge fixing conditions. Simultaneously less elegant and more difficult than general relativity, after more than two years of intensive work Einstein abandoned the theory in November, 1915 after realizing that the hole argument was mistaken.^[72]



Einstein at the Solvay Conference in 1911

Cosmology

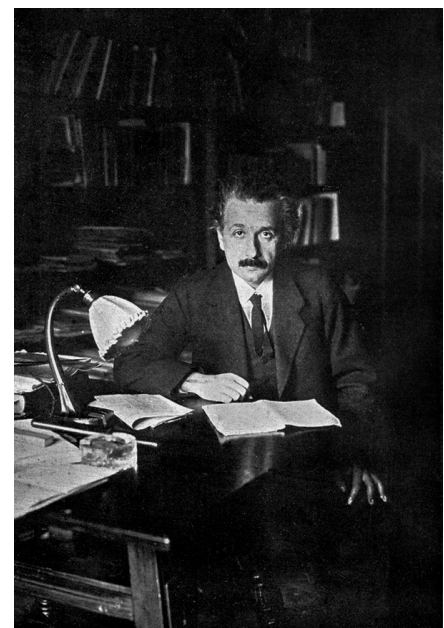
In 1917, Einstein applied the General theory of relativity to model the structure of the universe as a whole. He wanted the universe to be eternal and unchanging, but this type of universe is not consistent with relativity. To fix this, Einstein modified the general theory by introducing a new notion, the cosmological constant. With a positive cosmological constant, the universe could be an eternal static sphere^[73]

Einstein believed a spherical static universe is philosophically preferred, because it would obey Mach's principle. He had shown that general relativity incorporates Mach's principle to a certain extent in frame dragging by gravitomagnetic fields, but he knew that Mach's idea would not work if space goes on forever. In a closed universe, he believed that Mach's principle would hold.

Mach's principle has generated much controversy over the years.

Modern quantum theory

In 1917, at the height of his work on relativity, Einstein published an article in *Physikalische Zeitschrift* that proposed the possibility of stimulated emission, the physical process that makes possible the maser and the laser.^[74] This article showed that the statistics of absorption and emission of light would only be consistent with Planck's distribution law if the emission of light into a mode with n photons would be enhanced statistically compared to the emission of light into an empty mode. This paper was enormously influential in the later development of quantum mechanics, because it was the first paper to show that the statistics of atomic transitions had simple laws. Einstein discovered Louis de Broglie's work, and supported his ideas, which were received skeptically at first. In another major paper from this era, Einstein gave a wave equation for de Broglie waves, which Einstein suggested was the Hamilton–Jacobi equation of mechanics. This paper would inspire Schrödinger's work of 1926.



Einstein in his office at the University of Berlin.

Bose–Einstein statistics

In 1924, Einstein received a description of a statistical model from Indian physicist Satyendra Nath Bose, based on a counting method that assumed that light could be understood as a gas of indistinguishable particles. Einstein noted that Bose's statistics applied to some atoms as well as to the proposed light particles, and submitted his translation of Bose's paper to the *Zeitschrift für Physik*. Einstein also published his own articles describing the model and its implications, among them the Bose–Einstein condensate phenomenon that some particulates should appear at very low temperatures.^[75] It was not until 1995 that the first such condensate was produced experimentally by Eric Allin Cornell and Carl Wieman using ultra-cooling equipment built at the NIST–JILA laboratory at the University of Colorado at Boulder.^[76] Bose–Einstein statistics are now used to describe the behaviors of any assembly of bosons. Einstein's sketches for this project may be seen in the Einstein Archive in the library of the Leiden University.[□]

Energy momentum pseudotensor

General relativity includes a dynamical spacetime, so it is difficult to see how to identify the conserved energy and momentum. Noether's theorem allows these quantities to be determined from a Lagrangian with translation invariance, but general covariance makes translation invariance into something of a gauge symmetry. The energy and momentum derived within general relativity by Noether's prescriptions do not make a real tensor for this reason.

Einstein argued that this is true for fundamental reasons, because the gravitational field could be made to vanish by a choice of coordinates. He maintained that the non-covariant energy momentum pseudotensor was in fact the best description of the energy momentum distribution in a gravitational field. This approach has been echoed by Lev Landau and Evgeny Lifshitz, and others, and has become standard.

The use of non-covariant objects like pseudotensors was heavily criticized in 1917 by Erwin Schrödinger and others.

Unified field theory

Following his research on general relativity, Einstein entered into a series of attempts to generalize his geometric theory of gravitation to include electromagnetism as another aspect of a single entity. In 1950, he described his "unified field theory" in a *Scientific American* article entitled "On the Generalized Theory of Gravitation".^[77] Although he continued to be lauded for his work, Einstein became increasingly isolated in his research, and his efforts were ultimately unsuccessful. In his pursuit of a unification of the fundamental forces, Einstein ignored some mainstream developments in physics, most notably the strong and weak nuclear forces, which were not well understood until many years after his death. Mainstream physics, in turn, largely ignored Einstein's approaches to unification. Einstein's dream of unifying other laws of physics with gravity motivates modern quests for a theory of everything and in particular string theory, where geometrical fields emerge in a unified quantum-mechanical setting.

Wormholes

Einstein collaborated with others to produce a model of a wormhole. His motivation was to model elementary particles with charge as a solution of gravitational field equations, in line with the program outlined in the paper "Do Gravitational Fields play an Important Role in the Constitution of the Elementary Particles?". These solutions cut and pasted Schwarzschild black holes to make a bridge between two patches.

If one end of a wormhole was positively charged, the other end would be negatively charged. These properties led Einstein to believe that pairs of particles and antiparticles could be described in this way.

Einstein–Cartan theory

In order to incorporate spinning point particles into general relativity, the affine connection needed to be generalized to include an antisymmetric part, called the torsion. This modification was made by Einstein and Cartan in the 1920s.

Equations of motion

The theory of general relativity has a fundamental law – the Einstein equations which describe how space curves, the geodesic equation which describes how particles move may be derived from the Einstein equations.

Since the equations of general relativity are non-linear, a lump of energy made out of pure gravitational fields, like a black hole, would move on a trajectory which is determined by the Einstein equations themselves, not by a new law. So Einstein proposed that the path of a singular solution, like a black hole, would be determined to be a geodesic from general relativity itself.

This was established by Einstein, Infeld, and Hoffmann for pointlike objects without angular momentum, and by Roy Kerr for spinning objects.

Einstein's controversial beliefs in physics

In addition to his well-accepted results, some of Einstein's views are regarded as controversial:

- In the special relativity paper (in 1905), Einstein noted that, given a specific definition of the word "force" (a definition which he later agreed was not advantageous), and if we choose to maintain (by convention) the equation mass \times acceleration = force, then one arrives at $m/(1-v^2/c^2)$ as the expression for the transverse mass of a fast moving particle. This differs from the accepted expression today, because, as noted in the footnotes to Einstein's paper added in the 1913 reprint, "it is more to the point to define force in such a way that the laws of energy and momentum assume the simplest form", as was done, for example, by Max Planck in 1906, who gave the now familiar expression $m/\sqrt{1-v^2/c^2}$ for the transverse mass. As Miller points out, this is equivalent to the transverse mass predictions of both Einstein and Lorentz. Einstein had commented already in the 1905 paper that "With a different definition of force and acceleration, we should naturally obtain other expressions for the masses. This shows that in comparing different theories... we must proceed very cautiously."^[78]
- Einstein published (in 1922) a qualitative theory of superconductivity based on the vague idea of electrons shared in orbits. This paper predated modern quantum mechanics, and today is regarded as being incorrect. The current theory of low temperature superconductivity was only worked out in 1957, thirty years after the establishing of modern quantum mechanics. However, even today, superconductivity is not well understood, and alternative theories continue to be put forward, especially to account for high-temperature superconductors.
- After introducing the concept of gravitational waves in 1917, Einstein subsequently entertained doubts about whether they could be physically realized. In 1937 he published a paper saying that the focusing properties of geodesics in general relativity would lead to an instability which causes plane gravitational waves to collapse in on themselves. While this is true to a certain extent in some limits, because gravitational instabilities can lead to a concentration of energy density into black holes, for plane waves of the type Einstein and Rosen considered in their paper, the instabilities are under control. Einstein retracted this position a short time later.
- Einstein denied several times that black holes could form. In 1939 he published a paper that argues that a star collapsing would spin faster and faster, spinning at the speed of light with infinite energy well before the point where it is about to collapse into a black hole. This paper received no citations, and the conclusions are well understood to be wrong. Einstein's argument itself is inconclusive, since he only shows that stable spinning objects have to spin faster and faster to stay stable before the point where they collapse. But it is well understood today (and was understood well by some even then) that collapse cannot happen through stationary states the way Einstein imagined. Nevertheless, the extent to which the models of black holes in classical general relativity correspond to physical reality remains unclear, and in particular the implications of the central singularity implicit in these models are still not understood. Efforts to conclusively prove the existence of event horizons have still not been successful.
- Closely related to his rejection of black holes, Einstein believed that the exclusion of singularities might restrict the class of solutions of the field equations so as to force solutions compatible with quantum mechanics, but no such theory has ever been found.
- In the early days of quantum mechanics, Einstein tried to show that the uncertainty principle was not valid, but by 1927 he had become convinced that it was valid.
- In the EPR paper, Einstein argued that quantum mechanics cannot be a complete realistic and local representation of phenomena, given specific definitions of "realism", "locality", and "completeness". The modern consensus is that Einstein's concept of realism is too restrictive.
- Einstein himself considered the introduction of the cosmological term in his 1917 paper founding cosmology as a "blunder".^[79] The theory of general relativity predicted an expanding or contracting universe, but Einstein wanted a universe which is an unchanging three dimensional sphere, like the surface of a three dimensional ball in four dimensions. He wanted this for philosophical reasons, so as to incorporate Mach's principle in a reasonable way. He stabilized his solution by introducing a cosmological constant, and when the universe was shown to be expanding, he retracted the constant as a blunder. This is not really much of a blunder – the cosmological constant

is necessary within general relativity as it is currently understood, and it is widely believed to have a nonzero value today.

- Einstein did not immediately appreciate the value of Minkowski's four-dimensional formulation of special relativity, although within a few years he had adopted it as the basis for his theory of gravitation.
- Finding it too formal, Einstein believed that Heisenberg's matrix mechanics was incorrect. He changed his mind when Schrödinger and others demonstrated that the formulation in terms of the Schrödinger equation, based on Einstein's wave-particle duality was equivalent to Heisenberg's matrices.

Collaboration with other scientists

In addition to long time collaborators Leopold Infeld, Nathan Rosen, Peter Bergmann and others, Einstein also had some one-shot collaborations with various scientists.

Einstein-de Haas experiment

Einstein and De Haas demonstrated that magnetization is due to the motion of electrons, nowadays known to be the spin. In order to show this, they reversed the magnetization in an iron bar suspended on a torsion pendulum. They confirmed that this leads the bar to rotate, because the electron's angular momentum changes as the magnetization changes. This experiment needed to be sensitive, because the angular momentum associated with electrons is small, but it definitively established that electron motion of some kind is responsible for magnetization.

Schrödinger gas model

Einstein suggested to Erwin Schrödinger that he might be able to reproduce the statistics of a Bose–Einstein gas by considering a box. Then to each possible quantum motion of a particle in a box associate an independent harmonic oscillator. Quantizing these oscillators, each level will have an integer occupation number, which will be the number of particles in it.

This formulation is a form of second quantization, but it predates modern quantum mechanics. Erwin Schrödinger applied this to derive the thermodynamic properties of a semiclassical ideal gas. Schrödinger urged Einstein to add his name as co-author, although Einstein declined the invitation.^[80]

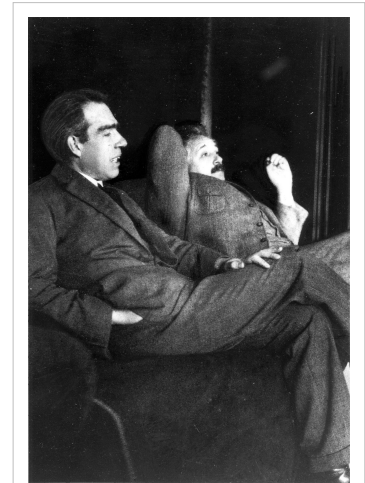
Einstein refrigerator

In 1926, Einstein and his former student Leó Szilárd co-invented (and in 1930, patented) the Einstein refrigerator. This absorption refrigerator was then revolutionary for having no moving parts and using only heat as an input.^[81] On 11 November 1930, U.S. Patent 1781541^[82] was awarded to Albert Einstein and Leó Szilárd for the refrigerator. Their invention was not immediately put into commercial production, as the most promising of their patents were quickly bought up by the Swedish company Electrolux to protect its refrigeration technology from competition.^[83]

Bohr versus Einstein

In the 1920s, quantum mechanics developed into a more complete theory. Einstein was unhappy with the Copenhagen interpretation of quantum theory developed by Niels Bohr and Werner Heisenberg, both in its outcomes and its instrumentalist methodology, Einstein being a scientific realist. In this interpretation, quantum phenomena are inherently probabilistic, with definite states resulting only upon interaction with classical systems. A public debate between Einstein and Bohr followed, lasting on and off for many years (including during the Solvay Conferences). Einstein formulated thought experiments against the Copenhagen interpretation, which were all rebutted by Bohr. In a 1926 letter to Max Born, Einstein wrote: "I, at any rate, am convinced that He [God] does not throw dice."^[84]

Einstein was never satisfied by what he perceived to be quantum theory's intrinsically incomplete description of nature, and in 1935 he further explored the issue in collaboration with Boris Podolsky and Nathan Rosen, noting that the theory seems to require non-local interactions; this is known as the EPR paradox.^[85] The EPR experiment has since been performed, with results confirming quantum theory's predictions.^[86] Repercussions of the Einstein–Bohr debate have found their way into philosophical discourse.



Einstein and Niels Bohr, 1925

Einstein–Podolsky–Rosen paradox

In 1935, Einstein returned to the question of quantum mechanics. He considered how a measurement on one of two entangled particles would affect the other. He noted, along with his collaborators, that by performing different measurements on the distant particle, either of position or momentum, different properties of the entangled partner could be discovered without disturbing it in any way.

He then used a hypothesis of local realism to conclude that the other particle had these properties already determined. The principle he proposed is that if it is possible to determine what the answer to a position or momentum measurement would be, without in any way disturbing the particle, then the particle actually has values of position or momentum.

This principle distilled the essence of Einstein's objection to quantum mechanics. As a physical principle, it has since been shown to be incompatible with experiments.

Political and religious views

Albert Einstein's political views emerged publicly in the middle of the 20th century due to his fame and reputation for genius. Einstein offered to and was called on to give judgments and opinions on matters often unrelated to theoretical physics or mathematics. (see main article)

Einstein's views on religious belief have been collected from interviews and original writings. These views covered theological determinism, agnosticism, humanism along with ethical culture, opting for Spinoza's god over belief in a personal god, religious belief, enlightenment and liberation, Jews, Christianity, Jesus, Pope Pius XII, and the Catholic Church. (see main article)

Non-scientific legacy

While travelling, Einstein wrote daily to his wife Elsa and adopted stepdaughters Margot and Ilse. The letters were included in the papers bequeathed to The Hebrew University. Margot Einstein permitted the personal letters to be made available to the public, but requested that it not be done until twenty years after her death (she died in 1986^[87]). Barbara Wolff, of The Hebrew University's Albert Einstein Archives, told the BBC that there are about 3,500 pages of private correspondence written between 1912 and 1955.^[88]

Einstein bequeathed the royalties from use of his image to The Hebrew University of Jerusalem. Corbis, successor to The Roger Richman Agency, licenses the use of his name and associated imagery, as agent for the university.^[89] ^[90]

In popular culture

In the period before World War II, Einstein was so well-known in America that he would be stopped on the street by people wanting him to explain "that theory". He finally figured out a way to handle the incessant inquiries. He told his inquirers "Pardon me, sorry! Always I am mistaken for Professor Einstein."^[91]

Einstein has been the subject of or inspiration for many novels, films, plays, and works of music.^[92] He is a favorite model for depictions of mad scientists and absent-minded professors; his expressive face and distinctive hairstyle have been widely copied and exaggerated. *Time* magazine's Frederic Golden wrote that Einstein was "a cartoonist's dream come true".^[93]



Albert Einstein, seen here with his wife Elsa Einstein and Zionist leaders, including future President of Israel Chaim Weizmann, his wife Dr. Vera Weizmann, Menahem Ussishkin, and Ben-Zion Mossinson on arrival in New York City in 1921.

Awards and honors

In 1922, Einstein was awarded the 1921 Nobel Prize in Physics,^[94] "for his services to Theoretical Physics, and especially for his discovery of the law of the photoelectric effect". This refers to his 1905 paper on the photoelectric effect, "On a Heuristic Viewpoint Concerning the Production and Transformation of Light", which was well supported by the experimental evidence by that time. The presentation speech began by mentioning "his theory of relativity [which had] been the subject of lively debate in philosophical circles [and] also has astrophysical implications which are being rigorously examined at the present time". (Einstein 1923)

It was long reported that, in accord with the divorce settlement,^[95] the Nobel Prize money had been deposited in a Swiss bank account for Maric to draw on the interest for herself and their two sons, while she could only use the capital by agreement with Einstein. However, personal correspondence made public in 2006^[96] shows that he invested much of it in the United States, and saw much of it wiped out in the Great Depression. However, ultimately he paid Maric more money than he received with the prize.^[97]

In 1929, Max Planck presented Einstein with the Max Planck medal of the German Physical Society in Berlin, for extraordinary achievements in theoretical physics.^[98]

In 1936, Einstein was awarded the Franklin Institute's Franklin Medal for his extensive work on relativity and the photo-electric effect.^[98]

The International Union of Pure and Applied Physics named 2005 the "World Year of Physics" in commemoration of the 100th anniversary of the publication of the annus mirabilis papers.^[99]

The *Albert Einstein Science Park* is located on the hill Telegrafenberg in Potsdam, Germany. The best known building in the park is the Einstein Tower which has a bronze bust of Einstein at the entrance. The Tower is an astrophysical observatory that was built to perform checks of Einstein's theory of General Relativity.^[100]

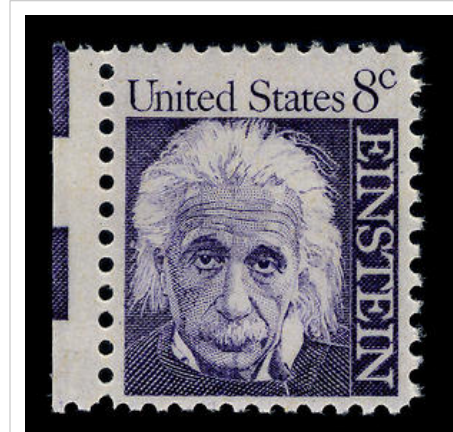
The *Albert Einstein Memorial* in central Washington, D.C. is a monumental bronze statue depicting Einstein seated with manuscript papers in hand. The statue, commissioned in 1979, is located in a grove of trees at the southwest corner of the grounds of the National Academy of Sciences on Constitution Avenue.

The chemical element 99, einsteinium, was named for him in August 1955, four months after Einstein's death.^[101] ^[102] 2001 Einstein is an inner main belt asteroid discovered on 5 March 1973.^[103]

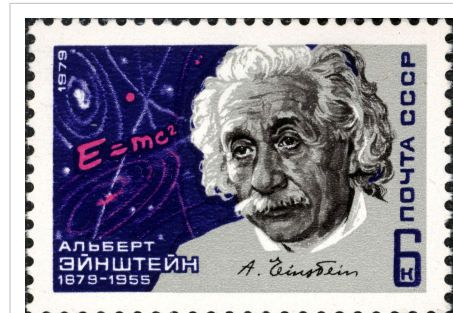
In 1999 *Time* magazine named him the Person of the Century,^[93] ^[104] ahead of Mahatma Gandhi and Franklin Roosevelt, among others. In the words of a biographer, "to the scientifically literate and the public at large, Einstein is synonymous with genius".^[105] Also in 1999, an opinion poll of 100 leading physicists ranked Einstein the "greatest physicist ever".^[106] A Gallup poll recorded him as the fourth most



Israeli postage stamp (1956).



U.S. postage stamp (1966).



Soviet postage stamp (1979).

admired person of the 20th century in the U.S.^[107]

In 1990, his name was added to the Walhalla temple for "laudable and distinguished Germans",^[108] which is located east of Regensburg, in Bavaria, Germany.^[109]

The United States Postal Service honored Einstein with a Prominent Americans series (1965–1978) 8¢ postage stamp.

Awards named after Einstein

The Albert Einstein Award (sometimes called the Albert Einstein Medal because it is accompanied with a gold medal) is an award in theoretical physics, established to recognize high achievement in the natural sciences. It was endowed by the Lewis and Rosa Strauss Memorial Fund in honor of Albert Einstein's 70th birthday. It was first awarded in 1951 and included a prize money of \$15,000,^[110] ^[111] which was later reduced to \$5,000.^[112] ^[113] The winner is selected by a committee (the first of which consisted of Einstein, Oppenheimer, von Neumann and Weyl^[114]) of the Institute for Advanced Study, which administers the award.^[111]

The Albert Einstein Medal is an award presented by the Albert Einstein Society in Bern, Switzerland. First given in 1979, the award is presented to people who have "rendered outstanding services" in connection with Einstein.^[115]

The Albert Einstein Peace Prize is given yearly by the Chicago, Illinois-based Albert Einstein Peace Prize Foundation. Winners of the prize receive \$50,000.^[116]

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- person in examining everything that engaged [Albert's] interest'. Talmud had Albert read and discuss many books with him. These included a series of twenty popular science books that convinced Albert 'a lot in the Bible stories could not be true', and a textbook of plane geometry that launched Albert on avid self-study of mathematics, years ahead of the school curriculum. Talmud even had Albert read Kant; as a result Einstein began preaching to his schoolmates about Kant, with 'forcefulness'. "Einstein as a Student (http://www.chem.harvard.edu/herschbach/Einstein_Student.pdf). (Herschbach, Dudley). from "Einstein for the 21st Century", Princeton University Press, 2008. Retrieved 18 April 2011.
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External links

- Works by Albert Einstein (public domain in Canada)
- *The MacTutor History of Mathematics archive* (<http://www-history.mcs.st-andrews.ac.uk/Biographies/Einstein.html>), School of Mathematics and Statistics, University of St Andrews, Scotland, April 1997, retrieved 14 June 2009
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 - MIT OpenCourseWare STS.042J / 8.225J : Einstein, Oppenheimer, Feynman: Physics in the 20th Century (<http://ocw.mit.edu/courses/science-technology-and-society/sts-042j-einstein-oppenheimer-feynman-physics-in-the-20th-century-spring-2006/>) Free, independent study course that explores the changing roles of physics and physicists during the 20th century.
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Emmy Noether

Emmy Noether	
 <p>Amalie Emmy Noether</p>	
Born	23 March 1882Erlangen, Bavaria, Germany
Died	14 April 1935 (aged 53)Bryn Mawr, Pennsylvania, USA
Citizenship	Germany
Fields	Mathematics and Physics
Institutions	University of Göttingen Bryn Mawr College
Alma mater	University of Erlangen
Doctoral advisor	Paul Gordan
Doctoral students	Max Deuring Hans Fitting Grete Hermann Zeng Jiongzhi Jacob Levitzki Hans Reichenbach Ernst Witt
Known for	Abstract algebra Theoretical physics

Amalie Emmy Noether, German pronunciation: [ˈnøːtɐ], (23 March 1882 – 14 April 1935) was an influential German mathematician known for her groundbreaking contributions to abstract algebra and theoretical physics. Described by David Hilbert, Albert Einstein and others as the most important woman in the history of mathematics,^[1] ^[2] she revolutionized the theories of rings, fields, and algebras. In physics, Noether's theorem explains the fundamental connection between symmetry and conservation laws.^[3]

She was born to a Jewish family in the Bavarian town of Erlangen; her father was the mathematician Max Noether. Emmy originally planned to teach French and English after passing the required examinations, but instead studied mathematics at the University of Erlangen, where her father lectured. After completing her dissertation in 1907 under the supervision of Paul Gordan, she worked at the Mathematical Institute of Erlangen without pay for seven years. In 1915 she was invited by David Hilbert and Felix Klein to join the mathematics department at the University of Göttingen, a world-renowned center of mathematical research. The philosophical faculty objected, however, and she spent four years lecturing under Hilbert's name. Her *habilitation* was approved in 1919, allowing her to obtain the rank of *privatdozent*.

Noether remained a leading member of the Göttingen mathematics department until 1933; her students were sometimes called the "Noether boys". In 1924, Dutch mathematician B. L. van der Waerden joined her circle and

soon became the leading expositor of Noether's ideas: her work was the foundation for the second volume of his influential 1931 textbook, *Moderne Algebra*. By the time of her plenary address at the 1932 International Congress of Mathematicians in Zürich, her algebraic acumen was recognized around the world. The following year, Germany's Nazi government dismissed Jews from university positions, and Noether moved to the United States to take up a position at Bryn Mawr College in Pennsylvania. In 1935 she underwent surgery for an ovarian cyst and, despite signs of a recovery, died four days later at the age of 53.

Noether's mathematical work has been divided into three "epochs".^[4] In the first (1908–1919), she made significant contributions to the theories of algebraic invariants and number fields. Her work on differential invariants in the calculus of variations, *Noether's theorem*, has been called "one of the most important mathematical theorems ever proved in guiding the development of modern physics".^[5] In the second epoch, (1920–1926), she began work that "changed the face of [abstract] algebra".^[6] In her classic paper *Idealtheorie in Ringbereichen* (*Theory of Ideals in Ring Domains*, 1921) Noether developed the theory of ideals in commutative rings into a powerful tool with wide-ranging applications. She made elegant use of the ascending chain condition, and objects satisfying it are named *Noetherian* in her honor. In the third epoch, (1927–1935), she published major works on noncommutative algebras and hypercomplex numbers and united the representation theory of groups with the theory of modules and ideals. In addition to her own publications, Noether was generous with her ideas and is credited with several lines of research published by other mathematicians, even in fields far removed from her main work, such as algebraic topology.

Biography

Emmy's father, Max Noether, was descended from a family of wholesale traders in Germany. He had been paralyzed by poliomyelitis at the age of fourteen. He regained mobility, but one leg remained affected. Largely self-taught, he was awarded a doctorate from the University of Heidelberg in 1868. After teaching there for seven years, he took a position in the Bavarian city of Erlangen, where he met and married Ida Amalia Kaufmann, the daughter of a prosperous merchant.^[7] Max Noether's mathematical contributions were to algebraic geometry mainly, following in the footsteps of Alfred Clebsch. His best known results are the *Brill–Noether theorem* and the residue, or *AF+BG theorem*; several other theorems are associated with him, including *Max Noether's theorem*.



Noether grew up in the Bavarian city of Erlangen, depicted here in a 1916 postcard

Emmy Noether was born on 23 March 1882, the first of four children. Her first name was "Amalie", after her mother and paternal grandmother, but she began using her middle name at a young age. As a girl, she was well-liked. She did not stand out academically although she was known for being clever and friendly. Emmy was near-sighted and talked with a minor lisp during childhood. A family friend recounted a story years later about young Emmy quickly solving a brain teaser at a children's party, showing logical acumen at that early age.^[8] Emmy was taught to cook and clean—as were most girls of the time—and she took piano lessons. She pursued none of these activities with passion, although she loved to dance.^[9]

Of her three brothers, only Fritz Noether, born in 1884, is remembered for his academic accomplishments. After studying in Munich he made a reputation for himself in applied mathematics. Her eldest brother, Alfred, was born in 1883, was awarded a doctorate in chemistry from Erlangen in 1909, but died nine years later. The youngest, Gustav Robert, was born in 1889. Very little is known about his life; he suffered from chronic illness and died in 1928.^[10]

University of Erlangen

Emmy Noether showed early proficiency in French and English. In the spring of 1900 she took the examination for teachers of these languages and received an overall score of *sehr gut* (very good). Her performance qualified her to teach languages at schools reserved for girls, but she chose instead to continue her studies at the University of Erlangen.

This was an unconventional decision; two years earlier, the Academic Senate of the university had declared that allowing coeducation would "overthrow all academic order".^[11] One of only two women students in a university of 986, Noether was forced to audit classes and required the permission of individual professors whose lectures she wished to attend. Despite the obstacles, on 14 July 1903 she passed the graduation exam at a *Realgymnasium* in Nuremberg.^[12]



Paul Gordan supervised Noether's doctoral dissertation on invariants of biquadratic forms

During the 1903–04 winter semester, she studied at the University of Göttingen, attending lectures given by astronomer Karl Schwarzschild and mathematicians Hermann Minkowski, Otto Blumenthal, Felix Klein, and David Hilbert. Soon thereafter, restrictions on women's rights in that university were rescinded.

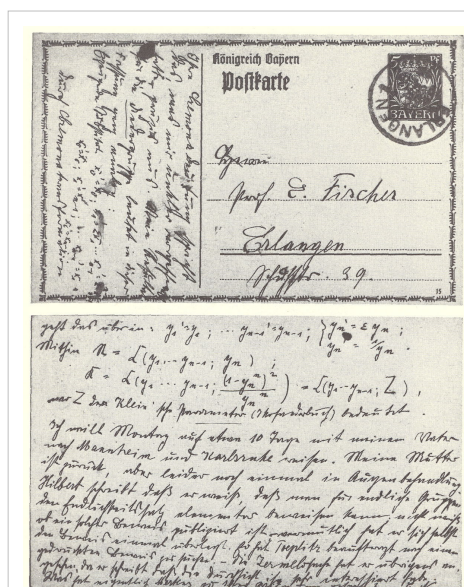
Noether returned to Erlangen. She officially reentered the university on 24 October 1904, and declared her intention to focus solely on mathematics. Under the supervision of Paul Gordan she wrote her dissertation, *Über die Bildung des Formensystems der ternären biquadratischen Form* (*On Complete Systems of Invariants for Ternary Biquadratic Forms*, 1907). Although it had been well received, Noether later described her thesis as "crap".^[13]

For the next seven years (1908–1915) she taught at the University of Erlangen's Mathematical Institute without pay, occasionally substituting for her father when he was too ill to lecture. In 1910 and 1911 she published an extension of her thesis work from three variables to n variables.

Gordan retired in the spring of 1910, but continued to teach occasionally with his successor, Erhard Schmidt, who left shortly afterward for a position in Breslau. Gordan retired from teaching altogether in 1911 with the arrival of his second successor, Ernst Fischer. Gordan died in December 1912.

According to Hermann Weyl, Fischer was an important influence on Noether, in particular by introducing her to the work of David Hilbert. From 1913 to 1916 Noether published several papers extending and applying Hilbert's methods to mathematical objects such as fields of rational functions and the invariants of finite groups. This phase marks the beginning of her engagement with abstract algebra, the field of mathematics to which she would make groundbreaking contributions.

Noether and Fischer shared lively enjoyment of mathematics and would often discuss lectures long after they were over; Noether is known to have sent postcards to Fischer continuing her train of mathematical thoughts.^[14]



Noether sometimes used postcards to discuss abstract algebra with her colleague, Ernst Fischer; this card is postmarked 10 April 1915

University of Göttingen

In the spring of 1915, Noether was invited to return to the University of Göttingen by David Hilbert and Felix Klein. Their effort to recruit her, however, was blocked by the philologists and historians among the philosophical faculty: women, they insisted, should not become *privatdozent*. One faculty member protested: "What will our soldiers think when they return to the university and find that they are required to learn at the feet of a woman?"^[15] Hilbert responded with indignation, stating, "I do not see that the sex of the candidate is an argument against her admission as *privatdozent*. After all, we are a university, not a bath house."^[15]



In 1915 David Hilbert invited Emmy Noether to join the mathematics department at the University of Göttingen, challenging the views of some of his colleagues that a woman should not be allowed to teach at a university

Noether left for Göttingen in late April; two weeks later her mother died suddenly in Erlangen. She had previously received medical care for an eye condition, but its nature and impact on her death is unknown. At about the same time Noether's father retired and her brother joined the German Army to serve in World War I. She returned to Erlangen for several weeks, mostly to care for her aging father.^[16]

During her first years teaching at Göttingen she did not have an official position and was not paid; her family paid for her room and board and supported her academic work. Her lectures often were advertised under Hilbert's name, and Noether would provide "assistance".

Soon after arriving at Göttingen, however, she demonstrated her capabilities by proving the theorem now known as **Noether's theorem**, which shows that a conservation law is associated with any differentiable symmetry of a physical system.^[17] American physicists Leon M. Lederman and Christopher T. Hill argue in their book *Symmetry and the Beautiful Universe* that Noether's theorem is "certainly one of the most important mathematical theorems ever proved in guiding the development of modern physics, possibly on a par with the Pythagorean theorem".^[5]

When World War I ended, the German Revolution of 1918–19 brought a significant change in social attitudes, including more rights for women. In 1919 the University of Göttingen allowed Noether to proceed with her *habilitation* (eligibility for tenure). Her oral examination was held in late May, and she successfully delivered her *habilitation* lecture in June.

Three years later she received a letter from the Prussian Minister for Science, Art, and Public Education, in which he conferred on her the title of *nicht beamteter ausserordentlicher Professor* (an untenured professor with limited internal administrative rights and functions^[18]). This was an unpaid "extraordinary" professorship, not the higher "ordinary" professorship, which was a civil-service position. Although it recognized the importance of her work, the position still provided no salary. Noether was not paid for her lectures until she was appointed to the special position of *Lehrauftrage für Algebra* a year later.^[19]



The mathematics department at the University of Göttingen allowed Noether's *habilitation* in 1919, four years after she had begun lecturing at the school

Seminal work in abstract algebra

Although Noether's theorem had a profound effect upon physics, among mathematicians she is best remembered for her seminal contributions to abstract algebra. As Nathan Jacobson says in his Introduction to Noether's *Collected Papers*,

The development of abstract algebra, which is one of the most distinctive innovations of twentieth century mathematics, is largely due to her – in published papers, in lectures, and in personal influence on her contemporaries.

Noether's groundbreaking work in algebra began in 1920. In collaboration with W. Schmeidler, she then published a paper about the theory of ideals in which they defined left and right ideals in a ring. The following year she published a landmark paper called, *Idealtheorie in Ringbereichen*, analyzing ascending chain conditions with regard to ideals. A noted algebraist, Irving Kaplansky, has called this work "revolutionary",^[20] and the publication gave rise to the term "Noetherian ring" and several other mathematical objects being dubbed, *Noetherian*.^[21]

In 1924, a young Dutch mathematician, B. L. van der Waerden, arrived at the University of Göttingen. He immediately began working with Noether, who provided invaluable methods of abstract conceptualization. van der Waerden later said that her originality was "absolute beyond comparison".^[22] In 1931 he published *Moderne Algebra*, a central text in the field; its second volume borrowed heavily from Noether's work. Although Emmy Noether did not seek recognition, he included as a note in the seventh edition "based in part on lectures by E. Artin and E. Noether".^[23] She sometimes allowed her colleagues and students to receive credit for her ideas, helping them develop their careers at the expense of her own.^[24]

van der Waerden's visit was part of a convergence of mathematicians from all over the world to Göttingen, which became a major hub of mathematical and physical research. From 1926 to 1930 the Russian topologist, Pavel Alexandrov, lectured at the university, and he and Noether quickly became good friends. He began referring to her as *der Noether*, using the masculine German article as a term of endearment to show his respect. She tried to arrange for him to obtain a position at Göttingen as a regular professor, but was only able to help him secure a scholarship from the Rockefeller Foundation.^[25] They met regularly and enjoyed discussions about the intersections of algebra and topology. In his 1935 memorial address, Alexandrov named Emmy Noether "the greatest woman mathematician of all time".^[26]

Lecturing and students

In Göttingen, Noether supervised more than a dozen doctoral students; her first was Grete Hermann, who defended her dissertation in February 1925. She later spoke reverently of her "dissertation-mother".^[27] Noether also supervised Max Deuring, who distinguished himself as an undergraduate and went on to contribute significantly to the field of arithmetic geometry; Hans Fitting, remembered for Fitting's theorem and the Fitting lemma; and Zeng Jiongzhi, who proved Tsen's theorem. She also worked closely with Wolfgang Krull, who greatly advanced commutative algebra with his *Hauptidealsatz* and his dimension theory for commutative rings.^[28]

In addition to her mathematical insight, Noether was respected for her consideration of others. Although she sometimes acted rudely toward those who disagreed with her, she nevertheless gained a reputation for constant helpfulness and patient guidance of new students. Her loyalty to mathematical precision caused one colleague to name her "a severe critic", but she combined this demand for accuracy with a nurturing attitude.^[29] A colleague later described her this way: "Completely unegotistical and free of vanity, she never claimed anything for herself, but promoted the works of her students above all."^[30]

Her frugal lifestyle at first was due to being denied pay for her work; however, even after the university began paying her a small salary in 1923, she continued to live a simple and modest life. She was paid more generously later in her life, but saved half of her salary to bequeath to her nephew, Gottfried E. Noether.^[31]

Mostly unconcerned about appearance and manners, she focused on her studies to the exclusion of romance and fashion. A distinguished algebraist Olga Taussky-Todd described a luncheon, during which Noether, wholly engrossed in a discussion of mathematics, "gesticulated wildly" as she ate and "spilled her food constantly and wiped it off from her dress, completely unperturbed".^[32] Appearance-conscious students cringed as she retrieved the handkerchief from her blouse and ignored the increasing disarray of her hair during a lecture. Two female students once approached her during a break in a two-hour class to express their concern, but were unable to break through the energetic mathematics discussion she was having with other students.^[33]

According to van der Waerden's obituary of Emmy Noether, she did not follow a lesson plan for her lectures, which frustrated some students. Instead, she used her lectures as a spontaneous discussion time with her students, to think through and clarify important cutting-edge problems in mathematics. Some of her most important results were developed in these lectures, and the lecture notes of her students formed the basis for several important textbooks, such as those of van der Waerden and Deuring.

Several of her colleagues attended her lectures, and she allowed some of her ideas, such as the crossed product (*verschränktes Produkt* in German) of associative algebras, to be published by others. Noether was recorded as having given at least five semester-long courses at Göttingen:^[34]

- Winter 1924/25: *Gruppentheorie und hyperkomplexe Zahlen* (Group Theory and Hypercomplex Numbers)
- Winter 1927/28: *Hyperkomplexe Grössen und Darstellungstheorie* (Hypercomplex Quantities and Representation Theory)
- Summer 1928: *Nichtkommutative Algebra* (Noncommutative Algebra)
- Summer 1929: *Nichtkommutative Arithmetik* (Noncommutative Arithmetic)
- Winter 1929/30: *Algebra der hyperkomplexen Grössen* (Algebra of Hypercomplex Quantities)

These courses often preceded major publications in these areas.

Noether spoke quickly—reflecting the speed of her thoughts, many said—and demanded great concentration from her students. Students who disliked her style often felt alienated; one wrote in a notebook with regard to a class that ended at 1:00 pm: "It's 12:50, thank God!"^[35] Some pupils felt that she relied too much on spontaneous discussions. Her most dedicated students, however, relished the enthusiasm with which she approached mathematics, especially since her lectures often built on earlier work they had done together.

She developed a close circle of colleagues and students who thought along similar lines and tended to exclude those who did not. "Outsiders" who occasionally visited Noether's lectures usually spent only 30 minutes in the room before leaving in frustration or confusion. A regular student said of one such instance: "The enemy has been defeated; he has cleared out."^[36]

Noether showed a devotion to her subject and her students that extended beyond the academic day. Once, when the building was closed for a state holiday, she gathered the class on the steps outside, led them through the woods, and lectured at a local coffee house.^[37] Later, after she had been dismissed by the Third Reich, she invited students into her home to discuss their future plans and mathematical concepts.^[38]

Moscow

In the winter of 1928–29 Noether accepted an invitation to Moscow State University, where she continued working with P. S. Alexandrov. In addition to carrying on with her research, she taught classes in abstract algebra and algebraic geometry. She worked with the topologists, Lev Pontryagin and Nikolai Chebotaryov, who later praised her contributions to the development of *Galois theory*.^[39]

Although politics was not central to her life, Noether took a keen interest in political matters and, according to Alexandrov, showed considerable support for the Russian Revolution (1917). She was especially happy to see Soviet advancements in the fields of science and mathematics, which she considered indicative of new opportunities made possible by the Bolshevik project. This attitude caused her problems in Germany, culminating in her eviction from a pension lodging building, after student leaders complained of living with "a Marxist-leaning Jewess".^[40]

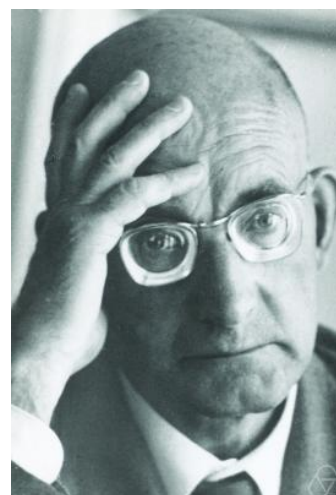
Noether planned to return to Moscow, an effort for which she received support from Alexandrov. After she left Germany in 1933 he tried to help her gain a chair at Moscow State University through the Soviet Education Ministry. Although this effort proved unsuccessful, they corresponded frequently during the 1930s, and in 1935 she made plans for a return to the Soviet Union.^[40] Meanwhile her brother, Fritz accepted a position at the Research Institute for Mathematics and Mechanics in Tomsk, in the Siberian Federal District of Russia, after losing his job in Germany.^[41]



Noether taught at the Moscow State University during the winter of 1928–29

Recognition

In 1932 Emmy Noether and Emil Artin received the Ackermann–Teubner Memorial Award for their contributions to mathematics.^[42] The prize carried a monetary reward of 500 Reichsmarks and was seen as a long-overdue official recognition of her considerable work in the field. Nevertheless, her colleagues expressed frustration at the fact that she was not elected to the Göttingen *Gesellschaft der Wissenschaften* (academy of sciences) and was never promoted to the position of *Ordentlicher Professor*^[43] (full professor).^[18]



Pavel Alexandrov.



Noether visited Zürich in 1932 to deliver a plenary address at the International Congress of Mathematicians

Noether's colleagues celebrated her fiftieth birthday in 1932, in typical mathematicians' style. Helmut Hasse dedicated an article to her in the *Mathematische Annalen*, wherein he confirmed her suspicion that some aspects of noncommutative algebra are simpler than those of commutative algebra, by proving a noncommutative reciprocity law.^[44] This pleased her immensely. He also sent her a mathematical riddle, the "μνϑ-riddle of syllables", which she solved immediately; the riddle has been lost.^[43]

In November of the same year, Noether delivered a plenary address (*großer Vortrag*) on "Hyper-complex systems in their relations to commutative algebra and to number theory" at the International Congress of Mathematicians in Zürich. The congress was attended by 800 people, including Noether's colleagues Hermann Weyl, Edmund Landau, and Wolfgang Krull. There were 420 official participants and twenty-one plenary

addresses presented. Apparently, Noether's prominent speaking position was a recognition of the importance of her contributions to mathematics. The 1932 congress is sometimes described as the high point of her career.^[45]

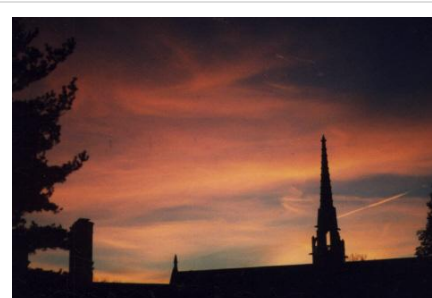
Expulsion from Göttingen

When Adolf Hitler became the German *Reichskanzler* in January 1933, Nazi activity around the country increased dramatically. At the University of Göttingen the German Student Association led the attack on the "un-German spirit" and was aided by a privatdozent named Werner Weber, a former student of Emmy Noether. Antisemitic attitudes created a climate hostile to Jewish professors. One young protester reportedly demanded: "Aryan students want Aryan mathematics and not Jewish mathematics."^[46]

One of the first actions of Hitler's administration was the Law for the Restoration of the Professional Civil Service which removed Jews and politically suspect government employees (including university professors) from their jobs unless they had "demonstrated their loyalty to Germany" by serving in World War I. In April 1933 Noether received a notice from the Prussian Ministry for Sciences, Art, and Public Education which read: "On the basis of paragraph 3 of the Civil Service Code of 7 April 1933, I hereby withdraw from you the right to teach at the University of Göttingen."^[47] Several of Noether's colleagues, including Max Born and Richard Courant, had their positions revoked.^[47] Noether accepted the decision calmly, providing support for others during this difficult time. Hermann Weyl later wrote that "Emmy Noether—her courage, her frankness, her unconcern about her own fate, her conciliatory spirit—was in the midst of all the hatred and meanness, despair and sorrow surrounding us, a moral solace."^[46] Typically, Noether remained focused on mathematics, gathering students in her apartment to discuss class field theory. When one of her students appeared in the uniform of the Nazi paramilitary organization *Sturmabteilung* (SA), she showed no sign of agitation and, reportedly, even laughed about it later.^[47]

Bryn Mawr

As dozens of newly unemployed professors began searching for positions outside of Germany, their colleagues in the United States sought to provide assistance and job opportunities for them. Albert Einstein and Hermann Weyl were appointed by the Institute for Advanced Study in Princeton, while others worked to find a sponsor required for legal immigration. Noether was contacted by representatives of two educational institutions, Bryn Mawr College in the United States and Somerville College at the University of Oxford in England. After a series of negotiations with the Rockefeller Foundation, a grant to Bryn Mawr was approved for Noether and she took a position there, starting in late 1933.^[48]



Bryn Mawr College provided a welcoming home for Noether during the last two years of her life

At Bryn Mawr, Noether met and befriended Anna Wheeler, who had studied at Göttingen just before Noether arrived there. Another source of support at the college was the Bryn Mawr president, Marion Edwards Park, who enthusiastically invited mathematicians in the area to "see Dr. Noether in action!"^[49] Noether and a small team of students worked quickly through van der Waerden's 1930 book *Moderne Algebra I* and parts of Erich Hecke's *Theorie der algebraischen Zahlen* (*Theory of algebraic numbers*, 1908).^[50]

In 1934, Noether began lecturing at the Institute for Advanced Study in Princeton upon the invitation of Abraham Flexner and Oswald Veblen. She also worked with and supervised Abraham Albert and Harry Vandiver.^[51] However, she remarked about Princeton University that she was not welcome at the "men's university, where nothing female is admitted".^[52]

Her time in the United States was pleasant, surrounded as she was by supportive colleagues and absorbed in her favorite subjects.^[53] In the summer of 1934 she briefly returned to Germany to see Emil Artin and her brother Fritz

before he left for Tomsk. Although many of her former colleagues had been forced out of the universities, she was able to use the library as a "foreign scholar".^[54]

Death

In April 1935 doctors discovered a tumor in Noether's pelvis. Worried about complications from surgery, they ordered two days of bed rest first. During the operation they discovered an ovarian cyst "the size of a large cantaloupe".^[55] Two smaller tumors in her uterus appeared to be benign and were not removed, to avoid prolonging surgery. For three days she appeared to convalesce normally, and recovered quickly from a circulatory collapse on the fourth. On 14 April, she fell unconscious, her temperature soared to 109 °F (42.8 °C), and she died. "[I]t is not easy to say what had occurred in Dr. Noether", one of the physicians wrote. "It is possible that there was some form of unusual and virulent infection, which struck the base of the brain where the heat centers are supposed to be located."^[55]



Noether's remains were placed under the walkway surrounding the cloisters of Bryn Mawr's M. Carey Thomas Library

A few days after Noether's death her friends and associates at Bryn Mawr held a small memorial service at President Park's house. Hermann Weyl and Richard Brauer traveled from Princeton and spoke with Wheeler and Taussky about their departed colleague. In the months which followed, written tributes began to appear around the globe: Albert Einstein joined van der Waerden, Weyl, and Pavel Alexandrov in paying their respects. Her body was cremated and the ashes interred under the walkway around the cloisters of the M. Carey Thomas Library at Bryn Mawr.^[56]

Contributions to mathematics and physics

First and foremost Noether is remembered by mathematicians as an algebraist and for her work in topology. Physicists appreciate her best for her famous theorem because of its far-ranging consequences for the study of subatomic particles and dynamic systems. She showed an acute propensity for abstract thought, which allowed her to approach problems of mathematics in fresh and original ways.^[57] Her friend and colleague Hermann Weyl described her scholarly output in three epochs:

"Emmy Noether's scientific production fell into three clearly distinct epochs:

- (1) the period of relative dependence, 1907–1919;
- (2) the investigations grouped around the general theory of ideals 1920–1926;
- (3) the study of the non-commutative algebras, their representations by linear transformations, and their application to the study of commutative number fields and their arithmetics." (Weyl 1935)

In the first epoch (1907–19), Noether dealt primarily with differential and algebraic invariants, beginning with her dissertation under Paul Gordan. Her mathematical horizons broadened, and her work became more general and abstract, as she became acquainted with the work of David Hilbert, through close interactions with a successor to Gordan, Ernst Sigismund Fischer. After moving to Göttingen in 1915, she produced her seminal work for physics, the two Noether's theorems.

In the second epoch (1920–26), Noether devoted herself to developing the theory of mathematical rings.^[58]

In the third epoch (1927–35), Noether focused on noncommutative algebra, linear transformations, and commutative number fields.^[59]

Historical context

In the century from 1832 to Noether's death in 1935, the field of mathematics—specifically algebra—underwent a profound revolution, whose reverberations are still being felt. Mathematicians of previous centuries had worked on practical methods for solving specific types of equations, e.g., cubic, quartic, and quintic equations, as well as on the related problem of constructing regular polygons using compass and straightedge. Beginning with Carl Friedrich Gauss' 1829 proof that prime numbers such as five can be factored in Gaussian integers, Évariste Galois's introduction of permutation groups in 1832 (although, because of his death, his papers were only published in 1846 by Liouville), William Rowan Hamilton's discovery of quaternions in 1843, and Arthur Cayley's more modern definition of groups in 1854, research turned to determining the properties of ever-more-abstract systems defined by ever-more-universal rules. Noether's most important contributions to mathematics were to the development of this new field, abstract algebra.^[60]

Abstract algebra and *begriffliche Mathematik* (conceptual mathematics)

Two of the most basic objects in abstract algebra are groups and rings.

A *group* consists of a set of elements and a single operation which combines a first and a second element and returns a third. The operation must satisfy certain constraints for it to determine a group: It must be closed (when applied to any pair of elements of the associated set, the generated element must also be a member of that set), it must be associative, there must be an identity element (an element which, when combined with another element using the operation, results in the original element, such as adding zero to a number or multiplying it by one), and for every element there must be an inverse element.

A *ring* likewise, has a set of elements, but now has *two* operations. The first operation must make the set a group, and the second operation is associative and distributive with respect to the first operation. It may or may not be commutative; this means that the result of applying the operation to a first and a second element is the same as to the second and first—the order of the elements does not matter. If every non-zero element has a multiplicative inverse (an element x such that $ax = xa = 1$), the ring is called a division ring. A field is defined as a commutative division ring.

Groups are frequently studied through *group representations*. In their most general form, these consist of a choice of group, a set, and an *action* of the group on the set, that is, an operation which takes an element of the group and an element of the set and returns an element of the set. Most often, the set is a vector space, and the group represents symmetries of the vector space. For example, there is a group which represents the rigid rotations of space. This is a type of symmetry of space, because space itself does not change when it is rotated even though the positions of objects in it do. Noether used these sorts of symmetries in her work on invariants in physics.

A powerful way of studying rings is through their *modules*. A module consists of a choice of ring, another set, usually distinct from the underlying set of the ring and called the underlying set of the module, an operation on pairs of elements of the underlying set of the module, and an operation which takes an element of the ring and an element of the module and returns an element of the module. The underlying set of the module and its operation must form a group. A module is a ring-theoretic version of a group representation: Ignoring the second ring operation and the operation on pairs of module elements determines a group representation. The real utility of modules is that the kinds of modules that exist and their interactions, reveal the structure of the ring in ways that are not apparent from the ring itself. An important special case of this is an *algebra*. (The word algebra means both a subject within mathematics as well as an object studied in the subject of algebra.) An algebra consists of a choice of two rings and an operation which takes an element from each ring and returns an element of the second ring. This operation makes the second ring into a module over the first. Often the first ring is a field.

Words such as "element" and "combining operation" are very general, and can be applied to many real-world and abstract situations. Any set of things that obeys all the rules for one (or two) operation(s) is, by definition, a group (or ring), and obeys all theorems about groups (or rings). Integer numbers, and the operations of addition and multiplication, are just one example. For example, the elements might be computer data words, where the first combining operation is exclusive or and the second is logical conjunction. Theorems of abstract algebra are powerful because they are general; they govern many systems. It might be imagined that little could be concluded about objects defined with so few properties, but precisely therein lay Noether's gift: **to discover the maximum that could be concluded from a given set of properties, or conversely, to identify the minimum set, the essential properties responsible for a particular observation.** Unlike most mathematicians, she did not make abstractions by generalizing from known examples; rather, she worked directly with the abstractions. As van der Waerden recalled in his obituary of her,^[61]

The maxim by which Emmy Noether was guided throughout her work might be formulated as follows:
 "Any relationships between numbers, functions, and operations become transparent, generally applicable, and fully productive only after they have been isolated from their particular objects and been formulated as universally valid concepts.

This is the *begriffliche Mathematik* (purely conceptual mathematics) that was characteristic of Noether. This style of mathematics was adopted by other mathematicians and, after her death, flowered into new forms, such as category theory.

Integers as an example of a ring

The integers form a commutative ring whose elements are the integers, and the combining operations are addition and multiplication. Any pair of integers can be added or multiplied, always resulting in another integer, and the first operation, addition, is commutative, i.e., for any elements a and b in the ring, $a + b = b + a$. The second operation, multiplication, also is commutative, but that need not be true for other rings, meaning that a combined with b might be different from b combined with a . Examples of noncommutative rings include matrices and quaternions. The integers do not form a division ring, because the second operation cannot always be inverted; there is no integer a such that $3 \times a = 1$.

The integers have additional properties which do not generalize to all commutative rings. An important example is the fundamental theorem of arithmetic, which says that every positive integer can be factored uniquely into prime numbers. Unique factorizations do not always exist in other rings, but Noether found a unique factorization theorem, now called the *Lasker–Noether theorem*, for the ideals of many rings. Much of Noether's work lay in determining what properties *do* hold for all rings, in devising novel analogs of the old integer theorems, and in determining the minimal set of assumptions required to yield certain properties of rings.

First epoch (1908–19)

Algebraic invariant theory

Much of Noether's work in the first epoch of her career was associated with invariant theory, principally algebraic invariant theory. Invariant theory is concerned with expressions that remain constant (invariant) under a group of transformations. As an everyday example, if a rigid yardstick is rotated, the coordinates (x, y, z) of its endpoints change, but its length L given by the formula $L^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$ remains the same. Invariant theory was an active area of research in the later nineteenth century, prompted in part by Felix Klein's Erlangen program, according to which different types of geometry should be characterized by their invariants under transformations, e.g., the cross-ratio of projective geometry. The archetypal example of an invariant is the discriminant $B^2 - 4AC$ of a binary quadratic form $Ax^2 + Bxy + Cy^2$. This is called an invariant because it is unchanged by

linear substitutions $x \rightarrow ax + by$, $y \rightarrow cx + dy$ with determinant $ad - bc = 1$. These substitutions form the special linear group SL_2 . (There are no invariants under the general linear group of all invertible linear transformations because these transformations can be multiplication by a scaling factor. To remedy this, classical invariant theory also considered *relative invariants*, which were forms invariant up to a scale factor.) One can ask for all polynomials in A , B , and C that are unchanged by the action of SL_2 ; these are called the invariants of binary quadratic forms, and turn out to be the polynomials in the discriminant. More generally, one can ask for the invariants of homogeneous polynomials $A_0 x^r y^0 + \dots + A_r x^0 y^r$ of higher degree, which will be certain polynomials in the coefficients A_0, \dots, A_r , and more generally still, one can ask the similar question for homogeneous polynomials in more than two variables.

One of the main goals of invariant theory was to solve the "finite basis problem". The sum or product of any two invariants is invariant, and the finite basis problem asked whether it was possible to get all the invariants by starting with a finite list of invariants, called *generators*, and then, adding or multiplying the generators together. For example, the discriminant gives a finite basis (with one element) for the invariants of binary quadratic forms. Noether's advisor, Paul Gordan, was known as the "king of invariant theory", and his chief contribution to mathematics was his 1870 solution of the finite basis problem for invariants of homogeneous polynomials in two variables.^[63] ^[64] He proved this by giving a constructive method for finding all of the invariants and their generators, but was not able to carry out this constructive approach for invariants in three or more variables. In 1890, David Hilbert proved a similar statement for the invariants of homogeneous polynomials in any number of variables.^[65] ^[66] Furthermore, his method worked, not only for the special linear group, but also for some of its subgroups such as the special orthogonal group.^[67] His first proof caused some controversy because it did not give a method for constructing the generators, although in later work he made his method constructive. For her thesis, Noether extended Gordan's computational proof to homogeneous polynomials in three variables. Noether's constructive approach made it possible to study the relationships among the invariants. Later, after she had turned to more abstract methods, Noether called her thesis *Mist* (crap) and *Formelngestrüpp* (a jungle of equations).

Table 2 from Noether's dissertation ^[62] on invariant theory. This table collects 202 of the 331 invariants of ternary biquadratic forms.

These forms are graded in two variables x and u . The horizontal direction of the table lists the invariants with increasing grades in x , while the vertical direction lists them with increasing grades in u .

Galois theory

Galois theory concerns transformations of number fields that permute the roots of an equation. Consider a polynomial equation of a variable x of degree n , in which the coefficients are drawn from some ground field, which might be, for example, the field of real numbers, rational numbers, or the integers modulo 7. There may or may not be choices of x , which make this polynomial evaluate to zero. Such choices, if they exist, are called roots. If the polynomial is $x^2 + 1$ and the field is the real numbers, then the polynomial has no roots, because any choice of x makes the polynomial greater than or equal to one. If the field is extended, however, then the polynomial may gain roots, and if it is extended enough, then it always has a number of roots equal to its degree. Continuing the previous example, if the field is enlarged to the complex numbers, then the polynomial gains two roots, i and $-i$, where i is the imaginary unit, that is, $i^2 = -1$. More generally, the extension field in which a polynomial can be factored into its roots is known as the splitting field of the polynomial.

The Galois group of a polynomial is the set of all ways of transforming the splitting field, while preserving the ground field and the roots of the polynomial. (In mathematical jargon, these transformations are called automorphisms.) The Galois group of $x^2 + 1$ consists of two elements: The identity transformation, which sends every complex number to itself, and complex conjugation, which sends i to $-i$. Since the Galois group does not change the ground field, it leaves the coefficients of the polynomial unchanged, so it must leave the set of all roots unchanged. Each root can move to another root, however, so transformation determines a permutation of the n roots among themselves. The significance of the Galois group derives from the fundamental theorem of Galois theory, which proves that the fields lying between the ground field and the splitting field are in one-to-one correspondence with the subgroups of the Galois group.

In 1918, Noether published a seminal paper on the inverse Galois problem.^[68] Instead of determining the Galois group of transformations of a given field and its extension, Noether asked whether, given a field and a group, it always is possible to find an extension of the field that has the given group as its Galois group. She reduced this to "Noether's problem", which asks whether the fixed field of a subgroup G of the permutation group S_n acting on the field $k(x_1, \dots, x_n)$ always is a pure transcendental extension of the field k . (She first mentioned this problem in a 1913 paper,^[69] where she attributed the problem to her colleague Fischer.) She showed this was true for $n = 2, 3$, or 4 . In 1969, R. G. Swan found a counter-example to Noether's problem, with $n = 47$ and G a cyclic group of order 47^[70] (although this group can be realized as a Galois group over the rationals in other ways). The inverse Galois problem remains unsolved.^[71]

Physics

Noether was brought to Göttingen in 1915 by David Hilbert and Felix Klein, who wanted her expertise in invariant theory to help them in understanding general relativity, a geometrical theory of gravitation developed mainly by Albert Einstein. Hilbert had observed that the conservation of energy seemed to be violated in general relativity, due to the fact that gravitational energy could itself gravitate. Noether provided the resolution of this paradox, and a fundamental tool of modern theoretical physics, with her **first Noether's theorem**, which she proved in 1915, but did not publish until 1918.^[72] She solved the problem not only for general relativity, but determined the conserved quantities for *every* system of physical laws that possesses some continuous symmetry.

Upon receiving her work, Einstein wrote to Hilbert: "Yesterday I received from Miss Noether a very interesting paper on invariants. I'm impressed that such things can be understood in such a general way. The old guard at Göttingen should take some lessons from Miss Noether! She seems to know her stuff."^[73]

For illustration, if a physical system behaves the same, regardless of how it is oriented in space, the physical laws that govern it are rotationally symmetric; from this symmetry, Noether's theorem shows the angular momentum of the system must be conserved.^[74] The physical system itself need not be symmetric; a jagged asteroid tumbling in space conserves angular momentum despite its asymmetry. Rather, the symmetry of the *physical laws* governing the system is responsible for the conservation law. As another example, if a physical experiment has the same outcome

at any place and at any time, then its laws are symmetric under continuous translations in space and time; by Noether's theorem, these symmetries account for the conservation laws of linear momentum and energy within this system, respectively.

Noether's theorem has become a fundamental tool of modern theoretical physics, both because of the insight it gives into conservation laws, and also, as a practical calculation tool.^[3] Her theorem allows researchers to determine the conserved quantities from the observed symmetries of a physical system. Conversely, it facilitates the description of a physical system based on classes of hypothetical physical laws. For illustration, suppose that a new physical phenomenon is discovered. Noether's theorem provides a test for theoretical models of the phenomenon: if the theory has a continuous symmetry, then Noether's theorem guarantees that the theory has a conserved quantity, and for the theory to be correct, this conservation must be observable in experiments.

Second epoch (1920–26)

Although the results of Noether's first epoch were impressive and useful, her fame as a mathematician rests more on the groundbreaking work she did in her second and third epochs, as noted by Hermann Weyl and B. L. van der Waerden in their obituaries of her.

In these epochs, she was not merely applying ideas and methods of earlier mathematicians; rather, she was crafting new systems of mathematical definitions that would be used by future mathematicians. In particular, she developed a completely new theory of ideals in rings, generalizing earlier work of Richard Dedekind. She also is renowned for developing ascending chain conditions, a simple finiteness condition that yielded powerful results in her hands. Such conditions and the theory of ideals enabled Noether to generalize many older results and to treat old problems from a new perspective, such as elimination theory and the algebraic varieties that had been studied by her father.

Ascending and descending chain conditions

In this epoch, Noether became famous for her deft use of ascending (*Teilerkettensatz*) or descending (*Vielfachenkettensatz*) chain conditions. A sequence of non-empty subsets A_1, A_2, A_3 , etc. of a set S is usually said to be *strictly ascending*, if each is a subset of the next

$$A_1 \subset A_2 \subset A_3 \subset \cdots$$

The ascending chain condition requires that such sequences break off after a finite number of steps; in other words, all such sequences of subsets must be finite. Conversely, with *strictly descending* sequences of subsets

$$A_1 \supset A_2 \supset A_3 \supset \cdots$$

the descending chain condition requires that such sequences break off after a finite number.

Ascending and descending chain conditions are general, meaning that they can be applied to many types of mathematical objects—and, on the surface, they might not seem very powerful. Noether showed how to exploit such conditions, however, to maximum advantage: for example, how to use them to show that every set of sub-objects has a maximal/minimal element or that a complex object can be generated by a smaller number of elements. These conclusions often are crucial steps in a proof.

Many types of objects in abstract algebra can satisfy chain conditions, and usually if they satisfy an ascending chain condition, they are called *Noetherian* in her honor. By definition, a Noetherian ring satisfies an ascending chain condition on its left and right ideals, whereas a Noetherian group is defined as a group in which every strictly ascending chain of subgroups is finite. A Noetherian module is a module in which every strictly ascending chain of submodules breaks off after a finite number. A Noetherian space is a topological space in which every strictly increasing chain of open subspaces breaks off after a finite number of terms; this definition is made so that the spectrum of a Noetherian ring is a Noetherian topological space.

The chain condition often is "inherited" by sub-objects. For example, all subspaces of a Noetherian space, are Noetherian themselves; all subgroups and quotient groups of a Noetherian group are likewise, Noetherian; and,

mutatis mutandis, the same holds for submodules and quotient modules of a Noetherian module. All quotient rings of a Noetherian ring are Noetherian, but that does not necessarily hold for its subrings. The chain condition also may be inherited by combinations or extensions of a Noetherian object. For example, finite direct sums of Noetherian rings are Noetherian, as is the ring of formal power series over a Noetherian ring.

Another application of such chain conditions is in Noetherian induction—also known as well-founded induction—which is a generalization of mathematical induction. It frequently is used to reduce general statements about collections of objects to statements about specific objects in that collection. Suppose that S is a partially ordered set. One way of proving a statement about the objects of S is to assume the existence of a counterexample and deduce a contradiction, thereby proving the contrapositive of the original statement. The basic premise of Noetherian induction is that the every non-empty subset of S contains a minimal element. In particular, the set of all counterexamples contains a minimal element, the *minimal counterexample*. In order to prove the original statement, therefore, it suffices to prove something seemingly much weaker: For any counterexample, there is a smaller counterexample.

Commutative rings, ideals, and modules

Noether's paper, *Idealtheorie in Ringbereichen* (*Theory of Ideals in Ring Domains*, 1921),^[75] is the foundation of general commutative ring theory, and gives one of the first general definitions of a commutative ring.^[76] Before her paper, most results in commutative algebra were restricted to special examples of commutative rings, such as polynomial rings over fields or rings of algebraic integers. Noether proved that in a ring which satisfies the ascending chain condition on ideals, every ideal is finitely generated. In 1943, French mathematician Claude Chevalley coined the term, *Noetherian ring*, to describe this property.^[76] A major result in Noether's 1921 paper is the **Lasker–Noether theorem**, which extends Lasker's theorem on the primary decomposition of ideals of polynomial rings to all Noetherian rings. The Lasker–Noether theorem can be viewed as a generalization of the fundamental theorem of arithmetic which states that any positive integer can be expressed as a product of prime numbers, and that this decomposition is unique.

Noether's work *Abstrakter Aufbau der Idealtheorie in algebraischen Zahl- und Funktionenkörpern* (*Abstract Structure of the Theory of Ideals in Algebraic Number and Function Fields*, 1927)^[77] characterized the rings in which the ideals have unique factorization into prime ideals as the Dedekind domains: integral domains that are Noetherian, 0 or 1-dimensional, and integrally closed in their quotient fields. This paper also contains what now are called the isomorphism theorems, which describe some fundamental natural isomorphisms, and some other basic results on Noetherian and Artinian modules.

Elimination theory

In 1923–24, Noether applied her ideal theory to elimination theory—in a formulation that she attributed to her student, Kurt Hentzelt—showing that fundamental theorems about the factorization of polynomials could be carried over directly.^[78] Traditionally, elimination theory is concerned with eliminating one or more variables from a system of polynomial equations, usually by the method of resultants. For illustration, the system of equations often can be written in the form of a matrix M (missing the variable x) times a vector v (having only different powers of x) equaling the zero vector, $M \cdot v = 0$. Hence, the determinant of the matrix M must be zero, providing a new equation in which the variable x has been eliminated.

Invariant theory of finite groups

Techniques such as Hilbert's original non-constructive solution to the finite basis problem could not be used to get quantitative information about the invariants of a group action, and furthermore, they did not apply to all group actions. In her 1915 paper,^[79] Noether found a solution to the finite basis problem for a finite group of transformations G acting on a finite dimensional vector space over a field of characteristic zero. Her solution shows that the ring of invariants is generated by homogenous invariants whose degree is less than, or equal to, the order of

the finite group; this is called, **Noether's bound**. Her paper gave two proofs of Noether's bound, both of which also work when the characteristic of the field is coprime to $|G|!$, the factorial of the order $|G|$ of the group G . The number of generators need not satisfy Noether's bound when the characteristic of the field divides the $|G|$,^[80] but Noether was not able to determine whether the bound was correct when the characteristic of the field divides $|G|!$ but not $|G|$. For many years, determining the truth or falsity of the bound in this case was an open problem called "Noether's gap". It finally was resolved independently by Fleischmann in 2000 and Fogarty in 2001, who both showed that the bound remains true.^[81]

In her 1926 paper,^[82] Noether extended Hilbert's theorem to representations of a finite group over any field; the new case that did not follow from Hilbert's work, is when the characteristic of the field divides the order of the group. Noether's result was later extended by William Haboush to all reductive groups by his proof of the Mumford conjecture.^[83] In this paper Noether also introduced the *Noether normalization lemma*, showing that a finitely generated domain A over a field k has a set x_1, \dots, x_n of algebraically independent elements such that A is integral over $k[x_1, \dots, x_n]$.

Contributions to topology

As noted by Pavel Alexandrov and Hermann Weyl in their obituaries, Noether's contributions to topology illustrate her generosity with ideas and how her insights could transform entire fields of mathematics. In topology, mathematicians study the properties of objects that remain invariant even under deformation, properties such as their connectedness. A common joke is that a topologist cannot distinguish a donut from a coffee mug, since they can be smoothly deformed into one another.

Noether is credited with the fundamental ideas that led to the development of algebraic topology from the earlier combinatorial topology, specifically, the idea of homology groups.^[84] According to the account of Alexandrov, Noether attended lectures given by Heinz Hopf and him in the summers of 1926 and 1927, where "she continually made observations, which were often deep and subtle"^[85] and he continues that,

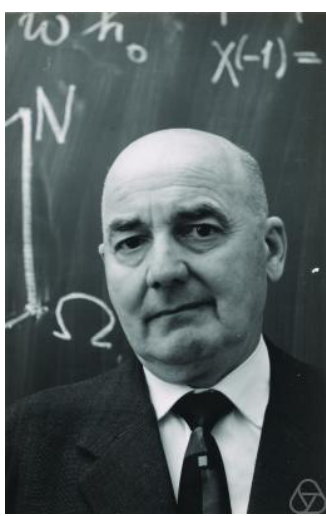
When ... she first became acquainted with a systematic construction of combinatorial topology, she immediately observed that it would be worthwhile to study directly the groups of algebraic complexes and cycles of a given polyhedron and the subgroup of the cycle group consisting of cycles homologous to zero; instead of the usual definition of Betti numbers, she suggested immediately defining the Betti group as the complementary (quotient) group of the group of all cycles by the subgroup of cycles homologous to zero. This observation now seems self-evident. But in those years (1925–1928) this was a completely new point of view.^[86]

Noether's suggestion that topology be studied algebraically, was adopted immediately by Hopf, Alexandrov, and others,^[86] and it became a frequent topic of discussion among the mathematicians of Göttingen.^[87] Noether observed that her idea of a Betti group makes the Euler–Poincaré formula simple to understand, and Hopf's own work on this subject^[88] "bears the imprint of these remarks of Emmy Noether".^[89] Noether mentions her own topology ideas only as an aside in one 1926 publication,^[90] where she cites it as an application of group theory.^[91]



A continuous deformation (homotopy) of a coffee cup into a doughnut (torus) and back

The algebraic approach to topology was developed independently in Austria. In a 1926–27 course given in Vienna, Leopold Vietoris defined a homology group, which was developed by Walther Mayer, into an axiomatic definition in 1928.^[92]



Helmut Hasse worked with Noether and others to found the theory of central simple algebras

Third epoch (1927–35)

Hypercomplex numbers and representation theory

Much work on hypercomplex numbers and group representations was carried out in the nineteenth and early twentieth centuries, but remained disparate. Noether united the results and gave the first general representation theory of groups and algebras.^[93] Briefly, Noether subsumed the structure theory of associative algebras and the representation theory of groups into a single arithmetic theory of modules and ideals in rings satisfying ascending chain conditions. This single work by Noether was of fundamental importance for the development of modern algebra.^[94]

Noncommutative algebra

Noether also was responsible for a number of other advancements in the field of algebra. With Emil Artin, Richard Brauer, and Helmut Hasse, she founded the theory of central simple algebras.^[95]

A seminal paper by Noether, Helmut Hasse, and Richard Brauer pertains to division algebras,^[96] which are algebraic systems in which division is possible. They proved two important theorems: a local-global theorem stating that if a finite dimensional central division algebra over a number field splits locally everywhere then it splits globally (so is trivial), and from this, deduced their *Hauptsatz* ("main theorem"): *every finite dimensional central division algebra over an algebraic number field F splits over a cyclic cyclotomic extension*. These theorems allow one to classify all finite dimensional central division algebras over a given number field. A subsequent paper by Noether showed, as a special case of a more general theorem, that all maximal subfields of a division algebra D are splitting fields.^[97] This paper also contains the Skolem–Noether theorem which states that any two embeddings of an extension of a field k into a finite dimensional central simple algebra over k , are conjugate. The Brauer–Noether theorem^[98] gives a characterization of the splitting fields of a central division algebra over a field.

Assessment, recognition, and memorials

Noether's work continues to be relevant for the development of theoretical physics and mathematics and she consistently is ranked as one of the greatest mathematicians of the twentieth century. In his obituary, fellow algebraist B. L. van der Waerden says that her mathematical originality was "absolute beyond comparison",^[99] and Hermann Weyl said that Noether "changed the face of algebra by her work".^[6] During her lifetime and even until today, Noether has been characterized as the greatest woman mathematician in recorded history by mathematicians [2] [100] such as Pavel Alexandrov,^[101] Hermann Weyl,^[102] and Jean Dieudonné.^[103]



The Emmy Noether Campus at the University of Siegen is home to its mathematics and physics departments

In a letter to *The New York Times*, Albert Einstein wrote:^[1]

In the judgment of the most competent living mathematicians, Fräulein Noether was the most significant creative mathematical genius thus far produced since the higher education of women began. In the realm of algebra, in which the most gifted mathematicians have been busy for centuries, she discovered methods which have proved of enormous importance in the development of the present-day younger generation of mathematicians.

On 2 January 1935, a few months before her death, mathematician Norbert Wiener wrote that ^[104]

Miss Noether is ... the greatest woman mathematician who has ever lived; and the greatest woman scientist of any sort now living, and a scholar at least on the plane of Madame Curie.

At an exhibition at the 1964 World's Fair devoted to Modern Mathematicians, Noether was the only woman represented among the notable mathematicians of the modern world.^[105]

Noether has been honored in several memorials,

- The Association for Women in Mathematics holds a Noether Lecture to honor women in mathematics every year; in its 2005 pamphlet for the event, the Association characterizes Noether as "one of the great mathematicians of her time, someone who worked and struggled for what she loved and believed in. Her life and work remain a tremendous inspiration".^[106]
- Consistent with her dedication to her students, the University of Siegen houses its mathematics and physics departments in buildings on *the Emmy Noether Campus*.^[107]
- The German Research Foundation (Deutsche Forschungsgemeinschaft) operates the *Emmy Noether Programme*, a scholarship providing funding to promising young post-doctorate scholars in their further research and teaching activities.^[108]
- A street in her hometown, Erlangen, has been named after Emmy Noether and her father, Max Noether.
- The successor to the secondary school she attended in Erlangen has been renamed as *the Emmy Noether School*.^[103]

In fiction, Emmy Nutter, the physics professor in "The God Patent" by Ransom Stephens, is based on Emmy Noether ^[109]

Farther from home,

- The crater Nöther on the far side of the Moon is named after her.
- The 7001 Noether asteroid also is named for Emmy Noether.^{[110] [111]}

List of doctoral students

Date	Student name	Dissertation title and English translation	University	Publication
1911.12.16	Falckenberg, Hans	Verzweigungen von Lösungen nichtlinearer Differentialgleichungen Ramifications of Solutions of Nonlinear Differential Equations [§]	Erlangen	Leipzig 1912
1916.03.04	Seidelmann, Fritz	Die Gesamtheit der kubischen und biquadratischen Gleichungen mit Affekt bei beliebigem Rationalitätsbereich Complete Set of Cubic and Biquadratic Equations with Affect in an Arbitrary Rationality Domain [§]	Erlangen	Erlangen 1916

1925.02.25	Hermann, Grete	Die Frage der endlich vielen Schritte in der Theorie der Polynomideale unter Benutzung nachgelassener Sätze von Kurt Hentzelt The Question of the Finite Number of Steps in the Theory of Ideals of Polynomials using Theorems of the Late Kurt Hentzelt [§]	Göttingen	Berlin 1926
1926.07.14	Grell, Heinrich	Beziehungen zwischen den Idealen verschiedener Ringe Relationships between the Ideals of Various Rings [§]	Göttingen	Berlin 1927
1927	Doräte, Wilhelm	Über einem verallgemeinerten Gruppenbegriff On a Generalized Conceptions of Groups [§]	Göttingen	Berlin 1927
died before defense	Hölzer, Rudolf	Zur Theorie der primären Ringe On the Theory of Primary Rings [§]	Göttingen	Berlin 1927
1929.06.12	Weber, Werner	Idealtheoretische Deutung der Darstellbarkeit beliebiger natürlicher Zahlen durch quadratische Formen Ideal-theoretic Interpretation of the Representability of Arbitrary Natural Numbers by Quadratic Forms [§]	Göttingen	Berlin 1930
1929.06.26	Levitski, Jakob	Über vollständig reduzible Ringe und Unterringe On Completely Reducible Rings and Subrings [§]	Göttingen	Berlin 1931
1930.06.18	Deuring, Max	Zur arithmetischen Theorie der algebraischen Funktionen On the Arithmetic Theory of Algebraic Functions [§]	Göttingen	Berlin 1932
1931.07.29	Fitting, Hans	Zur Theorie der Automorphismenringe Abelscher Gruppen und ihr Analogon bei nichtkommutativen Gruppen On the Theory of Automorphism-Rings of Abelian Groups and Their Analogs in Noncommutative Groups [§]	Göttingen	Berlin 1933
1933.07.27	Witt, Ernst	Riemann-Rochscher Satz und Zeta-Funktion im Hyperkomplexen The Riemann-Roch Theorem and Zeta Function in Hypercomplex Numbers [§]	Göttingen	Berlin 1934
1933.12.06	Tsen, Chiungtze	Algebren über Funktionenkörper Algebras over Function Fields [§]	Göttingen	Göttingen 1934

1934	Schilling, Otto	Über gewisse Beziehungen zwischen der Arithmetik hyperkomplexer Zahlssysteme und algebraischer Zahlkörper On Certain Relationships between the Arithmetic of Hypercomplex Number Systems and Algebraic Number Fields [§]	Marburg	Braunschweig 1935
1935	Stauffer, Ruth	The construction of a normal basis in a separable extension field	Bryn Mawr	Baltimore 1936
1935	Vorbeck, Werner	Nichtgaloissche Zerfällungskörper einfacher Systeme Non-Galois Splitting Fields of Simple Systems [§]	Göttingen	
1936	Wichmann, Wolfgang	Anwendungen der p -adischen Theorie im Nichtkommutativen Algebren Applications of the p -adic Theory in Noncommutative Algebras [§]	Göttingen	<i>Monatshefte für Mathematik und Physik</i> (1936) 44 , 203–224.

Eponymous mathematical topics

- Noetherian
- Noetherian induction
- Noether's theorem
- Noetherian group
- Noetherian scheme
- Noether's second theorem
- Noetherian ring
- Noether normalization lemma
- Lasker–Noether theorem
- Noetherian module
- Noether problem
- Skolem–Noether theorem
- Noetherian space
- Albert–Brauer–Hasse–Noether theorem

Notes

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- [11] Quoted in Kimberling 1981, p. 10.
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- [17] Osen 1974, pp. 144–145; Lederman & Hill 2004, p. 72.
- [18] Dick 1981, p. 188.
- [19] Kimberling 1981, pp. 14–18; Osen 1974, p. 145; Dick 1981, pp. 33–34.
- [20] Kimberling 1981, p. 18
- [21] Kimberling 1981, p. 18; Dick 1981, pp. 44–45; Osen 1974, pp. 145–146

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- [24] Lederman & Hill 2004, p. 74; Osen 1974, p. 148.
- [25] Kimberling 1981, pp. 24–25; Dick 1981, pp. 61–63.
- [26] Alexandrov 1981, pp. 100, 107.
- [27] Dick 1981, p. 51.
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- [33] Dick 1981, pp. 40–41.
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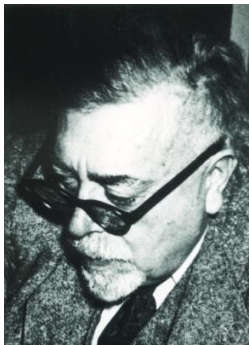
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Norbert Wiener

Norbert Wiener	
	
Born	November 26, 1894Columbia, Missouri, U.S.
Died	March 18, 1964 (aged 69)Stockholm, Sweden
Nationality	American
Fields	Mathematics Cybernetics
Institutions	Massachusetts Institute of Technology
Alma mater	Tufts College BA 1909 Harvard University PhD 1912
Doctoral advisor	Karl Schmidt Josiah Royce
Doctoral students	Amar Bose Colin Cherry Shikao Ikehara Norman Levinson

Norbert Wiener (November 26, 1894, Columbia, Missouri – March 18, 1964, Stockholm, Sweden) was an American mathematician.

A famous child prodigy, Wiener later became an early researcher in stochastic and noise processes, contributing work relevant to electronic engineering, electronic communication, and control systems.

Wiener is regarded as the originator of cybernetics, a formalization of the notion of feedback, with many implications for engineering, systems control, computer science, biology, philosophy, and the organization of society.

Biography

Youth

Wiener was the first child of Leo Wiener and Bertha Kahn, Jews of Polish and German descent, respectively. Employing teaching methods of his own invention, Leo educated Norbert at home until 1903, except for a brief interlude when Norbert was 7 years of age. Wiener became a child prodigy partly due to his father's tutoring. Earning his living teaching German and Slavic languages, Leo read widely and accumulated a personal library from which the young Norbert benefited greatly. Leo also had ample ability in mathematics, and tutored his son in the subject until he left home.

After graduating from Ayer High School in 1906 at 11 years of age, Wiener entered Tufts College. He was awarded a BA for mathematics in 1909 at the age of 14, whereupon he began graduate studies of zoology at Harvard. In 1910 he transferred to Cornell to study philosophy.

Harvard and World War I

The next year he returned to Harvard, while still continuing his philosophical studies. Back at Harvard, Wiener became influenced by Edward Vermilye Huntington, whose mathematical interests ranged from axiomatic foundations to engineering problems. Harvard awarded Wiener a Ph.D. in 1913, when he was merely 18 years old, for a dissertation on mathematical logic, supervised by Karl Schmidt, the essential results of which were published as Wiener (1914). In that dissertation, he was the first to state publicly that ordered pairs can be defined in terms of elementary set theory. Hence relations can be defined by set theory, thus the theory of relations does not require any axioms or primitive notions distinct from those of set theory. In 1921, Kazimierz Kuratowski proposed a simplification of Wiener's definition of ordered pairs, and that simplification has been in common use ever since.

In 1914, Wiener traveled to Europe, to be taught by Bertrand Russell and G. H. Hardy at Cambridge University, and by David Hilbert and Edmund Landau at the University of Göttingen. During 1915-16, he taught philosophy at Harvard, then worked as an engineer for General Electric and wrote for the *Encyclopedia Americana*. Wiener worked briefly as a journalist for the *Boston Herald*, where he wrote a feature story on the poor labor conditions for mill workers in Lawrence, Massachusetts, but he was fired soon afterwards for his reluctance to write favorable articles about a politician the newspaper's owners sought to promote.^[1]

Although Wiener eventually became a staunch pacifist, he eagerly contributed to the war effort in World War I. In 1916, with America's entry into the war drawing closer, Wiener attended a training camp for potential military officers, but failed to earn a commission. One year later Wiener again tried to join the military, but the government again rejected him due to his poor eyesight. In the summer of 1918, Oswald Veblen invited Wiener to work on ballistics at the Aberdeen Proving Ground in Maryland.^[2] Living and working with other mathematicians strengthened his interest in mathematics. However, Wiener was still eager to serve in uniform, and decided to make one more attempt to enlist, this time as a common soldier. Wiener wrote in a letter to his parents, "I should consider myself a pretty cheap kind of a swine if I were willing to be an officer but unwilling to be a soldier".^[3] This time the army accepted Wiener into its ranks and assigned him, by coincidence, to a unit stationed at Aberdeen, Maryland. World War I ended just days after Wiener's return to Aberdeen and Wiener was discharged from the military in February 1919.^[4]

After the war

Wiener was unable to secure a permanent position at Harvard, a situation he blamed largely on anti-semitism at the university and in particular on the antipathy of Harvard mathematician G. D. Birkhoff.^[5] He was also rejected for a position at the University of Melbourne. At W. F. Osgood's suggestion, Wiener became an instructor of mathematics at MIT, where he spent the remainder of his career, becoming promoted eventually to Professor.

In 1926, Wiener returned to Europe as a Guggenheim scholar. He spent most of his time at Göttingen and with Hardy at Cambridge, working on Brownian motion, the Fourier integral, Dirichlet's problem, harmonic analysis, and the Tauberian theorems.

In 1926, Wiener's parents arranged his marriage to a German immigrant, Margaret Engemann; they had two daughters.

During and after World War II

During World War II, his work on the automatic aiming and firing of anti-aircraft guns caused Wiener to study communication theory and eventually to formulate cybernetics. Unlike many of his contemporaries, Wiener was not invited to participate in the Manhattan Project.^[6] After the war, his fame helped MIT to recruit a research team in cognitive science, composed of researchers in neuropsychology and the mathematics and biophysics of the nervous system, including Warren Sturgis McCulloch and Walter Pitts. These men later made pioneering contributions to computer science and artificial intelligence. Soon after the group was formed, Wiener suddenly ended all contact with its members, mystifying his colleagues. In their biography of Wiener, Conway and Siegelman suggest that Wiener's wife Margaret, who detested McCulloch's bohemian lifestyle, engineered the breach.^[7]

Wiener later helped develop the theories of cybernetics, robotics, computer control, and automation. He shared his theories and findings with other researchers, and credited the contributions of others. These included Soviet researchers and their findings. Wiener's acquaintance with them caused him to be regarded with suspicion during the "Cold War". He was a strong advocate of automation to improve the standard of living, and to end economic underdevelopment. His ideas became influential in India, whose government he advised during the 1950s.

After the war, Wiener became increasingly concerned with what he believed was political interference with scientific research, and the militarization of science. His article "A Scientist Rebels" for the January 1947 issue of *The Atlantic Monthly*^[8] urged scientists to consider the ethical implications of their work. After the war, he refused to accept any government funding or to work on military projects. The way Wiener's beliefs concerning nuclear weapons and the Cold War contrasted with that of John von Neumann is the major theme of the book *John Von Neumann and Norbert Wiener* Heims (1980).^[9]

Awards and honors

- Wiener won the Bôcher Prize in 1933 and the National Medal of Science in 1963 (Presented by President Johnson at a White House Ceremony in January 1964.), soon before his death.
- The Norbert Wiener Prize in Applied Mathematics was endowed in 1967 in honor of Norbert Wiener by MIT's mathematics department and is provided jointly by the American Mathematical Society and Society for Industrial and Applied Mathematics.
- The Norbert Wiener Award for Social and Professional Responsibility awarded annually by CPSR, was established in 1987 in honor of Wiener to recognize contributions by computer professionals to socially responsible use of computers.
- The crater Wiener on the far side of the Moon is named after him.
- The Norbert Wiener Center for Harmonic Analysis and Applications, at the University of Maryland, College Park, is named in his honor.^[10]
- Robert A. Heinlein named a spaceship after him in his 1957 novel *Citizen of the Galaxy*; a 'Free Trader' ship called the *Norbert Wiener* mentioned in Chapter 14.

Work

Information is information, not matter or energy.

—Norbert Wiener, *Cybernetics: Or the Control and Communication in the Animal and the Machine*

Wiener was an early studier of stochastic and noise processes, contributing work relevant to electronic engineering, electronic communication, and control systems.

Wiener is regarded as the originator of cybernetics, a formalization of the notion of feedback, with many implications for engineering, systems control, computer science, biology, philosophy, and the organization of society.

Wiener's work with cybernetics influenced Gregory Bateson and Margaret Mead, and through them, Anthropology, Sociology, and Education.^[11]

Wiener equation

A simple mathematical representation of Brownian motion, the Wiener equation, named after Wiener, assumes the current velocity of a fluid particle fluctuates.

Wiener filter

For signal processing, the Wiener filter is a filter proposed by Wiener during the 1940s and published in 1949. Its purpose is to reduce the amount of noise present in a signal by comparison with an estimation of the desired noiseless signal.

In mathematics

Wiener took a great interest in the mathematical theory of Brownian motion (named after Robert Brown) proving many results now widely known such as the non-differentiability of the paths. As a result the one-dimensional version of Brownian motion became known as the Wiener process. It is the best known of the Lévy processes, càdlàg stochastic processes with stationary statistically independent increments, and occurs frequently in pure and applied mathematics, physics and economics (e.g. on the stock-market).

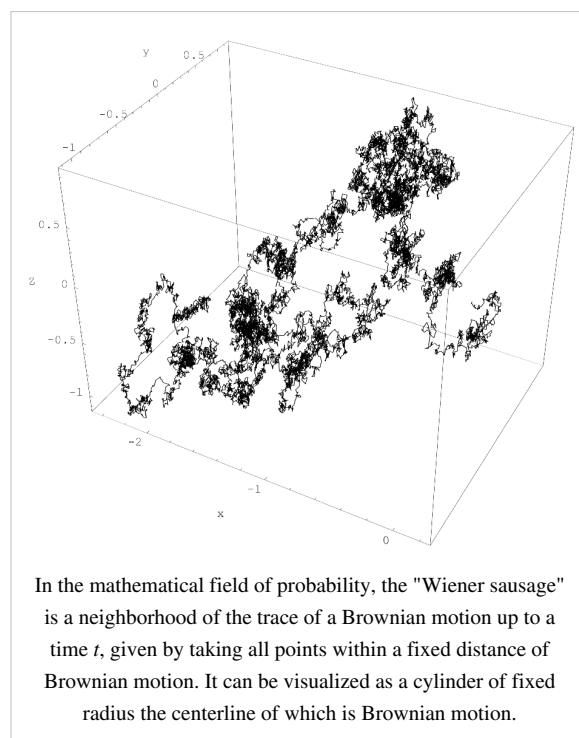
Wiener's Tauberian theorem, a 1932 result of Wiener, developed Tauberian theorems in summability theory, on the face of it a chapter of real analysis, by showing that most of the known results could be encapsulated in a principle taken from harmonic analysis. As now formulated, the theorem of Wiener does not have any obvious association with Tauberian theorems, which deal with infinite series; the translation from results formulated for integrals, or using the language of functional analysis and Banach algebras, is however a relatively routine process.

The Paley–Wiener theorem relates growth properties of entire functions on \mathbb{C}^n and Fourier transformation of Schwartz distributions of compact support.

The Wiener–Khinchin theorem, (or *Wiener – Khintchine theorem* or *Khinchin – Kolmogorov theorem*), states that the power spectral density of a wide-sense-stationary random process is the Fourier transform of the corresponding autocorrelation function.

An abstract Wiener space is a mathematical object in measure theory, used to construct a "decent", strictly positive and locally finite measure on an infinite-dimensional vector space. Wiener's original construction only applied to the space of real-valued continuous paths on the unit interval, known as classical Wiener space. Leonard Gross provided the generalization to the case of a general separable Banach space.

The notion of a Banach space itself was discovered independently by both Wiener and Stefan Banach at around the same time.^[12]



Publications

Wiener wrote many books and hundreds of articles.^[13]

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- 1959, *The Tempter*. Random House.

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Under the name "W. Norbert"

- 1952 *The Brain* and other short science fiction in Tech Engineering News

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External links


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Paul Dirac

Paul Adrien Maurice Dirac	
	
Born	Paul Adrien Maurice Dirac8 August 1902Bristol, England
Died	20 October 1984 (aged 82)Tallahassee, Florida, USA
Nationality	Switzerland (1902-1919) United Kingdom (1919-1984)
Fields	Physics (theoretical)
Institutions	University of Cambridge Florida State University
Alma mater	University of Bristol University of Cambridge
Doctoral advisor	Ralph Fowler
Doctoral students	Homi Bhabha Harish Chandra Mehta Dennis Sciama Behram Kurşunoğlu John Polkinghorne
Known for	Dirac equation Dirac comb Dirac delta function Fermi–Dirac statistics Dirac sea Dirac spinor Dirac measure Bra-ket notation Dirac adjoint Dirac large numbers hypothesis Dirac fermion Dirac string Dirac algebra Dirac operator Abraham-Lorentz-Dirac force Dirac bracket Fermi–Dirac integral Negative probability Dirac Picture Dirac-Coulomb-Breit Equation
Notable awards	Nobel Prize in Physics (1933) Copley Medal (1952)

Notes

He is the stepfather of Gabriel Andrew Dirac.

Paul Adrien Maurice Dirac, OM, FRS ( /dɪˈræk/ di-RAK; 8 August 1902 – 20 October 1984) was an English theoretical physicist who made fundamental contributions to the early development of both quantum mechanics and quantum electrodynamics. He held the Lucasian Chair of Mathematics at the University of Cambridge and spent the last fourteen years of his life at Florida State University.

Among other discoveries, he formulated the Dirac equation, which describes the behaviour of fermions, and predicted the existence of antimatter.

Dirac shared the Nobel Prize in physics for 1933 with Erwin Schrödinger, "for the discovery of new productive forms of atomic theory."^[1]

Early years

Paul Adrien Maurice Dirac was born at his parents home in Bristol, England on 8 August 1902,^[2] and grew up in the Bishopston area of the city.^[3] His father, Charles Adrien Ladislas Dirac, was an immigrant from Saint-Maurice in the Canton of Valais, Switzerland, who worked in Bristol as a French teacher. His mother, Florence Hannah Dirac née Holten, the daughter of a ship's captain, worked as a librarian at the Bristol Central Library. Paul had a younger sister, Béatrice Isabelle Marguerite, known as Betty, and an older brother, Reginald Charles Félix, known as Felix,^[4] ^[5] who committed suicide in March 1925.^[6] Dirac later recalled: "My parents were terribly distressed. I didn't know they cared so much. ... I never knew that parents were supposed to care for their children, but from then on I knew."^[7]

Charles and the children were officially Swiss nationals until they became naturalised on 22 October 1919.^[8] Dirac's father's was strict and authoritarian, although he disapproved of corporal punishment.^[9] Dirac had a strained relationship with his father, so much that after his death, he wrote, "I feel much freer now, and I am my own man." Charles forced his children to speak to him only in French, in order that they learn the language. When Dirac found that he could not express what he wanted to say in French, he chose to remain silent.^{[10] [11]}

Dirac was educated first at Bishop Road Primary School,^[12] and then at the all-boys Merchant Venturers' Technical College (later Cotham School), where his father was a French teacher.^[13] The school was an institution attached to the University of Bristol, which shared grounds and staff.^[14] It emphasised technical subjects like bricklaying, shoemaking and metal work, and modern languages.^[15] This was an unusual arrangement at a time when secondary education in Britain was still dedicated largely to the classics, and something for which Dirac would later express gratitude.^[14]

Dirac studied Electrical engineering on a City of Bristol University Scholarship at the University of Bristol's engineering faculty, which was co-located with the Merchant Venturers' Technical College.^[16] Shortly before he completed his degree in 1921, he sat the entrance examination for St John's College, Cambridge. He passed, and was awarded a £70 scholarship, but this fell short of the amount of money required to live and study at Cambridge. Despite graduating with a first class honours bachelor of science degree in engineering, the economic climate of the post-war depression was such that he was unable to find work as an engineer. Instead he took up an offer to study for bachelor of arts degree in mathematics at the University of Bristol free of charge. He was permitted to skip the first year of the course owing to his engineering degree.^[17]

In 1923, Dirac graduated, once again with first class honours, and received a £140 scholarship from the Department of Scientific and Industrial Research. Along with his £70 scholarship from St John's College, this was enough to live at Cambridge. There, Dirac pursued his interests in the theory of general relativity, an interest he gained earlier as a student in Bristol, and in the nascent field of quantum physics, under the supervision of Ralph Fowler.^[18]

Career

Dirac noticed an analogy between the Poisson brackets of classical mechanics and the recently proposed quantization rules in Werner Heisenberg's matrix formulation of quantum mechanics. This observation allowed Dirac to obtain the quantization rules in a novel and more illuminating manner. For this work, published in 1926, he received a Ph.D. from Cambridge.

In 1928, building on 2x2 spin matrices which he discovered independently (Abraham Pais quoted Dirac as saying "I believe I got these (matrices) independently of Pauli and possibly Pauli got these independently of me" ^[19]) of Wolfgang Pauli's work on non-relativistic spin systems, he proposed the Dirac equation as a relativistic equation of motion for the wavefunction of the electron.^[20] This work led Dirac to predict the existence of the positron, the electron's antiparticle, which he interpreted in terms of what came to be called the *Dirac sea*.^[21] The positron was observed by Carl Anderson in 1932. Dirac's equation also contributed to explaining the origin of quantum spin as a relativistic phenomenon.

The necessity of fermions i.e. matter being created and destroyed in Enrico Fermi's 1934 theory of beta decay, however, led to a reinterpretation of Dirac's equation as a "classical" field equation for any point particle of spin $\hbar/2$, itself subject to quantization conditions involving anti-commutators. Thus reinterpreted, in 1934 by Werner Heisenberg, as a (quantum) field equation accurately describing all elementary matter particles- today quarks and leptons - this Dirac field equation is as central to theoretical physics as the Maxwell, Yang-Mills and Einstein field equations. Dirac is regarded as the founder of quantum electrodynamics, being the first to use that term. He also introduced the idea of vacuum polarization in the early 1930s. This work was key to the development of quantum mechanics by the next generation of theorists, and in particular Schwinger, Feynman, Sin-Itiro Tomonaga and Dyson in their formulation of quantum electrodynamics.

Dirac's *Principles of Quantum Mechanics*, published in 1930, is a landmark in the history of science. It quickly became one of the standard textbooks on the subject and is still used today. In that book, Dirac incorporated the previous work of Werner Heisenberg on matrix mechanics and of Erwin Schrödinger on wave mechanics into a single mathematical formalism that associates measurable quantities to operators acting on the Hilbert space of vectors that describe the state of a physical system. The book also introduced the delta function. Following his 1939 article,^[22] he also included the bra-ket notation in the third edition of his book,^[23] thereby contributing to its universal use nowadays.

In 1933, following his 1931 paper on magnetic monopoles, Dirac showed that the existence of a single magnetic monopole in the universe would suffice to explain the observed quantization of electrical charge. In 1975,^[24] 1982,^[25] and 2009^[26] ^[27] ^[28] intriguing results suggested the possible detection of magnetic monopoles, but there is, to date, no direct evidence for their existence.

Dirac was the Lucasian Professor of Mathematics at Cambridge from 1932 to 1969. In 1937, he proposed a speculative cosmological model based on the so-called large numbers hypothesis. During World War II, he conducted important theoretical and experimental research on uranium enrichment by gas centrifuge.

Dirac's quantum electrodynamics made predictions that were - more often than not - infinite and therefore unacceptable. A workaround known as renormalization was developed, but Dirac never accepted this. "I must say that I am very dissatisfied with the situation," he said in 1975, "because this so-called 'good theory' does involve neglecting infinities which appear in its equations, neglecting them in an arbitrary way. This is just not sensible mathematics. Sensible mathematics involves neglecting a quantity when it is small — not neglecting it just because it is infinitely great and you do not want it!"^[29] His refusal to accept renormalization, resulted in his work on the subject moving increasingly out of the mainstream. However, from his once rejected notes he managed to work on putting quantum electrodynamics on "logical foundations" based on Hamiltonian formalism that he formulated. He found a rather novel way of deriving the anomalous magnetic moment "Schwinger term" and also the Lamb shift, afresh, using the Heisenberg picture and without using the joining method used by Weisskopf and French, the two pioneers of modern QED, Schwinger and Feynman, in 1963. That was two years before the

Tomonaga-Schwinger-Feynman QED was given formal recognition by an award of the Nobel Prize for physics. Weisskopf and French (FW) were the first to obtain the correct result for the Lamb shift and the anomalous magnetic moment of the electron. At first FW results did not agree with the incorrect but independent results of Feynman and Schwinger (Schweber SS 1994 "QED and the men who made it: Dyson, Feynman, Schwinger and Tomonaga", Princeton :PUP). The 1963-1964 lectures Dirac gave on quantum field theory at Yeshiva University were published in 1966 as the Belfer Graduate School of Science, Monograph Series Number, 3. After having relocated to Florida in order to be near his elder daughter, Mary, Dirac spent his last fourteen years (of both life and physics research) at the University of Miami in Coral Gables, Florida and Florida State University in Tallahassee, Florida.

In the 1950s in his search for a better QED, Paul Dirac developed the Hamiltonian theory of constraints (Canad J Math 1950 vol 2, 129; 1951 vol 3, 1) based on lectures that he delivered at the 1949 International Mathematical Congress in Canada. Dirac (1951 "The Hamiltonian Form of Field Dynamics" Canad Jour Math, vol 3 ,1) had also solved the problem of putting the Tomonaga-Schwinger equation into the Schrödinger representation (See Phillips R J N 1987 "Tributes to Dirac" p31 London:Adam Hilger) and given explicit expressions for the scalar meson field (spin zero pion or pseudoscalar meson), the vector meson field (spin one rho meson), and the electromagnetic field (spin one massless boson, photon).

The Hamiltonian of constrained systems is one of Dirac's many masterpieces. It is a powerful generalization of Hamiltonian theory that remains valid for curved spacetime. The equations for the Hamiltonian involve only six degrees of freedom described by g_{rs} , p^{rs} for each point of the surface on which the state is considered. The g_{m0} ($m = 0,1,2,3$) appear in the theory only through the variables g^{r0} , $(-g^{00})^{-1/2}$ which occur as arbitrary coefficients in the equations of motion. $H = \int d^3x [(-g^{00})^{-1/2} H_L - g^{r0} / g^{00} H_r]$ There are four constraints or weak equations for each point of the surface $x^0 = \text{constant}$. Three of them H_r form the four vector density in the surface. The fourth H_L is a 3-dimensional scalar density in the surface $H_L \approx 0$; $H_r \approx 0$ ($r=1,2,3$) In the late 1950s he applied the Hamiltonian methods he had developed to cast Einstein's general relativity in Hamiltonian form (Proc Roy Soc 1958,A vol 246, 333, Phys Rev 1959, vol 114, 924) and to bring to a technical completion the quantization problem of gravitation and bring it also closer to the rest of physics according to Salam and DeWitt. In 1959 also he gave an invited talk on "Energy of the Gravitational Field" at the New York Meeting of the American Physical Society later published in 1959 Phys Rev Lett 2, 368. In 1964 he published his "Lectures on Quantum Mechanics" (London:Academic) which deals with constrained dynamics of nonlinear dynamical systems including quantization of curved spacetime. He also published a paper entitled "Quantization of the Gravitational Field" in 1967 ICTP/IAEA Trieste Symposium on Contemporary Physics.

If one considers waves moving in the direction x^3 resolved into the corresponding Fourier components ($r,s = 1,2,3$), the variables in the degrees of freedom 13,23,33 are affected by the changes in the coordinate system whereas those in the degrees of freedom 12, (11-22) remain invariant under such changes. The expression for the energy splits up into terms each associated with one of these six degrees of freedom without any cross terms associated with two of them. The degrees of freedom 13, 23, 33 do not appear at all in the expression for energy of gravitational waves in the direction x^3 . The two degrees of freedom 12, (11-22) contribute a positive definite amount of such a form to represent the energy of gravitational waves. These two degrees of freedom correspond in the language of quantum theory, to the gravitational photons (gravitons) with spin +2 or -2 in their direction of motion. The degrees of freedom (11+22) gives rise to the Newtonian potential energy term showing the gravitational force between the two positive mass is attractive and the self energy of every mass is negative.

Amongst his many students was John Polkinghorne, who recalls that Dirac "was once asked what was his fundamental belief. He strode to a blackboard and wrote that the laws of nature should be expressed in beautiful equations."^[30]

Personal life

Family

Dirac married Eugene Wigner's sister, Margit, in 1937. He adopted Margit's two children, Judith and Gabriel. Paul and Margit Dirac had two children together, both daughters, Mary Elizabeth and Florence Monica.

Margit, known as Manci, visited her brother in 1934 in Princeton from her native Hungary and, while at dinner at the Annex Restaurant (1930s–2006^[31]), met the "lonely-looking man at the next table." This account came from a physicist from Korea who met and was influenced by Dirac, Y.S. Kim, who has also written: "It is quite fortunate for the physics community that Manci took good care of our respected Paul A.M. Dirac. Dirac published eleven papers during the period 1939-46.... Dirac was able to maintain his normal research productivity only because Manci was in charge of everything else."^[32]

A reviewer of the 2009 biography writes: "Dirac blamed his [emotional] frailties on his father, a Swiss immigrant who bullied his wife, chivvied his children and insisted Paul spoke only French at home, even though the Diracs lived in Bristol. 'I never knew love or affection when I was a child,' Dirac once said." She also writes that "[t]he problem lay with his genes. Both father and son had autism, to differing degrees. Hence the Nobel winner's reticence, literal-mindedness, rigid patterns of behaviour and self-centredness. [Quoting the biography:] 'Dirac's traits as a person with autism were crucial to his success as a theoretical physicist: his ability to order information about mathematics and physics in a systematic way, his visual imagination, his self-centredness, his concentration and determination.'"^[33]

Personality

Dirac was known among his colleagues for his precise and taciturn nature. His colleagues in Cambridge jokingly defined a unit of a dirac which was one word per hour.^[34] When Niels Bohr complained that he did not know how to finish a sentence in a scientific article he was writing, Dirac replied, "I was taught at school never to start a sentence without knowing the end of it."^[35] He criticized the physicist J. Robert Oppenheimer's interest in poetry: "The aim of science is to make difficult things understandable in a simpler way; the aim of poetry is to state simple things in an incomprehensible way. The two are incompatible."^[36]

Dirac himself wrote in his diary during his postgraduate years that he concentrated solely on his research, and only stopped on Sunday, when he took long strolls alone.

An anecdote recounted in a review of the 2009 biography tells of Werner Heisenberg and Dirac sailing on a cruise ship to a conference in Japan in August 1929. "Both still in their twenties, and unmarried, they made an odd couple. Heisenberg was a ladies' man who constantly flirted and danced, while Dirac—'an Edwardian geek', as [biographer] Graham Farmelo puts it—suffered agonies if forced into any kind of socialising or small talk. 'Why do you dance?' Dirac asked his companion. 'When there are nice girls, it is a pleasure,' Heisenberg replied. Dirac pondered this notion, then blurted out: 'But, Heisenberg, how do you know beforehand that the girls are nice?'"^[33]

According to a story told in different versions, a friend or student visited Dirac, not knowing of his marriage. Noticing the visitor's surprise at seeing an attractive woman in the house, Dirac said, "This is... this is Wigner's sister". Margit Dirac told both George Gamow and Anton Capri in the 1960s that her husband had actually said, "Allow me to present Wigner's sister, who is now my wife."^[37] ^[38]

Dirac was also noted for his personal modesty. He called the equation for the time evolution of a quantum-mechanical operator, which he was the first to write down, the "Heisenberg equation of motion". Most physicists speak of Fermi-Dirac statistics for half-integer-spin particles and Bose-Einstein statistics for integer-spin particles. While lecturing later in life, Dirac always insisted on calling the former "Fermi statistics". He referred to the latter as "Einstein statistics" for reasons, he explained, of "symmetry".

Religious views

Heisenberg recollected a conversation among young participants at the 1927 Solvay Conference about Einstein and Planck's views on religion. Wolfgang Pauli, Heisenberg and Dirac took part in it. Dirac's contribution was a criticism of the political purpose of religion, which was much appreciated for its lucidity by Bohr when Heisenberg reported it to him later. Among other things, Dirac said:

“I cannot understand why we idle discussing religion. If we are honest—and scientists have to be—we must admit that religion is a jumble of false assertions, with no basis in reality. The very idea of God is a product of the human imagination. It is quite understandable why primitive people, who were so much more exposed to the overpowering forces of nature than we are today, should have personified these forces in fear and trembling. But nowadays, when we understand so many natural processes, we have no need for such solutions. I can't for the life of me see how the postulate of an Almighty God helps us in any way. What I do see is that this assumption leads to such unproductive questions as why God allows so much misery and injustice, the exploitation of the poor by the rich and all the other horrors He might have prevented. If religion is still being taught, it is by no means because its ideas still convince us, but simply because some of us want to keep the lower classes quiet. Quiet people are much easier to govern than clamorous and dissatisfied ones. They are also much easier to exploit. Religion is a kind of opium that allows a nation to lull itself into wishful dreams and so forget the injustices that are being perpetrated against the people. Hence the close alliance between those two great political forces, the State and the Church. Both need the illusion that a kindly God rewards—in heaven if not on earth—all those who have not risen up against injustice, who have done their duty quietly and uncomplainingly. That is precisely why the honest assertion that God is a mere product of the human imagination is branded as the worst of all mortal sins.”

[39]

Heisenberg's view was tolerant. Pauli, raised as a Catholic, had kept silent after some initial remarks, but when finally he was asked for his opinion, said: "Well, our friend Dirac has got a religion and its guiding principle is 'There is no God and Paul Dirac is His prophet.'" Everybody, including Dirac, burst into laughter.^[40]

Death and commemoration

In 1984, Dirac died in Tallahassee, Florida and was buried at Tallahassee's Roselawn Cemetery.^[41] Dirac's childhood home in Bristol is commemorated with a blue plaque and the nearby Dirac Road is named in recognition of his links with the city. A plaque on the wall at the Bishop Road Primary School shows the Dirac equation.^[42] A commemorative stone was erected in a garden Saint-Maurice, Switzerland, the town of origin of his father's family, on 1 August 1991. On 13 November 1995 a commemorative marker, made from Burlington green slate and inscribed with the Dirac equation, was unveiled in Westminster Abbey.^[41] ^[43] Objections by the Dean of Westminster, Edward Carpenter, that Dirac was an atheist were brushed aside.^[44]

Dirac shared the 1933 Nobel Prize for physics with Erwin Schrödinger "for the discovery of new productive forms of atomic theory."^[1] Dirac was also awarded the Royal Medal in 1939 and both the Copley Medal and the Max Planck medal in 1952. He was elected a Fellow of the Royal Society in 1930, an Honorary Fellow of the American Physical Society in 1948, and an Honorary Fellow of the Institute of Physics, London in 1971. Dirac became a member of the Order of Merit, an outstanding recognition by the land of his birth, in 1973. He had previously turned down a knighthood, as he did not want to be addressed by his first name.^[45]

In 1975, Dirac gave a series of five lectures at the University of New South Wales which were subsequently published as a book, *Directions of Physics* (1978). He donated the royalties from this book to the university for the establishment of the Dirac Lecture Series. The Silver Dirac Medal for the Advancement of Theoretical Physics is awarded by the University of New South Wales on the occasion of the lecture.^[46]

Immediately after his death, two organisations of professional physicists established annual awards in Dirac's memory. The Institute of Physics, the United Kingdom's professional body for physicists, awards the Paul Dirac Medal and Prize for "outstanding contributions to theoretical (including mathematical and computational) physics".^[47] The first three recipients were Stephen Hawking (1987), John Stewart Bell (1988), and Roger Penrose (1989). The Abdus Salam International Centre for Theoretical Physics (ICTP) awards the Dirac Medal of the ICTP each year on Dirac's birthday (8 August). Also, the Dirac Prize is awarded by the International Centre for Theoretical

Physics in his memory. Dirac House in Bristol is the headquarters of Institute of Physics Publishing.

The Dirac-Hellmann Award at Florida State University was endowed by Dr Bruce P. Hellmann (Dirac's last doctoral student) in 1997 to reward outstanding work in theoretical physics by FSU researchers.^[48] The Paul A.M. Dirac Science Library at Florida State University, which Manci opened in December 1989, is named in his honour, and his papers are held there. Outside is a statue of him by Gabriella Bollobás.^[49] The street on which the National High Magnetic Field Laboratory in Tallahassee, Florida, is located was named Paul Dirac Drive. There is also a road named after him in his home town of Bristol, UK. The BBC named its video codec Dirac in his honour.

Legacy

Dirac is widely regarded as one of the world's greatest physicists. He was one of the founders of quantum mechanics and quantum electrodynamics.

His early contributions include the modern operator calculus for quantum mechanics, which he called transformation theory, and an early version of the path integral.^[50] He formulated a many-body formalism for quantum mechanics which allowed each particle to have its own proper time.

His relativistic wave equation for the electron was the first successful attack on the problem of relativistic quantum mechanics. Dirac founded quantum field theory with his reinterpretation of the Dirac equation as a many-body equation, which predicted the existence of antimatter and matter–antimatter annihilation. He was the first to formulate quantum electrodynamics, although he could not calculate arbitrary quantities because the short distance limit requires renormalization.

In an attempt to solve the quantum divergence problem, Dirac gave a classical point particle theory combining advanced and retarded waves to eliminate the classical electron self-energy. Although these classical methods did not immediately solve the problems in quantum electrodynamics, they did lead John Archibald Wheeler and Richard Feynman to formulate an alternative Green's function description for light, which eventually led to Feynman's point particle formulation of quantum field theory.

Dirac discovered the magnetic monopole solutions, the first topological configuration in physics, and used them to give the modern explanation of charge quantization. He developed constrained quantization in the 1960s, identifying the general quantum rules for arbitrary classical systems.

Dirac's quantum-field analysis of the vibrations of a membrane, in the early 1960s, proved extremely useful to modern practitioners of superstring theory and its closely related successor, M-Theory.^[51]

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- *Principles of Quantum Mechanics* (1930): This book summarizes the ideas of quantum mechanics using the modern formalism that was largely developed by Dirac himself. Towards the end of the book, he also discusses the relativistic theory of the electron (the Dirac equation), which was also pioneered by him. This work does not refer to any other writings then available on quantum mechanics.
- *Lectures on Quantum Mechanics* (1966): Much of this book deals with quantum mechanics in curved space-time.
- *Lectures on Quantum Field Theory* (1966): This book lays down the foundations of quantum field theory using the Hamiltonian formalism.
- *Spinors in Hilbert Space* (1974): This book based on lectures given in 1969 at the University of Miami, Coral Gables, Florida, USA, deals with the basic aspects of spinors starting with a real Hilbert space formalism. Dirac concludes with the prophetic words "We have boson variables appearing automatically in a theory that starts with only fermion variables, provided the number of fermion variables is infinite. There must be such boson variables connected with electrons..."
- *General Theory of Relativity* (1975): This 68-page work summarizes Einstein's general theory of relativity.

Notes

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- [5] Farmelo 2009, pp. 10–11
- [6] Farmelo 2009, pp. 77–78
- [7] Farmelo 2009, p. 79
- [8] Farmelo 2009, p. 34
- [9] Farmelo 2009, p. 22
- [10] Mehra 1972, p. 17
- [11] Kragh 1990, p. 2
- [12] Farmelo 2009, pp. 13–17
- [13] Farmelo 2009, pp. 20–21
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- [15] Farmelo 2009, p. 23
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- [17] Farmelo 2009, pp. 46–47
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Dirac videos

- Archival footage of Dirac in Princeton 1947 (<http://www.youtube.com/watch?v=PsIIr65-L4>)
- Dirac in 1927 (<http://www.youtube.com/watch?v=8GZdZUouzBY>)

External links

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- The Paul A. M. Dirac Collection Finding Aid at Florida State University (<http://diglib.lib.fsu.edu/findaids/dirachtml.html>)
- Photocopies of Dirac's papers from the Florida State University collection (<http://janus.lib.cam.ac.uk/>), held under Dirac's name in the Archive Centre (<http://www.chu.cam.ac.uk/archives/>) of Churchill College, Cambridge, UK
- Letters from Dirac (1932-36) and other papers (<http://janus.lib.cam.ac.uk/>), held in the Personal Papers archives (http://www.joh.cam.ac.uk/library/special_collections/personal_papers/) of St John's College, Cambridge, UK
- Free online access to Dirac's classic papers from Royal Society's Proceedings A (<http://royalsocietypublishing.org/search?submit=yes&submit=yes&submit=Submit&andorexacttitle=and&journalcode=royprsa&andorexacttitleabs=and&andorexactfulltext=and&author1=dirac&format=standard&hits=80&sortspec=relevance&submit=Go>)
- Annotated bibliography for Paul Dirac from the Alsos Digital Library for Nuclear Issues (<http://alsos.wlu.edu/qsearch.aspx?browse=people/Dirac,+Paul>)
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- Photos of Paul Dirac at the Emilio Segrè Visual Archives, American Institute of Physics (<http://photos.aip.org/veritySearch1.jsp?page=1&chapter=0&collection=storeTest&name=Dirac,+Paul+Adrien+Maurice&desc1=&search=SEARCH+>)
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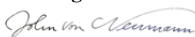
John von Neumann

John von Neumann



John von Neumann in the 1940s

Born	December 28, 1903Budapest, Austria-Hungary
Died	February 8, 1957 (aged 53)Washington, D.C., United States
Residence	United States
Nationality	Hungarian and American
Fields	Mathematics and computer science
Institutions	University of Berlin Princeton University Institute for Advanced Study Site Y, Los Alamos
Alma mater	University of Pázmány Péter ETH Zürich
Doctoral advisor	Lipót Fejér
Doctoral students	Donald B. Gillies Israel Halperin John P. Mayberry
Other notable students	Paul Halmos Clifford Hugh Dowker

Known for	<div>von Neumann Equation</div> <div>Abelian von Neumann algebra</div> <div>Duality Theorem</div> <div>Durbin–Watson statistic</div> <div>Game theory</div> <div>von Neumann algebra</div> <div>von Neumann architecture</div> <div>Von Neumann bicommutant theorem</div> <div>Von Neumann cellular automaton</div> <div>Von Neumann universal constructor</div> <div>Von Neumann entropy</div> <div>Von Neumann regular ring</div> <div>Von Neumann–Bernays–Gödel set theory</div> <div>Von Neumann universe</div> <div>Von Neumann conjecture</div> <div>Von Neumann's inequality</div> <div>Stone–von Neumann theorem</div> <div>Von Neumann stability analysis</div> <div>Minimax theorem</div> <div>Monte Carlo method</div> <div>Von Neumann extractor</div> <div>Von Neumann ergodic theorem</div> <div>Direct integral</div> <div>Ultrastrong topology</div>
Notable awards	<div>Enrico Fermi Award (1956)</div>
<div>Signature</div> <div></div>	

John von Neumann (English pronunciation: /vɒn ˈnoɪmən/) (December 28, 1903 – February 8, 1957) was a Hungarian American mathematician who made major contributions to a vast range of fields,^[1] including set theory, functional analysis, quantum mechanics, ergodic theory, continuous geometry, economics and game theory, computer science, numerical analysis, hydrodynamics (of explosions), and statistics, as well as many other mathematical fields. He is generally regarded as one of the greatest mathematicians in modern history.^[2] The mathematician Jean Dieudonné called von Neumann "the last of the great mathematicians",^[3] while Peter Lax described him as possessing the most "fearsome technical prowess" and "scintillating intellect" of the century.^[4] Even in Budapest, in the time that produced geniuses like Theodore von Kármán (b. 1881), Leó Szilárd (b. 1898), Eugene Wigner (b. 1902), and Edward Teller (b. 1908), his brilliance stood out.^[5]

Von Neumann was a pioneer of the application of operator theory to quantum mechanics, in the development of functional analysis, a principal member of the Manhattan Project and the Institute for Advanced Study in Princeton (as one of the few originally appointed), and a key figure in the development of game theory^[1] ^[6] and the concepts of cellular automata,^[1] the universal constructor, and the digital computer. In a short list of facts about his life he submitted to the National Academy of Sciences, he stated "The part of my work I consider most essential is that on quantum mechanics, which developed in Gottingen in 1926, and subsequently in Berlin in 1927-1929. Also, my work on various forms of operator theory, Berlin 1930 and Princeton 1935-1939; on the ergodic theorem, Princeton, 1931-1932." Along with Teller and Stanisław Ulam, von Neumann worked out key steps in the nuclear physics involved in thermonuclear reactions and the hydrogen bomb.

Biography

The eldest of three brothers, von Neumann was born - **Neumann János Lajos** (Hungarian pronunciation: [ˈnojmn̩ ˈjaːnoʃ ˈlɔjɔʃ]; in Hungarian the family name comes first) on December 28, 1903 in Budapest, Austro-Hungarian Empire, to wealthy Jewish parents.^{[7] [8] [9]} His father was Neumann Miksa (Max Neumann) who came to Budapest from Pécs at the end of 1880s, passed doctor of law examinations and worked for a bank. His mother was Kann Margit (Margaret Kann).^[10]

János, nicknamed "Jancsi" (Johnny), was a child prodigy, with an aptitude for languages, memorization, and mathematics. By the age of six, he could exchange jokes in Classical Greek, memorize telephone directories, and display prodigious mental calculation abilities.^[11] He entered the Hungarian-speaking Lutheran high school Fasori Evangelikus Gimnázium in Budapest in 1911. Although he attended school at the grade level appropriate to his age, his father hired private tutors to give him advanced instruction in those areas in which he had displayed an aptitude. Recognized as a mathematical prodigy, he began to study advanced calculus under Gábor Szegő at the age of 15. On their first meeting, Szegő was so astounded with the boy's mathematical talent that he was brought to tears.^[12] In 1913, his father was rewarded with ennoblement for his service to the Austro-Hungarian empire. (After becoming semi-autonomous in 1867, Hungary had found itself in need of a vibrant mercantile class.) The Neumann family thus acquiring the title *margittai*, Neumann János became margittai Neumann János (John Neumann of Margitta), which he later changed to the German Johann von Neumann. He received his Ph.D. in mathematics (with minors in experimental physics and chemistry) from Pázmány Péter University in Budapest at the age of 22.^[1] He simultaneously earned his diploma in chemical engineering from the ETH Zurich in Switzerland^[1] at the behest of his father, who wanted his son to invest his time in a more financially viable endeavour than mathematics. Between 1926 and 1930, he taught as a *Privatdozent* at the University of Berlin, the youngest in its history. By the end of year 1927 Neumann had published twelve major papers in mathematics, and by the end of year 1929 thirty-two, at a rate of nearly one major paper per month.^[10]

His father, Max von Neumann died in 1929. In 1930, von Neumann, his mother, and his brothers emigrated to the United States. He anglicized his first name to John, keeping the Austrian-aristocratic surname of von Neumann, whereas his brothers adopted surnames Vonneumann and Neumann (using the *de Neumann* form briefly when first in the U.S.).

Von Neumann was invited to Princeton University, New Jersey, in 1930, and, subsequently, was one of the first four people selected for the faculty of the Institute for Advanced Study (two of the others being Albert Einstein and Kurt Gödel), where he remained a mathematics professor from its formation in 1933 until his death.

In 1937, von Neumann became a naturalized citizen of the U.S. In 1938, he was awarded the Bôcher Memorial Prize for his work in analysis.

Funktionentheorie II, Dr. Neumann von Margitta, Mi So 9-11, p.	[634]
Analytische Zahlentheorie II, Prof. Schur, Mo Di Do Fr 11-12, p.	[635]
Axiomatik der Mengenlehre und mathematische Logik, Dr. Neumann von Margitta, Mo 16-17, Do 18-20, p.	[636]
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Besprechung neuerer Arbeiten zur Quantentheorie, Dr. Szilard mit Dr. Neumann von Margitta, Fr 15½-17, pg.	[689]
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Partielle Differentialgleichungen, Prof. Hammerstein, Mo Di Do Fr 12-13, p.	[661]
Kombinatorische Topologie, Dr. Hopf, Di Do Fr 10-11, p.	[662]
Hilbertsche Beweistheorie, Dr. Neumann von Margitta, Do 16-18, p.	[663]

Excerpt from the university calendar 1928 of the Friedrich-Wilhelms-Universität Berlin announcing Neumann's lectures on axiomatic set theory and logics, problems in quantum mechanics and special mathematical functions



Gravestone of John von Neumann

Von Neumann married twice. He married Mariette Kövesi in 1930, just prior to emigrating to the United States. They had one daughter (von Neumann's only child), Marina, who is now a distinguished professor of international trade and public policy at the University of Michigan. The couple divorced in 1937. In 1938, von Neumann married Klara Dan, whom he had met during his last trips back to Budapest prior to the outbreak of World War II. The von Neumanns were very active socially within the Princeton academic community, and it is from this aspect of his life that many of the anecdotes which are part of von Neumann's legend originate.

In 1955, von Neumann was diagnosed with what was either bone or pancreatic cancer.^[13] Von Neumann died a-year-and-a-half later. While at Walter Reed Hospital in Washington, D.C., he invited a Roman Catholic priest, Father Anselm Strittmatter, O.S.B., to visit him for consultation. This move shocked some of von Neumann's friends in view of his reputation as an agnostic.^[14] Von Neumann, however, is reported to have said in explanation that Pascal had a point, referring to Pascal's wager.^[15] Father Strittmatter administered the last sacraments to him.^[16] He died under military security lest he reveal military secrets while heavily medicated. John von Neumann was buried at Princeton Cemetery in Princeton, Mercer County, New Jersey.^[17]

Von Neumann wrote 150 published papers in his life; 60 in pure mathematics, 20 in physics, and 60 in applied mathematics. His last work, written while in the hospital and later published in book form as *The Computer and the Brain*, gives an indication of the direction of his interests at the time of his death.

Logic and set theory

The axiomatization of mathematics, on the model of Euclid's *Elements*, had reached new levels of rigor and breadth at the end of the 19th century, particularly in arithmetic (thanks to the axiom schema of Richard Dedekind and Charles Sanders Peirce) and geometry (thanks to David Hilbert). At the beginning of the twentieth century, efforts to base mathematics on naive set theory suffered a setback due to Russell's paradox (on the set of all sets that do not belong to themselves).

The problem of an adequate axiomatization of set theory was resolved implicitly about twenty years later (by Ernst Zermelo and Abraham Fraenkel). Zermelo and Fraenkel provided a series of principles that allowed for the construction of the sets used in the everyday practice of mathematics: But they did not explicitly exclude the possibility of the existence of a set that belong to itself. In his doctoral thesis of 1925, von Neumann demonstrated two techniques to exclude such sets: the *axiom of foundation* and the notion of *class*.

The axiom of foundation established that every set can be constructed from the bottom up in an ordered succession of steps by way of the principles of Zermelo and Fraenkel, in such a manner that if one set belongs to another then the first must necessarily come before the second in the succession (hence excluding the possibility of a set belonging to itself.) To demonstrate that the addition of this new axiom to the others did not produce contradictions, von Neumann introduced a method of demonstration (called the *method of inner models*) which later became an essential instrument in set theory.

The second approach to the problem took as its base the notion of class, and defines a set as a class which belongs to other classes, while a *proper class* is defined as a class which does not belong to other classes. Under the Zermelo/Fraenkel approach, the axioms impede the construction of a set of all sets which do not belong to themselves. In contrast, under the von Neumann approach, the class of all sets which do not belong to themselves can be constructed, but it is a *proper class* and not a set.

With this contribution of von Neumann, the axiomatic system of the theory of sets became fully satisfactory, and the next question was whether or not it was also definitive, and not subject to improvement. A strongly negative answer

arrived in September 1930 at the historic mathematical Congress of Königsberg, in which Kurt Gödel announced his first theorem of incompleteness: the usual axiomatic systems are incomplete, in the sense that they cannot prove every truth which is expressible in their language. This result was sufficiently innovative as to confound the majority of mathematicians of the time. But von Neumann, who had participated at the Congress, confirmed his fame as an instantaneous thinker, and in less than a month was able to communicate to Gödel himself an interesting consequence of his theorem: namely that the usual axiomatic systems are unable to demonstrate their own consistency. It is precisely this consequence which has attracted the most attention, even if Gödel originally considered it only a curiosity, and had derived it independently anyway (it is for this reason that the result is called *Gödel's second theorem*, without mention of von Neumann.)

Quantum mechanics

At the International Congress of Mathematicians of 1900, David Hilbert presented his famous list of twenty-three problems considered central for the development of the mathematics of the new century. The sixth of these was *the axiomatization of physical theories*. Among the new physical theories of the century the only one which had yet to receive such a treatment by the end of the 1930s was quantum mechanics. Quantum mechanics found itself in a condition of foundational crisis similar to that of set theory at the beginning of the century, facing problems of both philosophical and technical natures. On the one hand, its apparent non-determinism had not been reduced to an explanation of a deterministic form. On the other, there still existed two independent but equivalent heuristic formulations, the so-called matrix mechanical formulation due to Werner Heisenberg and the wave mechanical formulation due to Erwin Schrödinger, but there was not yet a single, unified satisfactory theoretical formulation.

After having completed the axiomatization of set theory, von Neumann began to confront the axiomatization of quantum mechanics. He immediately realized, in 1926, that a quantum system could be considered as a point in a so-called Hilbert space, analogous to the $6N$ dimension (N is the number of particles, 3 general coordinate and 3 canonical momentum for each) phase space of classical mechanics but with infinitely many dimensions (corresponding to the infinitely many possible states of the system) instead: the traditional physical quantities (e.g., position and momentum) could therefore be represented as particular linear operators operating in these spaces. The *physics* of quantum mechanics was thereby reduced to the *mathematics* of the linear Hermitian operators on Hilbert spaces.

For example, the famous uncertainty principle of Heisenberg, according to which the determination of the position of a particle prevents the determination of its momentum and vice versa, is translated into the *non-commutativity* of the two corresponding operators. This new mathematical formulation included as special cases the formulations of both Heisenberg and Schrödinger, and culminated in the 1932 classic *The Mathematical Foundations of Quantum Mechanics*. However, physicists generally ended up preferring another approach to that of von Neumann (which was considered elegant and satisfactory by mathematicians). This approach was formulated in 1930 by Paul Dirac.

Von Neumann's abstract treatment permitted him also to confront the foundational issue of determinism vs. non-determinism and in the book he demonstrated a theorem according to which quantum mechanics could not possibly be derived by statistical approximation from a deterministic theory of the type used in classical mechanics. This demonstration contained a conceptual error, but it helped to inaugurate a line of research which, through the work of John Stuart Bell in 1964 on Bell's Theorem and the experiments of Alain Aspect in 1982, demonstrated that quantum physics requires a *notion of reality* substantially different from that of classical physics.

Economics and game theory

Von Neumann raised the intellectual and mathematical level of economics in several stunning publications. Von Neumann's proved his minimax theorem in 1928. This theorem establishes that in matrix zero-sum games with perfect information (i.e., in which players know at each time all moves that have taken place so far), there exists a pair of strategies for both players that allows each to minimize his maximum losses (hence the name minimax). When examining every possible strategy, a player must consider all the possible responses of his adversary. The player then plays out the strategy which will result in the minimization of his maximum loss. Such strategies, which minimize the maximum loss for each player, are called optimal. Von Neumann showed that their minimaxes are equal (in absolute value) and contrary (in sign).

Von Neumann improved and extended the minimax theorem to include games involving imperfect information and games with more than two players, publishing this result in his 1944 *Theory of Games and Economic Behavior* (written with Oskar Morgenstern). The public interest in this work was such that The New York Times ran a front-page story. In this book, von Neumann declared that economic theory needed to use functional analytic methods, especially convex sets and topological fixed point theorem, rather than the traditional differential calculus, because the maximum-operator did not preserve differentiable functions. Independently, Leonid Kantorovich's functional analytic work on mathematical economics also focused attention on optimization theory, non-differentiability, and vector lattices. Von Neumann's functional-analytic program has dominated economic theory ever since.

For his model of an expanding economy, von Neumann proved the existence and uniqueness of an equilibrium using his generalization of Brouwer's fixed point theorem. Von Neumann's model of an expanding economy considered the matrix pencil $A - \lambda B$ with nonnegative matrices A and B ; von Neumann sought probability vectors p and q and a positive number λ that would solve the complementarity equation

$$p^T (A - \lambda B) q = 0,$$

along with two inequality systems expressing economic efficiency. In this model, the (transposed) probability vector p represents the prices of the goods while the probability vector q represents the "intensity" at which the production process would run. The unique solution λ represents the rate of growth of the economy, which equals the interest rate. Proving the existence of a positive growth rate and proving that the growth rate equals the interest rate were remarkable achievements, even for von Neumann.^{[18] [19] [20]} Von Neumann's results have been viewed as a special case of linear programming, where von Neumann's model uses only nonnegative matrices.^[21] The study of von Neumann's model of an expanding economy continues to interest mathematical economists with interests in computational economics.^{[22] [23] [24]} This paper has been called the greatest paper in mathematical economics by several authors, who recognized its introduction of fixed-point theorems, linear inequalities, complementary slackness, and saddlepoint duality.

Building on his results on matrix games and on his model of an expanding economy, Von Neumann also invented the theory of duality in linear programming, after George B. Dantzig described his work in a few minutes, after an impatient von Neumann asked him to get to the point. Then, Dantzig listened dumbfounded while von Neumann provided an hour lecture on convex sets, fixed-point theory, and duality, conjecturing the equivalence between matrix games and linear programming. Later, von Neumann suggested a new method of linear programming, using the homogeneous linear system of Gordan (1873) which was later popularized by Karmarkar's algorithm. Von Neumann's method used a pivoting algorithm between simplices, with the pivoting decision determined by a nonnegative least squares subproblem with a convexity constraint (projecting the zero-vector onto the convex hull of the active simplex). Von Neumann's algorithm was the first interior-point method of linear programming. However, it was not competitive with the simplex algorithm of Dantzig.^[25]

The lasting importance of the work on general equilibria and the methodology of fixed point theorems is underscored by the awarding of Nobel prizes in 1972 to Kenneth Arrow, in 1983 to Gérard Debreu, and in 1994 to John Nash who used fixed point theorems to establish equilibria for noncooperative games and for bargaining problems in his

Ph.D thesis. Arrow and Debreu also used linear programming, as did Nobel laureates Tjalling Koopmans, Leonid Kantorovich, Wassily Leontief, Paul Samuelson, Robert Dorfman, Robert Solow, and Leonid Hurwicz.

Von Neumann was also the inventor of the method of proof, used in game theory, known as backward induction (which he first published in 1944 in the book co-authored with Morgenstern, *Theory of Games and Economic Behaviour*).^[26]

Mathematical statistics and econometrics

Von Neumann made some fundamental contributions to mathematical statistics. In 1941, he derived the exact distribution of the ratio of mean square successive difference to the variance for normally distributed variables.^[27] This ratio was applied to the residuals from regression models and is commonly known as the Durbin-Watson statistic^[28] for testing the null hypothesis that the errors are serially independent against the alternative that they follow a stationary first order autoregression. Subsequently, John Denis Sargan and Alok Bhargava^[29] extended the results for testing if the errors on a regression model follow a Gaussian random walk (i.e. possess a unit root) against the alternative that they are a stationary first order autoregression. Von Neumann's contributions to statistics have had a major impact on econometric methodology.

Nuclear weapons

Beginning in the late 1930s, von Neumann began to take more of an interest in applied (as opposed to pure) mathematics. In particular, he developed an expertise in explosions—phenomena which are difficult to model mathematically. This led him to a large number of military consultancies, primarily for the Navy, which in turn led to his involvement in the Manhattan Project. The involvement included frequent trips by train to the project's secret research facilities in Los Alamos, New Mexico.^[1]

Von Neumann's principal contribution to the atomic bomb itself was in the concept and design of the explosive lenses needed to compress the plutonium core of the Trinity test device and the "Fat Man" weapon that was later dropped on Nagasaki. While von Neumann did not originate the "implosion" concept, he was one of its most persistent proponents, encouraging its continued development against the instincts of many of his colleagues, who felt such a design to be unworkable. The lens shape design work was completed by July 1944.



John von Neumann's wartime Los Alamos ID badge photo.

In a visit to Los Alamos in September 1944, von Neumann showed that the pressure increase from explosion shock wave reflection from solid objects was greater than previously believed if the angle of incidence of the shock wave was between 90° and some limiting angle. As a result, it was determined that the effectiveness of an atomic bomb would be enhanced with detonation some kilometers above the target, rather than at ground level.^[30]

Beginning in the spring of 1945, along with four other scientists and various military personnel, von Neumann was included in the target selection committee responsible for choosing the Japanese cities of Hiroshima and Nagasaki as the first targets of the atomic bomb. Von Neumann oversaw computations related to the expected size of the bomb blasts, estimated death tolls, and the distance above the ground at which the bombs should be detonated for optimum shock wave propagation and thus maximum effect.^[31] The cultural capital Kyoto, which had been spared the firebombing inflicted upon militarily significant target cities like Tokyo in World War II, was von Neumann's first choice, a selection seconded by Manhattan Project leader General Leslie Groves. However, this target was dismissed by Secretary of War Henry Stimson.^[32]

On July 16, 1945, with numerous other Los Alamos personnel, von Neumann was an eyewitness to the first atomic bomb blast, conducted as a test of the implosion method device, 35 miles (56 km) southeast of Socorro, New Mexico. Based on his observation alone, von Neumann estimated the test had resulted in a blast equivalent to 5 kilotons of TNT, but Enrico Fermi produced a more accurate estimate of 10 kilotons by dropping scraps of torn-up paper as the shock wave passed his location and watching how far they scattered. The actual power of the explosion had been between 20 and 22 kilotons.^[30]

After the war, Robert Oppenheimer remarked that the physicists involved in the Manhattan project had "known sin". Von Neumann's response was that "sometimes someone confesses a sin in order to take credit for it."

Von Neumann continued unperturbed in his work and became, along with Edward Teller, one of those who sustained the hydrogen bomb project. He then collaborated with Klaus Fuchs on further development of the bomb, and in 1946 the two filed a secret patent on "Improvement in Methods and Means for Utilizing Nuclear Energy", which outlined a scheme for using a fission bomb to compress fusion fuel to initiate a thermonuclear reaction.^[33] The Fuchs-von Neumann patent used radiation implosion, but not in the same way as is used in what became the final hydrogen bomb design, the Teller-Ulam design. Their work was, however, incorporated into the "George" shot of Operation Greenhouse, which was instructive in testing out concepts that went into the final design. The Fuchs-von Neumann work was passed on, by Fuchs, to the USSR as part of his nuclear espionage, but it was not used in the Soviet's own, independent development of the Teller-Ulam design. The historian Jeremy Bernstein has pointed out that ironically, "John von Neumann and Klaus Fuchs, produced a brilliant invention in 1946 that could have changed the whole course of the development of the hydrogen bomb, but was not fully understood until after the bomb had been successfully made."^[34]

Computer science

Von Neumann's hydrogen bomb work was also played out in the realm of computing, where he and Stanisław Ulam developed simulations on von Neumann's digital computers for the hydrodynamic computations. During this time he contributed to the development of the Monte Carlo method, which allowed complicated problems to be approximated using random numbers. Because using lists of "truly" random numbers was extremely slow, von Neumann developed a form of making pseudorandom numbers, using the middle-square method. Though this method has been criticized as crude, von Neumann was aware of this: he justified it as being faster than any other method at his disposal, and also noted that when it went awry it did so obviously, unlike methods which could be subtly incorrect.

While consulting for the Moore School of Electrical Engineering at the University of Pennsylvania on the EDVAC project, von Neumann wrote an incomplete *First Draft of a Report on the EDVAC*. The paper, which was widely distributed, described a computer architecture in which the data and the program are both stored in the computer's memory in the same address space. This architecture is to this day the basis of modern computer design, unlike the earliest computers that were 'programmed' by altering the electronic circuitry. Although the single-memory, stored program architecture is commonly called von Neumann architecture as a result of von Neumann's paper, the architecture's description was based partly^[35] on the work of J. Presper Eckert and John William Mauchly, inventors of the ENIAC at the University of Pennsylvania.^[36]

Von Neumann also created the field of cellular automata without the aid of computers, constructing the first self-replicating automata with pencil and graph paper. The concept of a universal constructor was fleshed out in his posthumous work *Theory of Self Reproducing Automata*.^[37] Von Neumann proved that the most effective way of performing large-scale mining operations such as mining an entire moon or asteroid belt would be by using self-replicating machines, taking advantage of their exponential growth.

He is credited with at least one contribution to the study of algorithms. Donald Knuth cites von Neumann as the inventor, in 1945, of the merge sort algorithm, in which the first and second halves of an array are each sorted recursively and then merged together.^[38] His algorithm for simulating a fair coin with a biased coin^[39] is used in the

"software whitening" stage of some hardware random number generators.

He also engaged in exploration of problems in numerical hydrodynamics. With R. D. Richtmyer he developed an algorithm defining *artificial viscosity* that improved the understanding of shock waves. It is possible that we would not understand much of astrophysics, and might not have highly developed jet and rocket engines without that work. The problem was that when computers solve hydrodynamic or aerodynamic problems, they try to put too many computational grid points at regions of sharp discontinuity (shock waves). The *artificial viscosity* was a mathematical trick to slightly smooth the shock transition without sacrificing basic physics.

Politics and social affairs

Von Neumann obtained at the age of 29 one of the first five professorships at the new Institute for Advanced Study in Princeton, New Jersey (another had gone to Albert Einstein). He was a frequent consultant for the Central Intelligence Agency, the United States Army, the RAND Corporation, Standard Oil, IBM, and others.

Throughout his life von Neumann had a respect and admiration for business and government leaders; something which was often at variance with the inclinations of his scientific colleagues.^[40] Von Neumann entered government service (Manhattan Project) primarily because he felt that, if freedom and civilization were to survive, it would have to be because the U.S. would triumph over totalitarianism from the right (Nazism and Fascism) and totalitarianism from the left (Soviet Communism).^[35]

As President of the Von Neumann Committee for Missiles, and later as a member of the United States Atomic Energy Commission, from 1953 until his death in 1957, he was influential in setting U.S. scientific and military policy. Through his committee, he developed various scenarios of nuclear proliferation, the development of intercontinental and submarine missiles with atomic warheads, and the controversial strategic equilibrium called mutual assured destruction. During a Senate committee hearing he described his political ideology as "violently anti-communist, and much more militaristic than the norm".

Von Neumann's interest in meteorological prediction led him to propose manipulating the environment by spreading colorants on the polar ice caps to enhance absorption of solar radiation (by reducing the albedo), thereby raising global temperatures. He also favored a preemptive nuclear attack on the Soviet Union, believing that doing so could prevent it from obtaining the atomic bomb.^[41]

Personality

Von Neumann invariably wore a conservative grey flannel business suit, once riding down the Grand Canyon astride a mule in a three-piece pin-stripe,^[35] and he enjoyed throwing large parties at his home in Princeton, occasionally twice a week.^[42] His white clapboard house at 26 Westcott Road was one of the largest in Princeton.^[43] Despite being a notoriously bad driver, he nonetheless enjoyed driving (frequently while reading a book) – occasioning numerous arrests as well as accidents. When Cuthbert Hurd hired him as a consultant to IBM, Hurd often quietly paid the fines for his traffic tickets.^[44]

Von Neumann liked to eat and drink; his wife, Klara, said that he could count everything except calories. He enjoyed Yiddish and "off-color" humor (especially limericks).^[16]

Honors

- The John von Neumann Theory Prize of the Institute for Operations Research and the Management Sciences (INFORMS, previously TIMS-ORSA) is awarded annually to an individual (or group) who have made fundamental and sustained contributions to theory in operations research and the management sciences.
- The IEEE John von Neumann Medal is awarded annually by the IEEE "for outstanding achievements in computer-related science and technology."
- The John von Neumann Lecture is given annually at the Society for Industrial and Applied Mathematics (SIAM) by a researcher who has contributed to applied mathematics, and the chosen lecturer is also awarded a monetary prize.
- The crater Von Neumann on the Moon is named after him.
- The John von Neumann Computing Center in Princeton, New Jersey (40°20'55"N 74°35'32"W) was named in his honour.
- The professional society of Hungarian computer scientists, John von Neumann Computer Society, is named after John von Neumann.^[45]
- On February 15, 1956, Neumann was presented with the Presidential Medal of Freedom by President Dwight Eisenhower.
- On May 4, 2005 the United States Postal Service issued the *American Scientists* commemorative postage stamp series, a set of four 37-cent self-adhesive stamps in several configurations. The scientists depicted were John von Neumann, Barbara McClintock, Josiah Willard Gibbs, and Richard Feynman.
- The John von Neumann Award of the Rajk László College for Advanced Studies was named in his honour, and has been given every year since 1995 to professors who have made an outstanding contribution to the exact social sciences and through their work have strongly influenced the professional development and thinking of the members of the college.

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Notes

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$$\mathbf{A} - \lambda \mathbf{I} \mathbf{q} = 0,$$

where the nonnegative matrix **A** must be square and where the diagonal matrix **I** is the identity matrix. Von Neumann's irreducibility condition was called the "whales and wranglers" hypothesis by David Champenowne, who provided a verbal and economic commentary on the English translation of von Neumann's article. Von Neumann's hypothesis implied that every economic process used a positive amount of every economic good. Weaker "irreducibility" conditions were given by David Gale and by John Kemeny, Oskar Morgenstern, and Gerald L. Thompson in the 1950s and then by Stephen M. Robinson in the 1970s.

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- Oral history interview with Alice R. Burks and Arthur W. Burks (<http://www.cbi.umn.edu/oh/display.phtml?id=43>), Charles Babbage Institute, University of Minnesota, Minneapolis. Alice Burks and Arthur Burks describe ENIAC, EDVAC, and IAS computers, and John von Neumann's contribution to the development of computers.
- Oral history interview with Eugene P. Wigner (<http://www.cbi.umn.edu/oh/display.phtml?id=77>), Charles Babbage Institute, University of Minnesota, Minneapolis. Wigner talks about his association with John von Neumann during their school years in Hungary, their graduate studies in Berlin, and their appointments to Princeton in 1930. Wigner discusses von Neumann's contributions to the theory of quantum mechanics, and von Neumann's interest in the application of theory to the atomic bomb project.
- Oral history interview with Nicholas C. Metropolis (<http://www.cbi.umn.edu/oh/display.phtml?id=81>), Charles Babbage Institute, University of Minnesota. Metropolis, the first director of computing services at Los Alamos National Laboratory, discusses John von Neumann's work in computing. Most of the interview concerns activity at Los Alamos: how von Neumann came to consult at the laboratory; his scientific contacts there, including Metropolis; von Neumann's first hands-on experience with punched card equipment; his contributions to shock-fitting and the implosion problem; interactions between, and comparisons of von Neumann and Enrico Fermi; and the development of Monte Carlo methods. Other topics include: the relationship between Alan Turing and von Neumann; work on numerical methods for non-linear problems; and the ENIAC calculations done for Los Alamos.
- Von Neumann vs. Dirac (<http://plato.stanford.edu/entries/qt-nvd/>) — from *Stanford Encyclopedia of Philosophy*.
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 - Annotated bibliography for John von Neumann from the Alsos Digital Library for Nuclear Issues (<http://alsos.wlu.edu/qsearch.aspx?browse=people/Neumann,+John+von>)
 - Budapest Tech Polytechnical Institution – John von Neumann Faculty of Informatics (<http://nik.bmf.hu/>)
 - John von Neumann speaking at the dedication of the NORD (<http://elm.eeng.dcu.ie/~alife/von-neumann-1954-NORD/>), December 2, 1954 (audio recording)
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 - John Von Neumann Memorial (<http://www.findagrave.com/cgi-bin/fg.cgi?page=gr&GRid=7333144>) at Find A Grave
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George Birkhoff

George David Birkhoff	
<div><div></div><div>George David Birkhoff</div></div>	
Born	21 March 1884Overisel, Michigan
Died	12 November 1944 (aged 60)Cambridge, Massachusetts
Nationality	 American
Fields	Mathematics
Institutions	Harvard University Yale University Princeton University Radcliffe College
Alma mater	University of Chicago
Doctoral advisor	E. H. Moore
Doctoral students	David Bourgin Robert D. Carmichael Hyman Ettlinger Bernard Koopman Rudolph Langer Marston Morse Marshall H. Stone Joseph L. Walsh Hassler Whitney David Widder
Known for	Ergodic theorem

George David Birkhoff (21 March 1884 – 12 November 1944) was an American mathematician, best known for what is now called the ergodic theorem. Birkhoff was one of the most important leaders in American mathematics in his generation, and during his prime he was considered by many to be the preeminent American mathematician. The mathematician Garrett Birkhoff (1911–1996) was his son.

Career

Birkhoff obtained his A.B. and A.M. from Harvard. He completed his Ph.D. in 1907, on differential equations, at the University of Chicago. While E. H. Moore was his supervisor, he was most influenced by the writings of Henri Poincaré. After teaching at the University of Wisconsin and Princeton University, he taught at Harvard University from 1912 until his death.

Awards and honors

In 1923, he was awarded the inaugural Bôcher Memorial Prize by the American Mathematical Society for his paper Birkhoff (1917) containing, among other things, what is now called the Birkhoff curve shortening flow.

He was elected to the National Academy of Sciences, the American Philosophical Society, the American Academy of Arts and Sciences, the Académie des Sciences in Paris, the Pontifical Academy, and the London and Edinburgh Mathematical Societies.

Service

- Vice-president of the American Mathematical Society, 1919.
- President of the American Mathematical Society, 1925–1926.
- Editor of Transactions of the American Mathematical Society, 1920–1924.

Work

In 1912, attempting to solve the four color problem, Birkhoff introduced the chromatic polynomial. Even though this line of attack did not prove fruitful, the polynomial itself became an important object of study in algebraic graph theory.

In 1913, he proved Poincaré's "Last Geometric Theorem," a special case of the three-body problem, a result that made him world famous. In 1927, he published his *Dynamical Systems* ^[1]. He wrote on the foundations of relativity and quantum mechanics, publishing (with R E Langer) the monograph *Relativity and Modern Physics* in 1923. In 1923, Birkhoff also proved that the Schwarzschild geometry is the unique spherically symmetric solution of the Einstein field equations. A consequence is that black holes are not merely a mathematical curiosity, but could result from any spherical star having sufficient mass.

Birkhoff's most durable result has been his 1931 discovery of what is now called the ergodic theorem. Combining insights from physics on the ergodic hypothesis with measure theory, this theorem solved, at least in principle, a fundamental problem of statistical mechanics. The ergodic theorem has also had repercussions for dynamics, probability theory, group theory, and functional analysis. He also worked on number theory, the Riemann–Hilbert problem, and the four colour problem. He proposed an axiomatization of Euclidean geometry different from Hilbert's (see Birkhoff's axioms); this work culminated in his text *Basic Geometry* (1941).

In his later years, Birkhoff published two curious speculative works. His 1933 *Aesthetic Measure* proposed a mathematical theory of aesthetics. While writing this book, he spent a year studying the art, music and poetry of various cultures around the world. His 1938 *Electricity as a Fluid* combined his ideas on philosophy and science. His 1943 theory of gravitation is also puzzling, since Birkhoff knew (but didn't seem to mind) that his theory allows as sources only matter which is a perfect fluid in which the speed of sound must equal the speed of light (which, needless to say, is quite inconsistent with experiment!).

Influence on hiring practices

Albert Einstein and Norbert Wiener, among others, accused Birkhoff of advocating anti-Semitic hiring practices. During the 1930s, when many Jewish mathematicians fled Europe and tried to obtain jobs in the USA, Birkhoff is alleged to have influenced the hiring process at American institutions to exclude Jews. While Birkhoff may have held anti-Semitic views, it was also the case that he had always been outspoken in his promotion of American mathematics and mathematicians. It has been argued that Birkhoff's actions were in good part motivated by a desire to assure jobs for home-grown American mathematicians. Saunders Mac Lane (1994), a close friend and collaborator of Birkhoff's son, argued that any anti-Semitic tendencies Birkhoff may have had were not unusual for his time.

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Further reading

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External links

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Stephen Weinberg

Steven Weinberg	
 <p>Steven Weinberg at the 2010 Texas Book Festival.</p>	
Born	May 3, 1933New York City, New York, USA
Residence	United States
Nationality	United States
Fields	Theoretical Physics
Institutions	University of California, Berkeley MIT Harvard University University of Texas at Austin
Alma mater	Cornell University Princeton University
Doctoral advisor	Sam Treiman
Doctoral students	Orlando Alvarez Claude Bernard Lay Nam Chang Bob Holdom Ubirajara van Kolck Rafael Lopez-Mobilia John Preskill Fernando Quevedo Mark G. Raizen Scott Willenbrock
Known for	Electromagnetism and Weak Force unification Weinberg-Witten theorem
Influenced	Alan Guth
Notable awards	Nobel Prize in Physics (1979)
Notes	He is married to the professor of law, Louise Weinberg.

Steven Weinberg (born May 3, 1933) is an American theoretical physicist and Nobel laureate in Physics for his contributions with Abdus Salam and Sheldon Glashow to the unification of the weak force and electromagnetic interaction between elementary particles.

Biography

Steven Weinberg was born in 1933 in New York City to Jewish immigrants Frederick and Eva Weinberg, but is an atheist.^[1] He graduated from Bronx High School of Science in 1950 in the same graduating class as Sheldon Glashow, whose own research, independent of Weinberg's, would result in them (and Abdus Salam) sharing the same 1979 Nobel in Physics (see below).

Weinberg received his bachelor's degree from Cornell University in 1954, living at the Cornell branch of the Telluride Association. He left Cornell and went to the Niels Bohr Institute in Copenhagen where he started his graduate studies and research. After one year, Weinberg returned to Princeton University where he earned his Ph.D. degree in Physics in 1957, studying under Sam Treiman.

Academic career

After completing his Ph.D., Weinberg worked as a post-doctoral researcher at Columbia University (1957–1959) and University of California, Berkeley (1959) and then he was promoted to faculty at Berkeley (1960–1966). He did research in a variety of topics of particle physics, such as the high energy behavior of quantum field theory, symmetry breaking, pion scattering, infrared photons and quantum gravity.^[2] It was also during this time that he developed the approach to quantum field theory that is described in the first chapters of his book *The Quantum Theory of Fields*^[3] and started to write his textbook *Gravitation and Cosmology*. Both textbooks, perhaps especially the second, are among the most influential texts in the scientific community in their subjects.

In 1966, Weinberg left Berkeley and accepted a lecturer position at Harvard. In 1967 he was a visiting professor at MIT. It was in that year at MIT that Weinberg proposed his model of unification of electromagnetism and of nuclear weak forces (such as those involved in beta-decay and kaon-decay),^[4] with the masses of the force-carriers of the weak part of the interaction being explained by spontaneous symmetry breaking. One of its fundamental aspects was the prediction of the existence of the Higgs boson. Weinberg's model, now known as the electroweak unification theory, had the same symmetry structure as that proposed by Glashow in 1961: hence both models included the then-unknown weak interaction mechanism between leptons, known as neutral current and mediated by the Z boson. The 1973 experimental discovery of this Z boson was one verification of the electroweak unification. The paper by Weinberg in which he presented this theory was one of the highest cited theoretical works ever in high energy physics as of 2009.^[5]

After his 1967 seminal work on the unification of weak and electromagnetic interactions, Steven Weinberg continued his work in many aspects of particle physics, quantum field theory, gravity, supersymmetry, superstrings and cosmology, as well as a theory called Technicolor.

In the years after 1967, the full Standard Model of elementary particle theory was developed through the work of many contributors. In it, the weak and electromagnetic interactions already unified by the work of Weinberg, Abdus Salam and Sheldon Glashow, are made consistent with a theory of the strong interactions between quarks, in one overarching theory. In 1973 Weinberg proposed a modification of the Standard Model which did not contain that model's fundamental Higgs boson.

Weinberg became Higgins Professor of Physics at Harvard University in 1973.

It is of special importance that in 1979 he pioneered the modern view on the renormalization aspect of quantum field theory that considers all quantum field theories as effective field theories and changed completely the viewpoint of previous work (including his own in his 1967 paper) that a sensible quantum field theory must be renormalizable.^[6] This approach allowed the development of effective theory of quantum gravity,^[7] low energy QCD, heavy quark effective field theory and other developments, and it is a topic of considerable interest in current research.

In 1979, some six years after the experimental discovery of the neutral currents — i.e. the discovery of the inferred existence of the Z boson — but following the 1978 experimental discovery of the theory's predicted amount of parity violation due to Z bosons' mixing with electromagnetic interactions, Weinberg was awarded the Nobel Prize in

Physics, together with Sheldon Glashow, and Abdus Salam who had independently proposed a theory of electroweak unification based on spontaneous symmetry breaking.

In 1982 Weinberg moved to the University of Texas at Austin as the Jack S. Josey-Welch Foundation Regents Chair in Science and founded the *Theory Group* of the Physics Department.

There is current (2008) interest in Weinberg's 1976 proposal of the existence of new strong interactions^[8] -- a proposal dubbed "Technicolor" by Leonard Susskind -- because of its chance of being observed in the LHC as an explanation of the hierarchy problem.

Steven Weinberg's influence and importance are confirmed by the fact that he is frequently among the top scientists with highest research effect indices, such as the h-index and the creativity index.^[9]

Other intellectual legacy

Besides his scientific research, Steven Weinberg has been a prominent public spokesman for science, testifying before Congress in support of the Superconducting Super Collider, writing articles for the *New York Review of Books*,^[10] and giving various lectures on the larger meaning of science. His books on science written for the public combine the typical scientific popularization with what is traditionally considered history and philosophy of science and atheism.

Weinberg was a major participant in what is known as the Science Wars, standing with Paul R. Gross, Norman Levitt, Alan Sokal, Lewis Wolpert, and Richard Dawkins, on the side arguing for the hard realism of science and scientific knowledge and against the constructionism proposed by such social scientists as Stanley Aronowitz, Barry Barnes, David Bloor, David Edge, Harry Collins, Steve Fuller, and Bruno Latour.

Weinberg is also known for his support of Israel. He wrote an essay titled "Zionism and Its Cultural Adversaries" to explain his views on the issue.

Weinberg has canceled trips to universities in the United Kingdom because of British boycotts directed towards Israel. He has explained:

"Given the history of the attacks on Israel and the oppressiveness and aggressiveness of other countries in the Middle East and elsewhere, boycotting Israel indicated a moral blindness for which it is hard to find any explanation other than antisemitism."^[11]

His views on religion were expressed in a speech from 1999 in Washington, D.C.:

"With or without religion, good people can behave well and bad people can do evil; but for good people to do evil—that takes religion."^[12]

He has also said:

"The more the universe seems comprehensible, the more it seems pointless."^[13]

He attended and was a speaker at the Beyond Belief symposium in November 2006.

Personal

He is married to Louise Weinberg and has one daughter, Elizabeth.

Honors and awards

The honors and awards that Professor Weinberg received include:

- Honorary Doctor of Science degrees from a dozen institutions: University of Chicago, Knox College, University of Rochester, Yale University, City University of New York, Dartmouth College, Weizmann Institute, Clark University, Washington College, Columbia University, Bates College.
 - American Academy of Arts and Sciences, elected 1968
-

- National Academy of Sciences, elected 1972
- J. R. Oppenheimer Prize, 1973
- Dannie Heineman Prize for Mathematical Physics, 1977
- Steel Foundation Science Writing Award, 1977, for authorship of *The First Three Minutes* (1977)
- Elliott Cresson Medal (Franklin Institute), 1979
- Nobel Prize in Physics, 1979
- Elected to American Philosophical Society, Royal Society of London (Foreign Honorary Member), Philosophical Society of Texas
- James Madison Medal of Princeton University, 1991
- National Medal of Science, 1991
- Lewis Thomas Prize for Writing about Science, 1999.
- 2002 Humanist of the Year, American Humanist Association
- James Joyce Award, University College Dublin, 2009

Selected publications

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- A Designer Universe? ^[18], a refutation of attacks on the theories of evolution and cosmology (e.g., those conducted under the rubric of intelligent design) is based on a talk given in April 1999 at the Conference on Cosmic Design of the American Association for the Advancement of Science in Washington, D.C. This and other works express Weinberg's strongly held position that scientists should be less passive in defending science against anti-science religiosity.

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External links

- Biography and Bibliographic Resources (<http://www.osti.gov/accomplishments/weinberg.html>), from the Office of Scientific and Technical Information, United States Department of Energy
- Home Page of Steven Weinberg at University of Texas at Austin (<http://www.ph.utexas.edu/~weintech/weinberg.html>)
- Steven Weinberg on LHC (<http://www.youtube.com/watch?v=Zl4W3DYTIKw>)
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- Publications (http://arxiv.org/find/hep-th/1/au:+Weinberg_S/0/1/0/all/0/1) on ArXiv

Claude Shannon

Claude Shannon	
 <p>Claude Elwood Shannon (1916-2001)</p>	
Born	April 30, 1916Petoskey, Michigan, United States
Died	February 24, 2001 (aged 84)Medford, Massachusetts, United States
Residence	United States
Nationality	American
Fields	Mathematics and electronic engineering
Institutions	Bell Laboratories Massachusetts Institute of Technology Institute for Advanced Study
Alma mater	University of Michigan Massachusetts Institute of Technology
Doctoral advisor	Frank Lauren Hitchcock
Doctoral students	Danny Hillis Ivan Edward Sutherland William Robert Sutherland Heinrich Ernst
Known for	Information Theory Shannon–Fano coding Shannon–Hartley law Nyquist–Shannon sampling theorem Noisy channel coding theorem Shannon switching game Shannon number Shannon index Shannon's source coding theorem Shannon's expansion Shannon-Weaver model of communication Whittaker–Shannon interpolation formula
Notable awards	IEEE Medal of Honor Kyoto Prize

Claude Elwood Shannon (April 30, 1916 – February 24, 2001) was an American mathematician, electronic engineer, and cryptographer known as "the father of information theory".^[1]

Shannon is famous for having founded information theory with one landmark paper published in 1948. But he is also credited with founding both digital computer and digital circuit design theory in 1937, when, as a 21-year-old master's student at MIT, he wrote a thesis demonstrating that electrical application of Boolean algebra could

construct and resolve any logical, numerical relationship. It has been claimed that this was the most important master's thesis of all time.^[2] Shannon contributed to the field of cryptanalysis during World War II and afterwards, including basic work on code breaking.

Biography

Shannon was born in Petoskey, Michigan. His father, Claude Sr (1862–1934), a descendant of early New Jersey settlers, was a businessman and for a while, Judge of Probate. His mother, Mabel Wolf Shannon (1890–1945), daughter of German immigrants, was a language teacher and for a number of years principal of Gaylord High School, Michigan. The first 16 years of Shannon's life were spent in Gaylord, Michigan, where he attended public school, graduating from Gaylord High School in 1932. Shannon showed an inclination towards mechanical things. His best subjects were science and mathematics, and at home he constructed such devices as models of planes, a radio-controlled model boat and a telegraph system to a friend's house half a mile away. While growing up, he worked as a messenger for Western Union. His childhood hero was Thomas Edison, who he later learned was a distant cousin. Both were descendants of John Ogden, a colonial leader and an ancestor of many distinguished people.^[3] ^[4]

Boolean theory

In 1932 he entered the University of Michigan, where he took a course that introduced him to the works of George Boole. He graduated in 1936 with two bachelor's degrees, one in electrical engineering and one in mathematics, then began graduate study at the Massachusetts Institute of Technology (MIT), where he worked on Vannevar Bush's differential analyzer, an analog computer.

While studying the complicated ad hoc circuits of the differential analyzer, Shannon saw that Boole's concepts could be used to great utility. A paper drawn from his 1937 master's thesis, *A Symbolic Analysis of Relay and Switching Circuits*,^[5] was published in the 1938 issue of the *Transactions of the American Institute of Electrical Engineers*. It also earned Shannon the Alfred Noble American Institute of American Engineers Award in 1940. Howard Gardner, of Harvard University, called Shannon's thesis "possibly the most important, and also the most famous, master's thesis of the century."

Victor Shestakov, at Moscow State University, had proposed a theory of electric switches based on Boolean logic earlier than Shannon, in 1935, but the first publication of Shestakov's result took place in 1941, after the publication of Shannon's thesis.

In this work, Shannon proved that Boolean algebra and binary arithmetic could be used to simplify the arrangement of the electromechanical relays then used in telephone routing switches, then expanded the concept and also proved that it should be possible to use arrangements of relays to solve Boolean algebra problems. Exploiting this property of electrical switches to do logic is the basic concept that underlies all electronic digital computers. Shannon's work became the foundation of practical digital circuit design when it became widely known among the electrical engineering community during and after World War II. The theoretical rigor of Shannon's work completely replaced the *ad hoc* methods that had previously prevailed.

Flush with this success, Vannevar Bush suggested that Shannon work on his dissertation at Cold Spring Harbor Laboratory, funded by the Carnegie Institution headed by Bush, to develop similar mathematical relationships for Mendelian genetics, which resulted in Shannon's 1940 PhD thesis at MIT, *An Algebra for Theoretical Genetics*.^[6]

In 1940, Shannon became a National Research Fellow at the Institute for Advanced Study in Princeton, New Jersey. At Princeton, Shannon had the opportunity to discuss his ideas with influential scientists and mathematicians such as Hermann Weyl and John von Neumann, and even had the occasional encounter with Albert Einstein. Shannon worked freely across disciplines, and began to shape the ideas that would become information theory.^[7]

Wartime research

Shannon then joined Bell Labs to work on fire-control systems and cryptography during World War II, under a contract with section D-2 (Control Systems section) of the National Defense Research Committee (NDRC).

For two months early in 1943, Shannon came into contact with the leading British cryptanalyst and mathematician Alan Turing. Turing had been posted to Washington to share with the US Navy's cryptanalytic service the methods used by the British Government Code and Cypher School at Bletchley Park to break the ciphers used by the German U-boats in the North Atlantic.^[8] He was also interested in the encipherment of speech and to this end spent time at Bell Labs. Shannon and Turing met at teatime in the cafeteria.^[8] Private archives from Bell Labs suggest that a Visual Binary encoding system was developed via their collaboration at this time.

VISUAL BINARY				(C) 1941 BELL LABS			
ZEROPOINT							
1 (bits)							
	0	1		1			
2	00	01		10	11		
3	000	001	010	011	100	101	110
4							
5							
6							
7							
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Turing showed Shannon his seminal 1936 paper that defined what is now known as the "Universal Turing machine"^{[9] [10]} which impressed him, as many of its ideas were complementary to his own.

In 1945, as the war was coming to an end, the NDRC was issuing a summary of technical reports as a last step prior to its eventual closing down. Inside the volume on fire control a special essay titled *Data Smoothing and Prediction in Fire-Control Systems*, coauthored by Shannon, Ralph Beebe Blackman, and Hendrik Wade Bode, formally treated the problem of smoothing the data in fire-control by analogy with "the problem of separating a signal from interfering noise in communications systems."^[11] In other words it modeled the problem in terms of data and signal processing and thus heralded the coming of the information age.

His work on cryptography was even more closely related to his later publications on communication theory.^[12] At the close of the war, he prepared a classified memorandum for Bell Telephone Labs entitled "A Mathematical Theory of Cryptography," dated September, 1945. A declassified version of this paper was subsequently published in 1949 as "Communication Theory of Secrecy Systems" in the *Bell System Technical Journal*. This paper incorporated many of the concepts and mathematical formulations that also appeared in his *A Mathematical Theory of Communication*. Shannon said that his wartime insights into communication theory and cryptography developed simultaneously and "they were so close together you couldn't separate them".^[13] In a footnote near the beginning of the classified report, Shannon announced his intention to "develop these results ... in a forthcoming memorandum on

the transmission of information."^[14]

While at Bell Labs, he proved that the one-time pad is unbreakable in his World War II research that was later published in October 1949. He also proved that any unbreakable system must have essentially the same characteristics as the one-time pad: the key must be truly random, as large as the plaintext, never reused in whole or part, and kept secret.^[15]

Postwar contributions

In 1948 the promised memorandum appeared as "A Mathematical Theory of Communication", an article in two parts in the July and October issues of the *Bell System Technical Journal*. This work focuses on the problem of how best to encode the information a sender wants to transmit. In this fundamental work he used tools in probability theory, developed by Norbert Wiener, which were in their nascent stages of being applied to communication theory at that time. Shannon developed information entropy as a measure for the uncertainty in a message while essentially inventing the field of information theory.

The book, co-authored with Warren Weaver, *The Mathematical Theory of Communication*, reprints Shannon's 1948 article and Weaver's popularization of it, which is accessible to the non-specialist. Shannon's concepts were also popularized, subject to his own proofreading, in John Robinson Pierce's *Symbols, Signals, and Noise*.

Information theory's fundamental contribution to natural language processing and computational linguistics was further established in 1951, in his article "Prediction and Entropy of Printed English", proving that treating whitespace as the 27th letter of the alphabet actually lowers uncertainty in written language, providing a clear quantifiable link between cultural practice and probabilistic cognition.

Another notable paper published in 1949 is "Communication Theory of Secrecy Systems", a declassified version of his wartime work on the mathematical theory of cryptography, in which he proved that all theoretically unbreakable ciphers must have the same requirements as the one-time pad. He is also credited with the introduction of sampling theory, which is concerned with representing a continuous-time signal from a (uniform) discrete set of samples. This theory was essential in enabling telecommunications to move from analog to digital transmissions systems in the 1960s and later.

He returned to MIT to hold an endowed chair in 1956.

Hobbies and inventions

Outside of his academic pursuits, Shannon was interested in juggling, unicycling, and chess. He also invented many devices, including rocket-powered flying discs, a motorized pogo stick, and a flame-throwing trumpet for a science exhibition. One of his more humorous devices was a box kept on his desk called the "Ultimate Machine", based on an idea by Marvin Minsky. Otherwise featureless, the box possessed a single switch on its side. When the switch was flipped, the lid of the box opened and a mechanical hand reached out, flipped off the switch, then retracted back inside the box. Renewed interest in the "Ultimate Machine" has emerged on YouTube and Thingiverse. In addition he built a device that could solve the Rubik's cube puzzle.^[3]

He is also considered the co-inventor of the first wearable computer along with Edward O. Thorp.^[16] The device was used to improve the odds when playing roulette.

Legacy and tributes

Shannon came to MIT in 1956 to join its faculty and to conduct work in the Research Laboratory of Electronics (RLE). He continued to serve on the MIT faculty until 1978. To commemorate his achievements, there were celebrations of his work in 2001, and there are currently six statues of Shannon sculpted by Eugene L. Daub: one at the University of Michigan; one at MIT in the Laboratory for Information and Decision Systems; one in Gaylord, Michigan; one at the University of California, San Diego; one at Bell Labs; and another at AT&T Shannon Labs.^[17]

After the breakup of the Bell system, the part of Bell Labs that remained with AT&T was named Shannon Labs in his honor.

Robert Gallager has called Shannon the greatest scientist of the 20th century. According to Neil Sloane, an AT&T Fellow who co-edited Shannon's large collection of papers in 1993, the perspective introduced by Shannon's communication theory (now called information theory) is the foundation of the digital revolution, and every device containing a microprocessor or microcontroller is a conceptual descendant of Shannon's 1948 publication:^[18] "He's one of the great men of the century. Without him, none of the things we know today would exist. The whole digital revolution started with him."^[19]

Shannon developed Alzheimer's disease, and spent his last few years in a Massachusetts nursing home. He was survived by his wife, Mary Elizabeth Moore Shannon; a son, Andrew Moore Shannon; a daughter, Margarita Shannon; a sister, Catherine S. Kay; and two granddaughters.^[20] ^[21]

Shannon was oblivious to the marvels of the digital revolution because his mind was ravaged by Alzheimer's disease. His wife mentioned in his obituary that had it not been for Alzheimer's "he would have been bemused" by it all.^[19]

Other work

Shannon's mouse

Theseus, created in 1950, was a magnetic mouse controlled by a relay circuit that enabled it to move around a maze of 25 squares. Its dimensions were the same as an average mouse.^[1] The maze configuration was flexible and it could be modified at will.^[1] The mouse was designed to search through the corridors until it found the target. Having travelled through the maze, the mouse would then be placed anywhere it had been before and because of its prior experience it could go directly to the target. If placed in unfamiliar territory, it was programmed to search until it reached a known location and then it would proceed to the target, adding the new knowledge to its memory thus learning.^[1] Shannon's mouse appears to have been the first learning device of its kind.^[1]



Shannon and his famous electromechanical mouse *Theseus* (named after Theseus from Greek mythology) which he tried to have solve the maze in one of the first experiments in artificial intelligence

Shannon's computer chess program

In 1950 Shannon published a groundbreaking paper on computer chess entitled *Programming a Computer for Playing Chess*. It describes how a machine or computer could be made to play a reasonable game of chess. His process for having the computer decide on which move to make is a minimax procedure, based on an evaluation function of a given chess position. Shannon gave a rough example of an evaluation function in which the value of the black position was subtracted from that of the white position. *Material* was counted according to the usual relative chess piece relative value (1 point for a pawn, 3 points for a knight or bishop, 5 points for a rook, and 9 points for a queen).^[22] He considered some positional factors, subtracting $\frac{1}{2}$ point for each doubled pawns, backward pawn, and isolated pawn. Another positional factor in the evaluation function was *mobility*, adding 0.1 point for each legal move available. Finally, he considered checkmate to be the capture of the king, and gave the king the artificial value of 200 points. Quoting from the paper:

The coefficients .5 and .1 are merely the writer's rough estimate. Furthermore, there are many other terms that should be included. The formula is given only for illustrative purposes. Checkmate has been artificially included here by giving the king the large value 200 (anything greater than the maximum of all other terms

would do).

The evaluation function is clearly for illustrative purposes, as Shannon stated. For example, according to the function, pawns that are doubled as well as isolated would have no value at all, which is clearly unrealistic.

The Las Vegas connection: Information theory and its applications to game theory

Shannon and his wife Betty also used to go on weekends to Las Vegas with M.I.T. mathematician Ed Thorp,^[23] and made very successful forays in blackjack using game theory type methods co-developed with fellow Bell Labs associate, physicist John L. Kelly Jr. based on principles of information theory.^[24] They made a fortune, as detailed in the book *Fortune's Formula* by William Poundstone and corroborated by the writings of Elwyn Berlekamp,^[25] Kelly's research assistant in 1960 and 1962.^[2] Shannon and Thorp also applied the same theory, later known as the *Kelly criterion*, to the stock market with even better results.^[26] Over the decades, Kelly's scientific formula has become a part of mainstream investment theory^[27] and the most prominent users, well-known and successful billionaire investors Warren Buffett,^[28] [29] Bill Gross^[30] and Jim Simons use Kelly methods. Warren Buffett met Thorp the first time in 1968. It's said that Buffett uses a form of the Kelly criterion in deciding how much money to put into various holdings. Also Elwyn Berlekamp had applied the same logical algorithm for Axcom Trading Advisors, an alternative investment management company, that he had founded. Berlekmap's company was acquired by Jim Simons and his Renaissance Technologies Corp hedge fund in 1992, whereafter its investment instruments were either subsumed into (or essentially renamed as) Renaissance's flagship Medallion Fund. But as Kelly's original paper demonstrates, the criterion is only valid when the investment or "game" is played many times over, with the same probability of winning or losing each time, and the same payout ratio.^[31]

The theory was also exploited by the famous *MIT Blackjack Team*, which was a group of students and ex-students from the Massachusetts Institute of Technology, Harvard Business School, Harvard University, and other leading colleges who used card-counting techniques and other sophisticated strategies to beat casinos at blackjack worldwide. The team and its successors operated successfully from 1979 through the beginning of the 21st century. Many other blackjack teams have been formed around the world with the goal of beating the casinos.

Claude Shannon's card count techniques were explained in *Bringing Down the House*, the best-selling book published in 2003 about the MIT Blackjack Team by Ben Mezrich. In 2008 the book was adapted into a drama film titled *21*.

Shannon's maxim

Shannon formulated a version of Kerckhoffs' principle as "the enemy knows the system". In this form it is known as "Shannon's maxim".

Biographical notes

He met his wife Betty when she was a numerical analyst at Bell Labs.

Awards and honors list

- Alfred Noble Prize, 1939
- Morris Liebmann Memorial Prize of the Institute of Radio Engineers, 1949^[32]
- Yale University (Master of Science), 1954
- Stuart Ballantine Medal of the Franklin Institute, 1955
- Research Corporation Award, 1956
- University of Michigan, honorary doctorate, 1961
- Rice University Medal of Honor, 1962
- Princeton University, honorary doctorate, 1962
- Marvin J. Kelly Award, 1962
- University of Edinburgh, honorary doctorate, 1964
- University of Pittsburgh, honorary doctorate, 1964
- Medal of Honor of the Institute of Electrical and Electronics Engineers, 1966^[33]
- National Medal of Science, 1966, presented by President Lyndon B. Johnson
- Golden Plate Award, 1967
- Northwestern University, honorary doctorate, 1970
- Harvey Prize, the Technion of Haifa, Israel, 1972
- Royal Netherlands Academy of Arts and Sciences (KNAW), foreign member, 1975
- University of Oxford, honorary doctorate, 1978
- Joseph Jacquard Award, 1978
- Harold Pender Award, 1978
- University of East Anglia, honorary doctorate, 1982
- Carnegie Mellon University, honorary doctorate, 1984
- Audio Engineering Society Gold Medal, 1985
- Kyoto Prize, 1985
- Tufts University, honorary doctorate, 1987
- University of Pennsylvania, honorary doctorate, 1991
- Basic Research Award, Eduard Rhein Foundation, Germany, 1991^[34]
- National Inventors Hall of Fame inducted, 2004

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Shannon videos

- Shannon's video machines (<http://www.youtube.com/watch?v=sBHGzRxfeJY>)
- Shannon - father of the information age (http://www.youtube.com/watch?v=z2Whj_nL-x8)
- AT&T Tech Channel's Tech Icons - Claude Shannon (<http://techchannel.att.com/play-video.cfm/2011/4/19/Tech-Icons-Claude-Shannon>)

External links

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- Shannon's math genealogy (<http://www.genealogy.math.ndsu.nodak.edu/id.php?id=42920>)

- Shannon's NNDB profile (<http://www.nndb.com/people/934/000023865/>)
 - Works by or about Claude Shannon (<http://worldcat.org/identities/lccn-n92-78142>) in libraries (WorldCat catalog)
 - *A Mathematical Theory of Communication* (<http://cm.bell-labs.com/cm/ms/what/shannonday/paper.html>)
 - *Communication Theory of Secrecy Systems* (<http://netlab.cs.ucla.edu/wiki/files/shannon1949.pdf>)
 - *Communication in the Presence of Noise* (<http://www.stanford.edu/class/ee104/shannonpaper.pdf>)
 - Summary of Shannon's life and career (<http://www.lucent.com/minds/infotheory/who.html>)
 - Biographical summary from Shannon's collected papers (<http://www.research.att.com/~njas/doc/shannonbio.html>)
 - Video documentary: "Claude Shannon - Father of the Information Age" (<http://www.ucsd.tv/search-details.asp?showID=6090>)
 - Mathematical Theory of Claude Shannon (<http://web.mit.edu/6.933/www/Fall2001/Shannon1.pdf>) In-depth MIT class paper on the development of Shannon's work to 1948.
 - Retrospective at the University of Michigan (<http://www.engin.umich.edu/150th/alum-legends/shannon.html>)
 - Shannon's University of Michigan profile (<http://www.engin.umich.edu/alumni/engineer/04SS/achievements/advances.html#shannon>)
 - Notes on Computer-Generated Text (<http://www.nightgarden.com/infosci.htm>)
 - Shannon's Juggling Theorem and Juggling Robots (<http://www2.bc.edu/~lewbcl/Shannon.html>)
 - Color Photo of Shannon, Juggling (http://www.stanstudio.com/Boston_Photographers/portfolio/nw_8.htm)
 - Shannon's paper on computer chess, text (<http://www.pi.infn.it/~carosi/chess/shannon.txt>)
 - Shannon's paper on computer chess (<http://www.ascotti.org/programming/chess/Shannon - Programming a computer for playing chess.pdf>)PDF (175 KiB)
 - Shannon's paper on computer chess, text, alternate source (<http://www.dcc.uchile.cl/~cgutierr/cursos/IA/shannon.txt>)
 - A Bibliography of His Collected Papers (<http://www.research.att.com/~njas/doc/shannonbib.html>)
 - A Register of His Papers in the Library of Congress ([http://memory.loc.gov/cgi-bin/query/r?faid/faid:@field\(DOCID+ms003071\)](http://memory.loc.gov/cgi-bin/query/r?faid/faid:@field(DOCID+ms003071)))
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Ludwig von Bertalanffy

Ludwig von Bertalanffy	
Born	19 September 1901Vienna, Austria
Died	12 June 1972 (aged 70)Buffalo, New York, USA
Fields	Biology and systems theory
Alma mater	University of Vienna
Known for	General System Theory
Influences	Rudolf Carnap, Gustav Theodor Fechner, Nicolai Hartmann, Otto Neurath, Moritz Schlick
Influenced	Russell L. Ackoff, Kenneth E. Boulding, Peter Checkland, C. West Churchman, Jay Wright Forrester, Ervin László, James Grier Miller, Anatol Rapoport

Karl Ludwig von Bertalanffy (September 19, 1901, Atzgersdorf near Vienna, Austria – June 12, 1972, Buffalo, New York, USA) was an Austrian-born biologist known as one of the founders of general systems theory (GST). GST is an interdisciplinary practice that describes systems with interacting components, applicable to biology, cybernetics, and other fields. Bertalanffy proposed that the laws of thermodynamics applied to closed systems, but not necessarily to "open systems," such as living things. His mathematical model of an organism's growth over time, published in 1934, is still in use today.

Von Bertalanffy grew up in Austria and subsequently worked in Vienna, London, Canada and the USA.

Biography

Ludwig von Bertalanffy was born and grew up in the little village of Atzgersdorf (now Liesing) near Vienna. The Bertalanffy family had roots in the 16th century nobility of Hungary which included several scholars and court officials.^[1] His grandfather Charles Joseph von Bertalanffy (1833–1912) had settled in Austria and was a state theatre director in Klagenfurt, Graz, and Vienna, which were important positions in imperial Austria. Ludwig's father Gustav von Bertalanffy (1861–1919) was a prominent railway administrator. On his mother's side Ludwig's grandfather Joseph Vogel was an imperial counsellor and a wealthy Vienna publisher. Ludwig's mother Charlotte Vogel was seventeen when she married the thirty-four year old Gustav. They divorced when Ludwig was ten, and both remarried outside the Catholic Church in civil ceremonies.^[2]

Ludwig von Bertalanffy grew up as an only child educated at home by private tutors until he was ten. When he went to the gymnasium/grammar school he was already well trained in self study, and kept studying on his own. His neighbour, the famous biologist Paul Kammerer, became a mentor and an example to the young Ludwig.^[3] In 1918 he started his studies at the university level with the philosophy and art history, first at the University of Innsbruck and then at the University of Vienna. Ultimately, Bertalanffy had to make a choice between studying philosophy of science and biology, and chose the latter because, according to him, one could always become a philosopher later, but not a biologist. In 1926 he finished his PhD thesis (translated title: Fechner and the problem of integration of higher order) on the physicist and philosopher Gustav Theodor Fechner.^[3]

Von Bertalanffy met his future wife Maria in April 1924 in the Austrian Alps, and were almost never apart for the next forty-eight years.^[4] She wanted to finish studying but never did, instead devoting her life to Bertalanffy's career. Later in Canada she would work both for him and with him in his career, and after his death she compiled two of Bertalanffy's last works. They had one child, who would follow in his father's footsteps by making his profession in the field of cancer research.

Von Bertalanffy was a professor at the University of Vienna from 1934–48, University of London (1948–49), Université de Montréal (1949), University of Ottawa (1950–54), University of Southern California (1955–58), the Menninger Foundation (1958–60), University of Alberta (1961–68), and State University of New York at Buffalo (SUNY) (1969–72). In 1972, he died from a sudden heart attack.

Work

Today, Bertalanffy is considered to be a founder and one of the principal authors of the interdisciplinary school of thought known as general systems theory. According to Weckowicz (1989), he "occupies an important position in the intellectual history of the twentieth century. His contributions went beyond biology, and extended into cybernetics, education, history, philosophy, psychiatry, psychology and sociology. Some of his admirers even believe that this theory will one day provide a conceptual framework for all these disciplines".^[1] Spending most of his life in semi-obscure, Ludwig von Bertalanffy may well be the least known intellectual titan of the twentieth century.^[5]

The individual growth model

The individual growth model published by von Bertalanffy in 1934 is widely used in biological models and exists in a number of permutations.

In its simplest version the so-called von Bertalanffy growth equation is expressed as a differential equation of length (L) over time (t):

$$L'(t) = r_B (L_{\infty} - L(t))$$

when r_B is the von Bertalanffy growth rate and L_{∞} the ultimate length of the individual. This model was proposed earlier by **A. Pütter** in 1920 (*Arch. Gesamte Physiol. Mensch. Tiere*, **180**: 298-340).

The **Dynamic Energy Budget theory** provides a mechanistic explanation of this model in the case of isomorphs that experience a constant food availability. The inverse of the von Bertalanffy growth rate appears to depend linearly on the ultimate length, when different food levels are compared. The intercept relates to the maintenance costs, the slope to the rate at which reserve is mobilized for use by metabolism. The ultimate length equals the maximum length at high food availabilities.^[6]

Bertalanffy Module

To honor Bertalanffy, ecological systems engineer and scientist Howard T. Odum named the storage symbol of his General Systems Language as the Bertalanffy module (see image right).^[7]

General System Theory (GST)

The biologist is widely recognized for his contributions to science as a systems theorist; specifically, for the development of a theory known as General System Theory (GST). The theory attempted to provide alternatives to conventional models of organization. GST defined new foundations and developments as a generalized theory of systems with applications to numerous areas of study, emphasizing holism over reductionism, organism over mechanism.

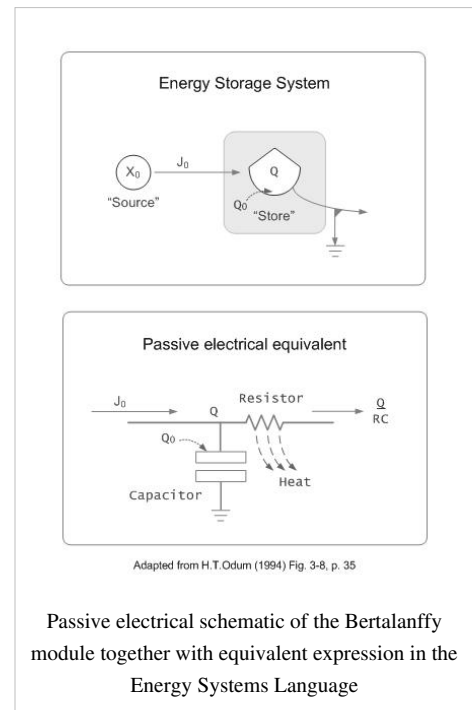
Open systems

Bertalanffy's contribution to systems theory is best known for his theory of open systems. The system theorist argued that traditional closed system models based on classical science and the second law of thermodynamics were untenable. Bertalanffy maintained that "the conventional formulation of physics are, in principle, inapplicable to the living organism being open system having steady state. We may well suspect that many characteristics of living systems which are paradoxical in view of the laws of physics are a consequence of this fact."^[8] However, while closed physical systems were questioned, questions equally remained over whether or not open physical systems could justifiably lead to a definitive science for the application of an open systems view to a general theory of systems.

In Bertalanffy's model, the theorist defined general principles of open systems and the limitations of conventional models. He ascribed applications to biology, information theory and cybernetics. Concerning biology, examples from the open systems view suggested they "may suffice to indicate briefly the large fields of application" that could be the "outlines of a wider generalization;"^[9] from which, a hypothesis for cybernetics. Although potential applications exist in other areas, the theorist developed only the implications for biology and cybernetics. Bertalanffy also noted unsolved problems, which included continued questions over thermodynamics, thus the unsubstantiated claim that there are physical laws to support generalizations (particularly for information theory), and the need for further research into the problems and potential with the applications of the open system view from physics.

Systems in the social sciences

In the social sciences, Bertalanffy did believe that general systems concepts were applicable, e.g. theories that had been introduced into the field of sociology from a modern systems approach that included "the concept of general system, of feedback, information, communication, etc."^[10] The theorist critiqued classical "atomistic" conceptions of social systems and ideation "such as 'social physics' as was often attempted in a reductionist spirit."^[11] Bertalanffy also recognized difficulties with the application of a new general theory to social science due to the complexity of the intersections between natural sciences and human social systems. However, the theory still encouraged for new developments from sociology, to anthropology, economics, political science, and psychology among other areas. Today, Bertalanffy's GST remains a bridge for interdisciplinary study of systems in the social sciences.



Publications

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- 1930, *Lebenswissenschaft und Bildung*, Stenger, Erfurt 1930
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- 1953, *Biophysik des Fließgleichgewichts*, Braunschweig: Vieweg. 2nd rev. ed. by W. Beier and R. Laue, East Berlin: Akademischer Verlag, 1977
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- [15] <http://www.bertalanffy.org>
- [16] <http://www.richardjung.cz/bert1.pdf>

External links

- International Society for the Systems Sciences' (<http://www.issss.org/lumLVB.htm>) biography of Ludwig von Bertalanffy
- Bertalanffy Center for the Study of Systems Science (<http://www.bertalanffy.org/>) BCSSS in Vienna.
- Ludwig von Bertalanffy (1901-1972): A Pioneer of General Systems Theory (<http://www.richardjung.cz/bert1.pdf>) working paper by T.E. Weckowicz, University of Alberta Center for Systems Research.

Stephen Smale

Stephen Smale	
Born	July 15, 1930
Nationality	 United States
Fields	Mathematics
Institutions	City University of Hong Kong University of Chicago Columbia University University of California, Berkeley
Alma mater	University of Michigan
Doctoral advisor	Raoul Bott
Doctoral students	Rufus Bowen John Guckenheimer Jacob Palis Themistocles M. Rassias
Notable awards	Wolf Prize (2006/07) National Medal of Science (1996) Fields Medal (1966)

Steven Smale a.k.a. **Steve Smale**, **Stephen Smale** (born July 15, 1930) is an American mathematician from Flint, Michigan. He was awarded the Fields Medal in 1966, and spent more than three decades on the mathematics faculty of the University of California, Berkeley (1960–61 and 1964–1995).

Education and career

He entered the University of Michigan in 1948. Initially, Smale was a good student, placing into an honors calculus sequence taught by Bob Thrall and earning himself A's. However, his sophomore and junior years were marred with mediocre grades, mostly Bs, Cs and even an F in nuclear physics. However, with some luck, Smale was accepted as a graduate student at the University of Michigan's mathematics department. Yet again, Smale performed poorly his first years, earning a C average as a graduate student. It was only when the department chair, Hildebrandt, threatened to kick out Smale, that he began to work hard. Smale finally earned his Ph.D. in 1957, under Raoul Bott.

Smale began his career as an instructor at the college at the University of Chicago. In 1958, he astounded the mathematical world with a proof of a sphere eversion. He then cemented his reputation with a proof of the Poincaré conjecture for all dimensions greater than or equal to 5; he later generalized the ideas in a 107 page paper that established the h-cobordism theorem.

After having made great strides in topology, he then turned to the study of dynamical systems, where he made significant advances as well. His first contribution is the Smale horseshoe that jumpstarted significant research in dynamical systems. He also outlined a research program carried out by many others. Smale is also known for injecting Morse theory into mathematical economics, as well as recent explorations of various theories of computation.

In 1998 he compiled a list of 18 problems in mathematics to be solved in the 21st century, known as Smale's problems. This list was compiled in the spirit of Hilbert's famous list of problems produced in 1900. In fact, Smale's list contains some of the original Hilbert problems, including the Riemann hypothesis and the second half of Hilbert's sixteenth problem, both of which are still unsolved. Other famous problems on his list include the Poincaré conjecture, the $P = NP$ problem, and the Navier-Stokes equations, all of which have been designated Millennium

Prize Problems by the Clay Mathematics Institute.

Earlier in his career, Smale was involved in controversy over remarks he made regarding his work habits while proving the higher dimensional Poincaré conjecture. He said that his best work had been done "on the beaches of Rio". This led to the withholding of his grant money from the NSF. He has been politically active in various movements in the past, such as the Free Speech movement. At one time he was subpoenaed by the House Un-American Activities Committee.

In 1960 Smale was appointed an associate professor of mathematics at the University of California, Berkeley, moving to a professorship at Columbia University the following year. In 1964 he returned to a professorship at UC Berkeley where he has spent the main part of his career. He retired from UC Berkeley in 1995 and took up a post as professor at the City University of Hong Kong. He also amassed over the years one of the finest private mineral collections in existence. Many of Smale's mineral specimens can be seen in the book - *The Smale Collection: Beauty in Natural Crystals*. [1]

Since 2002 Smale is a Professor at the Toyota Technological Institute at Chicago; starting August 1, 2009, he is also a Distinguished University Professor at the City University of Hong Kong.^[2]

In 2007, Smale was awarded the Wolf Prize in mathematics.^[3]

Important publications

- S. Smale, *Generalized Poincaré's conjecture in dimensions greater than four*, Annals of Mathematics, 2nd Ser., 74 (1961), no. 2, 391 – 406. (via JSTOR ^[4])
- S. Smale, *Differentiable dynamical systems*, Bulletin of the American Mathematical Society, 73 (1967), 747 – 817. ([5])
- F. Cucker & R Wong, *The Collected Papers of Stephen Smale*, ISBN 978-981-02-4307-4
- L. Blum, F. Cucker, M. Shub and S. Smale, *Complexity and Real Computation*, ISBN 0-387-98281-7.

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- [3] Press release (http://www.huji.ac.il/cgi-bin/dovrut/dovrut_search_eng_dev.pl?mesge116895485932688760)
- [4] <http://links.jstor.org/sici?sici=0003-486X%28196109%292%3A74%3A2%3C391%3AGPCIDG%3E2.0.CO%3B2-B>
- [5] <http://www.ams.org/bull/1967-73-06/S0002-9904-1967-11797-X/home.html>


External links

- Stephen Smale (<http://www.genealogy.ams.org/id.php?id=5086>) at the Mathematics Genealogy Project
- O'Connor, John J.; Robertson, Edmund F., "Stephen Smale" (<http://www-history.mcs.st-andrews.ac.uk/Biographies/Smale.html>), *MacTutor History of Mathematics archive*, University of St Andrews.
- Weisstein, Eric W., "Smale's Problems" (<http://mathworld.wolfram.com/SmaleProblems.html>) from MathWorld.
- Robion Kirby, *Stephen Smale: The Mathematician Who Broke the Dimension Barrier* (<http://www.ams.org/notices/200011/rev-kirby.pdf>), a book review of a biography in the Notices of the AMS.

Personal Website at Universities

- Steven Smale (<http://www.ee.cityu.edu.hk/~cccn/smale.htm>) at the City University of Hong Kong
 - Stephen Smale (<http://www.cityu.edu.hk/ma/staff/smale.html>) at the City University of Hong Kong
 - Stephen Smale (<http://ttic.uchicago.edu/~smale/vita.html>) at the University of Chicago
 - Steve Smale (<http://math.berkeley.edu/~smale/>) at the University of California, Berkeley
-

Yakov Sinai

Yakov G. Sinai	
 <p>Yakov G. Sinai</p>	
Born	September 21, 1935Moscow, Russian Soviet Federative Socialist Republic, USSR
Residence	Princeton, New Jersey, United States
Nationality	Russian / American
Fields	Mathematician
Institutions	Moscow State University, Princeton University
Alma mater	Moscow State University
Doctoral advisor	Andrey Kolmogorov
Doctoral students	Leonid Bunimovich Grigory Margulis Marina Ratner
Known for	works on dynamical systems, mathematical and statistical physics, probability theory, mathematical fluid dynamics
Notable awards	Boltzmann Medal (1986) Dannie Heineman Prize (1990) Dirac Prize (1992) Wolf Prize (1997) Nemmers Prize (2002) Henri Poincaré Prize (2009)

Yakov Grigorevich Sinai (Russian: Яков Григорьевич Сина́й; born September 21, 1935) is an influential mathematician working in the theory of dynamical systems, in mathematical physics and in probability theory. His work has shaped the modern metric theory of dynamical systems (also often called after Kolmogorov the theory of stochasticity of dynamical systems). Sinai has created bridges connecting the world of deterministic (dynamical) systems with the world of probabilistic (stochastic) systems.

Biography

Sinai was born in Moscow, USSR (now Russia) into a Jewish family that played a prominent role in Russia's scientific and cultural life since the nineteenth century. His grandfather Veniamin Kagan was a Russian geometer, and Sinai's parents were prominent researchers in the medical and biological sciences.

Yakov Sinai received his Ph.D. from Moscow State University in 1960; his advisor was Andrey Kolmogorov. In 1971 he became a Professor at Moscow State University and a senior researcher at the Landau Institute of Theoretical Physics. Since 1993 he has been a Professor of Mathematics at Princeton University.

Sinai is a member of the United States National Academy of Sciences, Russian Academy of Sciences and others. Among his awards are the Boltzmann Medal (1986), Dannie Heineman Prize for Mathematical Physics (1990), Dirac Medal (1992), the Wolf Prize in Mathematics (1997), Nemmers Prize (2002), and the Henri Poincaré Prize (2009). Sinai has worked, among other topics, on Kolmogorov–Sinai entropy, Sinai Billiards, Sinai's random walk, Sinai–Ruelle–Bowen measures, Pirogov–Sinai theory.

Sinai is highly respected in the physics community, where, as well as in mathematics, Kolmogorov-Sinai entropy, Sinai's billiards, Sinai's random walk, Sinai-Ruelle-Bowen measures, Pirogov-Sinai theory and his other achievements are basic notions that shaped the understanding of many fundamental physical phenomena.



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- Sinai ^[1] biography
- Another biography ^[2]
- Sinai ^[3] on scholarpedia
- Yakov Sinai ^[4] at the Mathematics Genealogy Project

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 [4] <http://www.genealogy.ams.org/id.php?id=10481>

Marston Morse

H. C. Marston Morse	
<div></div> <div>Marston Morse in 1965 (courtesy MFO)</div>	
Born	24 March 1892Waterville, Maine
Died	22 June 1977 (aged 85)Princeton, New Jersey
Nationality	 American
Fields	Mathematics
Institutions	Harvard University
Alma mater	Colby College Harvard University
Doctoral advisor	G. D. Birkhoff
Doctoral students	Emilio Baiada Gustav Hedlund Walter Leighton Sumner Myers
Known for	Morse theory

Harold Calvin Marston Morse (24 March 1892 – 22 June 1977) was an American mathematician best known for his work on the calculus of variations in the large, a subject where he introduced the technique of differential topology now known as Morse theory. In 1933 he was awarded the Bôcher Memorial Prize for his work in mathematical analysis.

He was born in Waterville, Maine to Ella Phoebe Marston and Howard Calvin Morse in 1892. He received his bachelor's degree from Colby College (also in Waterville) in 1914. At Harvard University, he received both his master's degree in 1915 and his Ph.D. in 1917.

He taught at Harvard, Brown, and Cornell Universities before accepting a position in 1935 at the Institute for Advanced Study in Princeton, where he remained until his retirement in 1962.

He spent most of his career on a single subject, eponymously titled Morse Theory, a branch of differential topology. Morse Theory is a very important subject in modern mathematical physics, such as string theory.

Quotes

"Mathematics are the result of mysterious powers which no one understands, and which the unconscious recognition of beauty must play an important part. Out of an infinity of designs a mathematician chooses one pattern for beauty's sake and pulls it down to earth."

Publications

- Morse, Marston (1981), Bott, Raoul, ed., *Selected papers* ^[1], Berlin, New York: Springer-Verlag, ISBN 978-0-387-90532-7, MR635124
- Morse, Marston (1987), Montgomery, Deane; Bott, Raoul, eds., *Collected papers. Vol. 1--6* ^[2], Singapore: World Scientific Publishing Co., ISBN 9789971978945, MR889255

External links

- O'Connor, John J.; Robertson, Edmund F., "Marston Morse" ^[3], *MacTutor History of Mathematics archive*, University of St Andrews.
- Marston Morse ^[4] at the Mathematics Genealogy Project


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- [4] <http://www.genealogy.ams.org/id.php?id=4926>

G. A. Hedlund

Gustav Arnold Hedlund, an American mathematician, was one of the founders of symbolic and topological dynamics. He was a student of Marston Morse.

William Ross Ashby

W. Ross Ashby	
	
Born	6 September 1903London, England
Died	15 November 1972 (aged 69)
Fields	Psychiatry, Cybernetics, Systems theory
Known for	Cybernetics, Law of Requisite Variety, Principle of Self-Organization
Influenced	Norbert Wiener, Ludwig von Bertalanffy, Herbert Simon, Stafford Beer and Stuart Kauffman

W. Ross Ashby (London, 6 September 1903 – 15 November 1972) was an English psychiatrist and a pioneer in cybernetics, the study of complex systems. His first name was not used: he was known as *Ross Ashby*.

His two books, *Design for a brain* and *An introduction to cybernetics*, were landmark works. They introduced exact, logical, thinking to the nascent discipline, and were highly influential.

Biography

William Ross Ashby was born in 1903 in London, where his father was working at an advertising agency.^[1] From 1917 to 1921 William studied at the Edinburgh Academy in Scotland, and from 1921 at Sidney Sussex College, Cambridge, where he received his B.A. in 1924 and his M.B. and B.Ch. in 1928. From 1924 to 1928 he worked at the St. Bartholomew's Hospital in London. Later on he also received a Diploma in Psychological Medicine in 1931, and an M.A. 1930 and M.D. from Cambridge in 1935.

Ross Ashby started working in 1930 as a Clinical Psychiatrist in the London County Council. From 1936 until 1947 he was a Research Pathologist in the St Andrew's Hospital in Northampton in England. From 1945 to 1947 he served in India where he was a Major in the Royal Army Medical Corps.

When he returned to England he served as Director of Research of the Barnwood House Hospital in Gloucester from 1947 until 1959. For a year he was Director of the Burden Neurological Institute in Bristol. In 1960 he went to the United States and became Professor, Depts. of Biophysics and Electrical Engineering, University of Illinois at Urbana-Champaign, until his retirement in 1970.^[2]

Ashby was president of the Society for General Systems Research from 1962 to 1964. He became a fellow of the Royal College of Psychiatry in 1971.

On March 4–6, 2004, a W. Ross Ashby centenary conference was held at the University of Illinois at Urbana-Champaign to mark the 100th anniversary of his birth. Presenters at the conference included Stuart Kauffman, Stephen Wolfram and George Klir.^[3] In February 2009 a special issue of the International Journal of General Systems was specifically devoted to Ashby and his work, containing papers from leading scholars such as Klaus Krippendorff, Stuart Umpleby and Kevin Warwick.^[4]

Work

Despite being widely influential within cybernetics, systems theory and, more recently, complex systems, he is not as well known as many of the notable scientists his work influenced including Herbert Simon, Norbert Wiener, Ludwig von Bertalanffy, Stafford Beer and Stuart Kauffman.^[5]

Journal

Ashby kept a journal for over 44 years in which he recorded his ideas about new theories. He started May 1928, when he was medical student at St. Bartholomew's Hospital in London. Over the years he wrote down a series of 25 volumes with intotal 7,400 pages. In 2003 these journals were given to The British Library, London, and since 2008, they were made available online as The W. Ross Ashby Digital Archive.^[6]

Cybernetics

Ross Ashby was one of the original members of the Ratio Club, a small informal dining club of young psychologists, physiologists, mathematicians and engineers who met to discuss issues in cybernetics. The club was founded in 1949 by the neurologist John Bates and continued to meet until 1958.

Earlier, in 1946, Alan Turing wrote a letter^[7] to Ashby suggesting he use Turing's Automatic Computing Engine (ACE) for his experiments instead of building a special machine. In 1948 Ashby made the Homeostat.^[8]

Variety

In *An Introduction to Cybernetics* Ashby formulated his Law of Requisite Variety^[9] stating that "variety absorbs variety, defines the minimum number of states necessary for a controller to control a system of a given number of states." This law can be applied for example to the number of bits necessary in a digital computer to produce a required description or model.

In response Conant (1970) produced his so called "Good Regulator theorem" stating that "every Good Regulator of a System Must be a Model of that System".^[10]

Stafford Beer applied Variety to found management cybernetics and the Viable System Model. Working independently Gregory Chaitin followed this with algorithmic information theory.

Publications

Books

- 1952. *Design for a Brain*^[11], Chapman & Hall.
- 1956. *An Introduction to Cybernetics*^[12], Chapman & Hall.
- 1981. Conant, Roger C. (ed.). *Mechanisms of Intelligence: Ross Ashby's Writings on Cybernetics*, Intersystems Publishers.

Articles, a selection

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- 1962. "Principles of the Self-Organizing System". In: Heinz Von Foerster and George W. Zopf, Jr. (eds.), *Principles of Self-Organization* (Sponsored by Information Systems Branch, U.S. Office of Naval Research). Republished as a PDF^[13] in *Emergence: Complexity and Organization* (E:CO) Special Double Issue Vol. 6, Nos.

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About W. Ross Ashby

- Asaro, Peter (2008). "From Mechanisms of Adaptation to Intelligence Amplifiers: The Philosophy of W. Ross Ashby,"^[14] in Michael Wheeler, Philip Husbands and Owen Holland (eds.) *The Mechanical Mind in History*, Cambridge, MA: MIT Press.

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- [3] W. Ross Ashby Centenary Conference (<http://www.rossashby.info/centenary.html>) The W. Ross Ashby Digital Archive, 2008
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- [12] <http://pespmc1.vub.ac.be/ASHBBOOK.html>
- [13] http://emergence.org/ECO_site/ECO_Archive/Issue_6_1-2/Ashby.pdf
- [14] <http://peterasaro.org/writing/Asaro%20Ashby.pdf>

External links


- The W. Ross Ashby Digital Archive (<http://www.rossashby.info/index.html>) includes an extensive biography, bibliography, letters, photographs, movies, and fully-indexed images of all 7,400 pages of Ashby's 25 volume journal.
- Homepage of William Ross Ashby (<http://www.gwu.edu/~asc/biographies/ashby/ashby.html>) with a short text from the *Encyclopædia Britannica Yearbook* 1973, and some links.
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- 159 Aphorisms from Ashby and further links at the Cybernetics Society (<http://www.cybsoc.org/ross.htm>)
- W. Ross Ashby, Cybernetics and Requisite Variety (<http://www.panarchy.org/ashby/variety.1956.html>) (1956) from *An Introduction to Cybernetics*
- W. Ross Ashby, Feedback, Adaptation and Stability (<http://www.panarchy.org/ashby/adaptation.1960.html>) (1960) from *Design for a Brain*
- What is Cybernetics? (http://www.youtube.com/watch?v=_hjAXkNbPfk) Livas short introductory videos on YouTube

Robert Rosen

Robert Rosen may refer to:

- Robert Rosén (born 1987), Swedish ice hockey player
 - Robert Rosen (theoretical biologist) (1934 – 1998), American theoretical biologist
 - Robert Rosen (writer) (born 1952), American author
 - Robert Ozn (born Robert M. Rosen), American producer, screenwriter, and entertainer
-

Edward Norton Lorenz

Edward Norton Lorenz	
<div><div>Edward Norton Lorenz</div></div>	
Born	May 23, 1917West Hartford, Connecticut, United States
Died	April 16, 2008 (aged 90)Cambridge, Massachusetts, United States
Residence	United States
Fields	Mathematics and Meteorology
Institutions	Massachusetts Institute of Technology
Alma mater	Dartmouth College (BA, 1938) Harvard University (Master's, 1940) Massachusetts Institute of Technology (SM, 1943; ScD, 1948)
Doctoral advisor	James Murdoch Austin
Doctoral students	Kevin E. Trenberth
Known for	Chaos theory Lorenz attractor Butterfly effect
Notable awards	Kyoto Prize (1991)

Edward Norton Lorenz (May 23, 1917 - April 16, 2008) was an American mathematician and meteorologist, and a pioneer of chaos theory.^[1] He discovered the strange attractor notion and coined the term *butterfly effect*.

Biography

Lorenz was born in West Hartford, Connecticut.^[2] He studied mathematics at both Dartmouth College in New Hampshire and Harvard University in Cambridge, Massachusetts. From 1942 until 1946, he served as a weather forecaster for the United States Army Air Corps. After his return from the war, he decided to study meteorology.^[1] Lorenz earned two degrees in the area from the Massachusetts Institute of Technology where he later was a professor for many years. He was a Professor Emeritus at MIT from 1987 until his death.^[1]

During the 1950s, Lorenz became skeptical of the appropriateness of the linear statistical models in meteorology, as most atmospheric phenomena involved in weather forecasting are non-linear.^[1] His work on the topic culminated in the publication of his 1963 paper *Deterministic Nonperiodic Flow* in Journal of the Atmospheric Sciences, and with it, the foundation of Chaos theory.^{[1] [3]} His description of the Butterfly effect followed in 1969,^{[1] [4] [5]} Kyoto Prize for basic sciences, in the field of earth and planetary sciences, in 1991,^[6] the Buys Ballot Award in 2004, and the Tomassoni Award in 2008. In his later years, he lived in Cambridge, Massachusetts. He was an avid outdoorsman, who enjoyed hiking, climbing, and cross-country skiing. He kept up with these pursuits until very late in his life, and

managed to continue most of his regular activities until only a few weeks before his death. According to his daughter, Cheryl Lorenz, Lorenz had "finished a paper a week ago with a colleague."^[7] On April 16, 2008, Lorenz died at his home in Cambridge at the age of 90, having suffered from cancer.^[8]

Awards

- 1969 Carl Gustaf Rossby Research Medal, American Meteorological Society.
- 1973 Symons Memorial Gold Medal, Royal Meteorological Society.
- 1975 Fellow, National Academy of Sciences (U.S.A.).
- 1981 Member, Norwegian Academy of Science and Letters.
- 1983 Crafoord Prize, Royal Swedish Academy of Sciences.
- 1984 Honorary Member, Royal Meteorological Society.
- 1989 Elliott Cresson Medal, The Franklin Institute
- 1991 Kyoto Prize for '... *his boldest scientific achievement in discovering "deterministic chaos"* '.
- 2004 Buys Ballot medal.
- 2004 Lomonosov Gold Medal

Work

Lorenz built a mathematical model of the way air moves around in the atmosphere. As Lorenz studied weather patterns he began to realize that they did not always change as predicted. Minute variations in the initial values of variables in his twelve variable computer weather model (c. 1960) would result in grossly divergent weather patterns.^[1] This sensitive dependence on initial conditions came to be known as the butterfly effect (it also meant that weather predictions from more than about a week out are generally fairly inaccurate).^[9]

Lorenz went on to explore the underlying mathematics and published his conclusions in a seminal work titled *Deterministic Nonperiodic Flow*, in which he described a relatively simple system of equations that resulted in a very complicated dynamical object now known as the Lorenz attractor.^[3]

Publications

Lorenz published several books and articles. A selection:

- 1955 Available potential energy and the maintenance of the general circulation. *Tellus*. Vol.7
- 1963 Deterministic nonperiodic flow. *Journal of Atmospheric Sciences*. Vol.20 : 130—141 link ^[10] ^[11]
- 1967 The nature and theory of the general circulation of atmosphere. *World Meteorological Organization*. No.218
- 1969 Three approaches to atmospheric predictability. *American Meteorological Society*. Vol.50
- 1972 Predictability: Does the Flap of a Butterfly's Wings in Brazil Set Off a Tornado in Texas?
- 1976 Nondeterministic theories of climate change. *Quaternary Research*. Vol.6
- 1990 Can chaos and intransitivity lead to interannual variability? *Tellus*. Vol.42A
- 2005 Designing Chaotic Models. *Journal of the Atmospheric Sciences*: Vol. 62, No. 5, pp. 1574—1587.

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- [10] http://eapsweb.mit.edu/research/Lorenz/Deterministic_63.pdf
- [11] According to the Web of Science online academic database, this paper has received at least 4000 unique citations by subsequent authors, making it one of the most-cited papers of all time.

External links

- List of Lorenz's publications, including downloadable pdfs (MIT website) (<http://eapsweb.mit.edu/research/Lorenz/publications.htm>)
- Video clip of Edward N. Lorenz speaking at the International Conference on Complex Systems, hosted by the New England Complex Systems Institute (NECSI) (<http://www.necsi.edu/events/iccs/video/iccs2002wednesday/3-lorenzclip.html>)
- Obituary (<http://www.telegraph.co.uk/news/main.jhtml?xml=/news/2008/04/18/db1801.xml>), *Daily Telegraph*, 18 April 2008.

Otto Rössler

Otto E. Rössler (born 20 May 1940) is a German biochemist and is notable for his work on chaos theory and his theoretical equation known as the Rössler attractor.

Biography


Rössler was born in Berlin. He was awarded his MD in 1966. Rössler then began his post doc at the Max Planck Institute for Behavioral Physiology, in Bavaria. In 1969, he started a visiting appointment at the Center for Theoretical Biology at SUNY-Buffalo. Later that year, he became Professor for Theoretical Biochemistry at the University of Tübingen. In 1976, he became a tenured University Docent. In 1994, he became Professor of Chemistry by decree.

Rössler has held visiting positions at the University of Guelph (Mathematics) in Canada, the Center for Nonlinear Studies ^[1] of the University of California at Los Alamos, the University of Virginia (Chemical Engineering), the Technical University of Denmark (Theoretical Physics), and the Santa Fe Institute (Complexity Research) in New Mexico.

In June 2008 Rössler emerged in the public eye ^[2] as a critic of the Large Hadron Collider (LHC) proton collision experiment supervised by the European Organization for Nuclear Research in Geneva and was involved in a failed law suit to halt its start up.

Rössler has authored around 300 scientific papers in fields as wide-ranging as biogenesis, the origin of language, differentiable automata, chaotic attractors, endophysics, micro relativity, artificial universes, the hypertext encyclopedia, and world-changing technology.

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- with Peter Weibel: *Aussenwelt – Innenwelt – Überwelt. Ein Gespräch*. 1997, (ISBN 3-87877-628-4)
- with Wilfried Kriese: *Mut zu Lampsacus. Das Internet als Chance*. 1998
- with Artur P. Schmidt: *Medium des Wissens. Das Menschenrecht auf Information*. 2000, (PDF ^[3]; 1,61 MB )

as well as the audio book CD *Descartes' Traum*, a compilation of his short lectures read by himself. 2002, (ISBN 3-932513-28-2)


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Paul Koebe

Paul Koebe	
Born	February 15, 1882Luckenwalde
Died	August 6, 1945 (aged 63)
Nationality	 Germany
Fields	Mathematics
Institutions	University of Leipzig University of Jena
Alma mater	University of Berlin
Academic advisors	Hermann Schwarz Friedrich Schottky
Notable students	Alfred Fischer Karl Georgi Georg Feigl C. Herbert Grötzsch Ernst Graeser Walter Brödel Jaroslav Tagamlitski
Known for	Koebe function Koebe 1/4 theorem
Notable awards	Ackermann–Teubner Memorial Award (1922)

Paul Koebe (February 15, 1882, Luckenwalde, Brandenburg – August 6, 1945) was a 20th-century German mathematician. His work dealt exclusively with the complex numbers, his most important results being on the uniformization of Riemann surfaces in a series of four papers in 1907–1909. He did his thesis at Berlin, where he worked under Herman Schwarz. He was an extraordinary professor at Leipzig from 1910 to 1914, then an ordinary professor at the University of Jena before returning to Leipzig in 1926 as an ordinary professor. He died in Leipzig.

Awards

- 1922, Ackermann–Teubner Memorial Award^[1]

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 - O'Connor, John J.; Robertson, Edmund F., "Paul Koebe" (<http://www-history.mcs.st-andrews.ac.uk/Biographies/Koebe.html>), *MacTutor History of Mathematics archive*, University of St Andrews.
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Jakob Nielsen (mathematician)

For other people with similar names see Jakob Nielsen.

Jakob Nielsen (October 15, 1890, Mjels–August 3, 1959, Helsingør) was a Danish mathematician known for his work on automorphisms of surfaces. He was born in the village Mjels on the island of Als in North Schleswig, in modern day Denmark. His mother died when he was 3, and in 1900 he went to live with his aunt and was enrolled in the Realgymnasium. In 1907 he was expelled for membership to an illicit student club. Nevertheless, he matriculated at the University of Kiel in 1908.

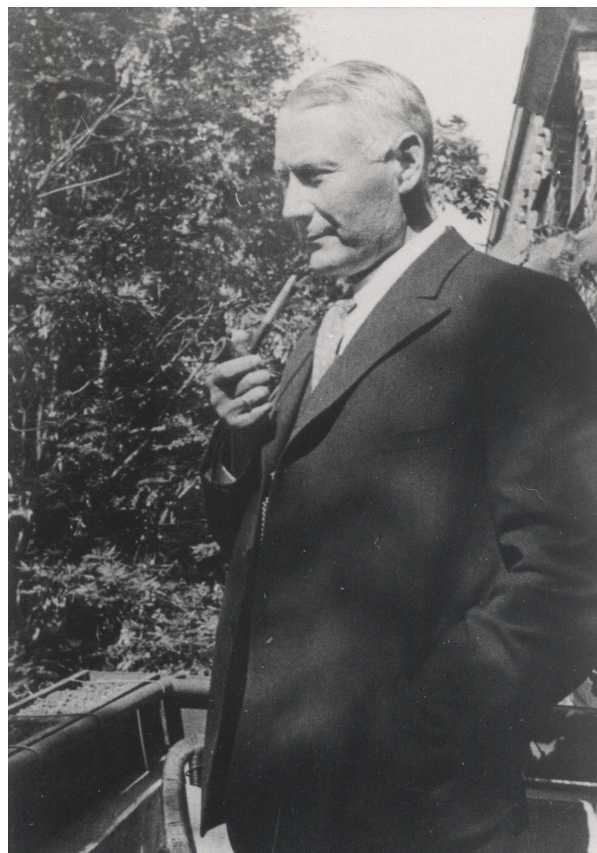
Nielsen completed his doctoral dissertation in 1913. Soon thereafter, he was drafted into the German Imperial Navy. He was assigned to coastal defense. In 1915 he was sent to Constantinople as a military adviser to the Turkish Government. After the war, in the spring of 1919, Nielsen married Carola von Pieverling, a German medical doctor.

In 1920 Nielsen took a position at the Technical University of Breslau. The next year he published a paper in *Mathematisk Tidsskrift* in which he proved that any subgroup of a finitely generated free group is free. In 1926 Otto Schreier would generalize this result by removing the condition that the free group be

finitely generated; this result is now known as the Nielsen–Schreier theorem. Also in 1921 Nielsen moved to the Royal Veterinary and Agricultural University in Copenhagen, where he would stay until 1925, when he moved to the Technical University in Copenhagen. He also proved the Dehn–Nielsen theorem on mapping class groups.

During World War II some efforts were made to bring Nielsen to the United States as it was feared that he would be assaulted by the Nazis. Nielsen would, in fact, stay in Denmark during the war without being harassed by the Nazis.

In 1951 Nielsen became professor of mathematics at the University of Copenhagen, taking the position vacated by the death of Harald Bohr. He resigned this position in 1955 because of his international undertakings, in particular with UNESCO, where he served on the executive board from 1952 to 1958.



Jakob Nielsen

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External links

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- Jakob Nielsen (mathematician) ^[4] at the Mathematics Genealogy Project

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Benoît Mandelbrot

Benoît Mandelbrot	
 <p>Mandelbrot in 2007</p>	
Born	20 November 1924Warsaw, Poland
Died	14 October 2010 (aged 85)Cambridge, Massachusetts, United States
Residence	Poland; France; United States
Nationality	Polish, French, American
Fields	Mathematics, Aerodynamics
Institutions	Yale University International Business Machines(IBM) Pacific Northwest National Laboratory
Alma mater	École Polytechnique California Institute of Technology University of Paris
Known for	Mandelbrot set, Fractals, Chaos Theory, Zipf–Mandelbrot law
Notable awards	Wolf Prize (1993) Japan Prize (2003) The Franklin Medal Légion d'honneur (the Legion of Honour)

Benoît B. Mandelbrot^[1] ^[2] (20 November 1924 – 14 October 2010) was a French American mathematician. Born in Poland, he moved to France with his family when he was a child. Mandelbrot spent much of his life living and working in the United States, acquiring dual French and American citizenship.

Mandelbrot worked on a wide range of mathematical problems, including mathematical physics and quantitative finance, but is best known as the father of fractal geometry. He coined the term fractal and described the Mandelbrot set. Mandelbrot extensively popularized his work, writing books and giving lectures aimed at the general public.

Mandelbrot spent most of his career at IBM's Thomas J. Watson Research Center, and was appointed as an IBM Fellow. He later became a Sterling Professor of Mathematical Sciences at Yale University. Mandelbrot also held positions at the Pacific Northwest National Laboratory, Université Lille Nord de France, Institute for Advanced Study and Centre National de la Recherche Scientifique.

Early years

Mandelbrot was born in Warsaw into a Jewish family from Lithuania.^[3] Mandelbrot was born into a family with a strong academic tradition - his mother was a physician and he was introduced to mathematics by two of his uncles, one of whom, Szolem Mandelbrojt, was a mathematician who resided in Paris. However, his father made his living trading clothing.^[4] Anticipating the threat posed by Nazi Germany, the family fled from Poland to France in 1936 when he was 11.^[5] Mandelbrot attended the Lycée Rolin in Paris until the start of World War II, when his family moved to Tulle, France. He was helped by Rabbi David Feuerwerker, the Rabbi of Brive-la-Gaillarde, to continue his studies.^[6] In 1944 he returned to Paris. He studied at the Lycée du Parc in Lyon and in 1945 - 47 attended the École Polytechnique, where he studied under Gaston Julia and Paul Lévy. From 1947 to 1949 he studied at California Institute of Technology, where he earned a master's degree in aeronautics.^[1] Returning to France, he obtained his Ph.D. degree in Mathematical Sciences at the University of Paris in 1952.^[4]

From 1949 to 1958 Mandelbrot was a staff member at the Centre National de la Recherche Scientifique. During this time he spent a year at the Institute for Advanced Study in Princeton, New Jersey, where he was sponsored by John von Neumann. In 1955 he married Aliette Kagan and moved to Geneva, Switzerland, and later to the Université Lille Nord de France.^[7] In 1958 the couple moved to the United States where Mandelbrot joined the research staff at the IBM Thomas J. Watson Research Center in Yorktown Heights, New York.^[7] He remained at IBM for 35 years, becoming an IBM Fellow, and later Fellow Emeritus.^[4]

Research career

From 1951 onward, Mandelbrot worked on problems and published papers not only in mathematics but in applied fields such as information theory, economics, and fluid dynamics. He became convinced that two key themes, fat tails and self-similar structure, ran through a multitude of problems encountered in those fields.

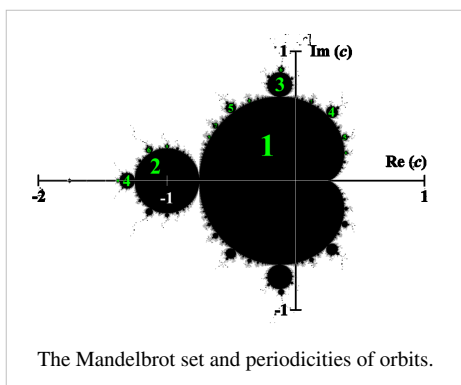
Mandelbrot found that price changes in financial markets did not follow a Gaussian distribution, but rather Lévy stable distributions having theoretically infinite variance. He found, for example, that cotton prices followed a Lévy stable distribution with parameter α equal to 1.7 rather than 2 as in a Gaussian distribution. "Stable" distributions have the property that the sum of many instances of a random variable follows the same distribution but with a larger scale parameter.^[8]

Mandelbrot also put his ideas to work in cosmology. He offered in 1974 a new explanation of Olbers' paradox (the "dark night sky" riddle), demonstrating the consequences of fractal theory as a sufficient, but not necessary, resolution of the paradox. He postulated that if the stars in the universe were fractally distributed (for example, like Cantor dust), it would not be necessary to rely on the Big Bang theory to explain the paradox. His model would not rule out a Big Bang, but would allow for a dark sky even if the Big Bang had not occurred.^[9]

In 1975, Mandelbrot coined the term *fractal* to describe these structures, and published his ideas in *Les objets fractals, forme, hasard et dimension* (1975; an English translation *Fractals: Form, Chance and Dimension* was published in 1977).^[10] Mandelbrot developed here ideas from the article *Deux types fondamentaux de distribution statistique*^[11] (1938; an English translation *Two Basic Types of Statistical Distribution*) of Czech geographer, demographer and statistician Jaromír Korčák.



Mandelbrot speaking in 2007



While on secondment as Visiting Professor of Mathematics at Harvard University in 1979, Mandelbrot began to study fractals called Julia sets that were invariant under certain transformations of the complex plane. Building on previous work by Gaston Julia and Pierre Fatou, Mandelbrot used a computer to plot images of the Julia sets of the formula $z^2 - \mu$. While investigating how the topology of these Julia sets depended on the complex parameter μ he studied the Mandelbrot set fractal that is now named after him. (Note that the Mandelbrot set is now usually defined in terms of the formula $z^2 + c$, so Mandelbrot's early plots in terms of the earlier parameter μ are left-right mirror images of more recent plots in terms of the parameter c .)

In 1982, Mandelbrot expanded and updated his ideas in *The Fractal Geometry of Nature*.^[12] This influential work brought fractals into the mainstream of professional and popular mathematics, as well as silencing critics, who had dismissed fractals as "program artifacts".

Mandelbrot left IBM in 1987, after 35 years and 12 days, when IBM decided to end pure research in his division.^[13] He joined the Department of Mathematics at Yale, and obtained his first tenured post in 1999, at the age of 75.^[14] At the time of his retirement in 2005, he was Sterling Professor of Mathematical Sciences. His awards include the Wolf Prize for Physics in 1993, the Lewis Fry Richardson Prize of the European Geophysical Society in 2000, the Japan Prize in 2003, and the Einstein Lectureship of the American Mathematical Society in 2006.

The small asteroid 27500 Mandelbrot was named in his honor. In November 1990, he was made a Knight in the French Legion of Honour. In December 2005, Mandelbrot was appointed to the position of Battelle Fellow at the Pacific Northwest National Laboratory.^[15] Mandelbrot was promoted to Officer of the Legion of Honour in January 2006.^[16] An honorary degree from Johns Hopkins University was bestowed on Mandelbrot in the May 2010 commencement exercises.^[17]



Mandelbrot speaking about the Mandelbrot set, during his acceptance speech for the Légion d'honneur in 2006

Fractals and regular roughness

Although Mandelbrot coined the term *fractal*, some of the mathematical objects he presented in *The Fractal Geometry of Nature* had been described by other mathematicians. Before Mandelbrot, they had been regarded as isolated curiosities with unnatural and non-intuitive properties. Mandelbrot brought these objects together for the first time and turned them into essential tools for the long-stalled effort to extend the scope of science to non-smooth objects in the real world. He highlighted their common properties, such as self-similarity (linear, non-linear, or statistical), scale invariance, and a (usually) non-integer Hausdorff dimension.



A limb of a maple tree, illustrating organic fractal branching.

He also emphasized the use of fractals as realistic and useful models of many "rough" phenomena in the real world.

Natural

fractals

include

the shapes of mountains, coastlines and river basins; the structures of plants, blood vessels and lungs; the clustering of galaxies; and Brownian motion. Fractals are found in human pursuits, such as music, painting, architecture, and stock market prices. Mandelbrot believed that fractals, far from being unnatural, were in many ways more intuitive and natural than the artificially smooth objects of traditional Euclidean geometry:

Clouds are not spheres, mountains are not cones,
coastlines are not circles, and bark is not smooth, nor does
lightning travel in a straight line.

—Mandelbrot, in his introduction to *The Fractal
Geometry of Nature*



Natural water frost crystal growth on cold glass,
showing fractal branching growth in a purely
physical system.

Mandelbrot has been called a visionary^[18] and a maverick.^[19] His informal and passionate style of writing and his emphasis on visual and geometric intuition (supported by the inclusion of numerous illustrations) made *The Fractal Geometry of Nature* accessible to non-specialists. The book sparked widespread popular interest in fractals and contributed to chaos theory and other fields of science and mathematics.

When visiting the Museu de la Ciència de Barcelona in 1988, he told its director that the painting *The Face of War* had given him "the intuition about the transcendence of the fractal geometry when making intelligible the omnipresent similitude in the forms of nature".^[20] He also said that, fractally, Gaudí was superior to Van der Rohe.^[20]

Death

Mandelbrot died in a hospice in Cambridge, Massachusetts, on 14 October 2010 from pancreatic cancer, at the age of 85.^[21] ^[22] Reacting to news of his death, mathematician Heinz-Otto Peitgen said "if we talk about impact inside mathematics, and applications in the sciences, he is one of the most important figures of the last 50 years."^[21] Chris Anderson described Mandelbrot as "an icon who changed how we see the world."^[23] French President Nicolas Sarkozy said Mandelbrot had "a powerful, original mind that never shied away from innovating and shattering preconceived notions". Sarkozy also added, "His work, developed entirely outside mainstream research, led to modern information theory."^[24] Mandelbrot's obituary in *The Economist* points out his fame as "celebrity beyond the academy" and lauds him as the "father of fractal geometry."^[25]

Honors and awards

A partial list of awards received by Mandelbrot:^[26]

- 2004 Best Business Book of the Year Award
 - AMS Einstein Lectureship
 - Barnard Medal
 - Caltech Service
 - Casimir Funk Natural Sciences Award
 - Charles Proteus Steinmetz Medal
 - Franklin Medal
 - Harvey Prize
 - Honda Prize
 - Humboldt Preis
 - Fellow, American Geophysical Union
 - IBM Fellowship
 - Japan Prize
 - John Scott Award
 - Légion d'honneur (Legion of Honour)
 - Lewis Fry Richardson Medal
 - Medaglia della Presidenza della Repubblica Italiana
 - Médaille de Vermeil de la Ville de Paris
 - Nevada Prize
 - Science for Art
 - Sven Berggren-Priset
 - Władysław Orlicz Prize
 - Wolf Foundation Prize for Physics
- Member of the Norwegian Academy of Science and Letters.^[27]

Notes

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External links

- Mandelbrot's page at Yale (<http://www.math.yale.edu/mandelbrot/>)
- Ted talk: "Benoît Mandelbrot: Fractals and the art of roughness" (http://www.ted.com/talks/benoit_mandelbrot_fractals_the_art_of_roughness.html)
- Interview at FT.com (<http://video.ft.com/v/63078298001/Why-efficient-markets-collapse-Mandelbrot>), in which Mandelbrot discusses his early work with stock market behavior, 2009-09-30

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